Transform Calculus and its Applications in Differential Equations
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Lecture - 45
Solution of Heat Equation and Wave Equation using Laplace Transform

In the last lecture, we had started the solution of partial differential equations using Laplace transform. We have seen the different kinds of partial differential equations present. Then we have discussed about the second order linear equations which maybe of the parabolic, hyperbolic or elliptic type, we have made the classification also. So, let us continue with that and let us take one more example and that of the heat equation.

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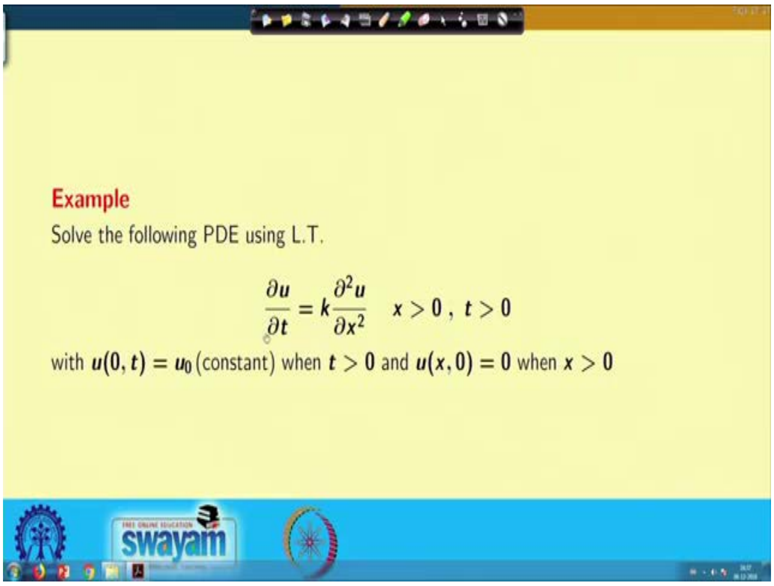

Let us solve the heat equation using Laplace transform.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad , \quad x > 0, t > 0$$

with $u(0,t) = u_0$(constant) when $t > 0$ and $u(x,0) = 0$ when $x > 0$.

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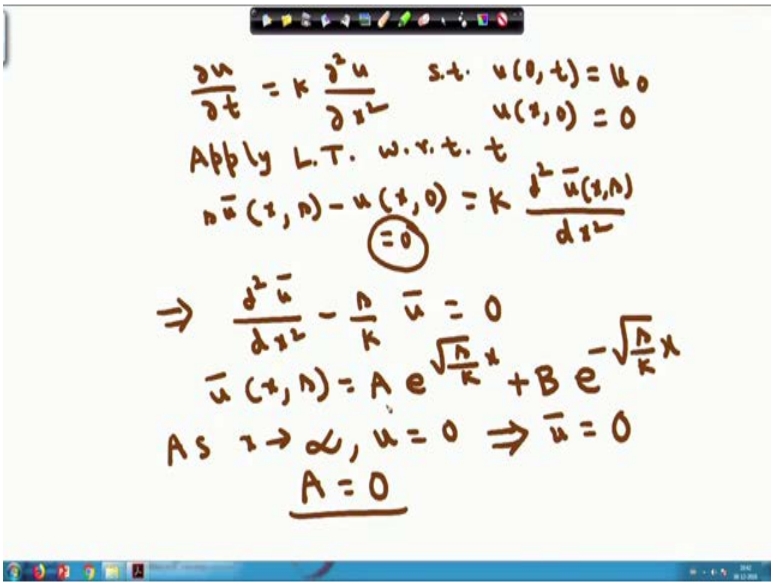

Here, both the variables have range from 0 to ∞. Since $u_x(0,t)$ is not given, we will take Laplace transform on both sides with respect to t. Now using the formulae for the Laplace transform for partial derivatives, we will obtain,

$$L\left\{\frac{\partial u}{\partial t}\right\} = kL\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow s\,\bar{u}(x,s) - u(x,0) = k\frac{d^2\bar{u}}{dx^2}$$

$$\Rightarrow \frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u} = 0 \quad (\because u(x,0) = 0\,)$$

where, $\bar{u}(x,s) = L\{u(x,t)\}$. Therefore, from the given PDE, we obtained an ODE.

Auxiliary equation for the above ODE is,

$$m^2 - \frac{s}{k} = 0 \Rightarrow m = \pm\sqrt{\frac{s}{k}}$$

Therefore, the general solution of the above ODE is given as,

$$\bar{u}(x,s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x} \tag{1}$$

where, A and B are the constants of integration. From the initial condition, we have $u(x,0) = 0$ when $x > 0$. So, we can say that, $u(x,t) \to 0$ as $x \to \infty$.

$$\therefore \bar{u}(x,s) \to 0 \text{ as } x \to \infty$$

$$\therefore (1) \Rightarrow A = 0$$

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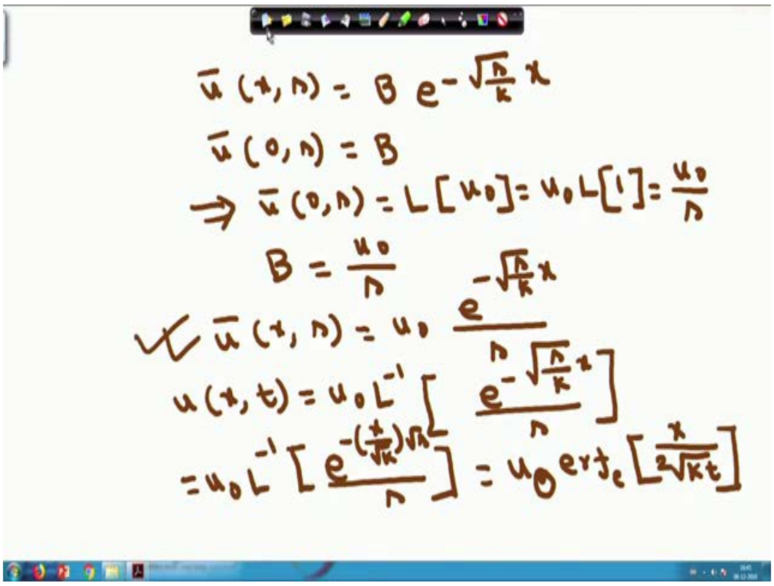

Again $u = u_0$ at $x = 0$.

$$\therefore \bar{u}(0,s) = L\{u_0\} = \frac{u_0}{s}$$

$$\therefore (1) \Rightarrow B = \frac{u_0}{s}$$

Now we have both the constants A and B.

$$\therefore \bar{u}(x,s) = u_0 \frac{e^{-\sqrt{\frac{s}{k}}x}}{s}$$

Now taking inverse Laplace transform, we can obtain $u(x,t)$ as,

$$u(x,t) = u_0 L^{-1}\left\{\frac{e^{-\sqrt{\frac{s}{k}}x}}{s}\right\}$$

Again, we know that,

$$L^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \mathrm{erf}_c\left(\frac{a}{2\sqrt{t}}\right)$$

$$\therefore u(x,t) = u_0 \, \mathrm{erf}_c\left(\frac{x}{2\sqrt{kt}}\right)$$

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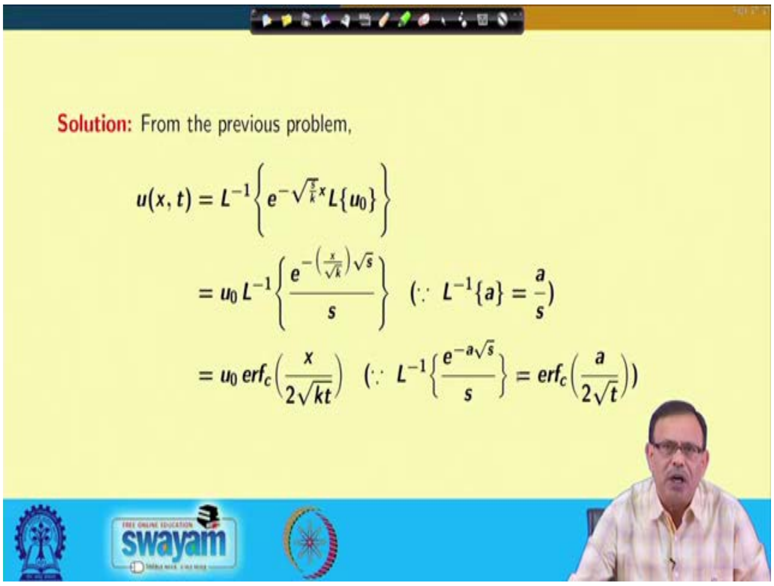

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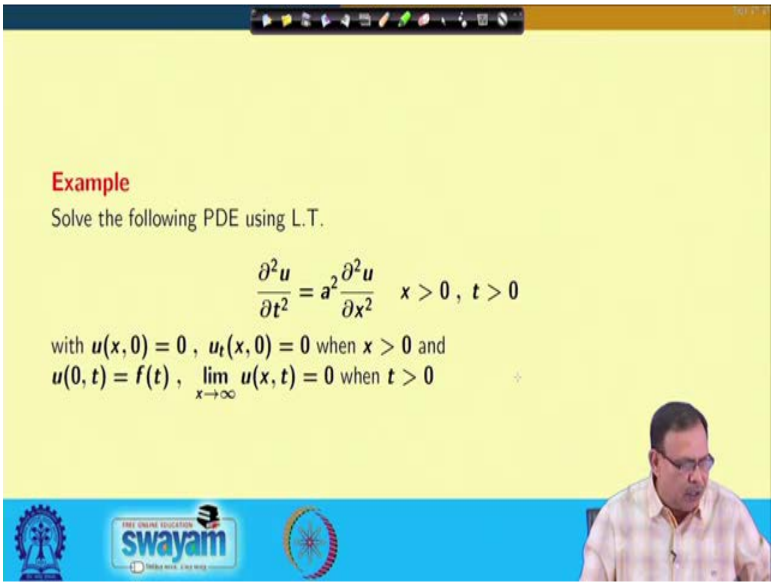

Now let us take another problem:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad x > 0, t > 0$$

with $u(x,0) = 0,\ u_t(x,0) = 0$ when $x > 0$ and $u(0,t) = f(t),\ \lim_{x\to\infty} u(x,t) = 0$ when $t > 0$. This is nothing but the wave equation. So, basically we are trying to find out the solution of wave equation using the Laplace transform.

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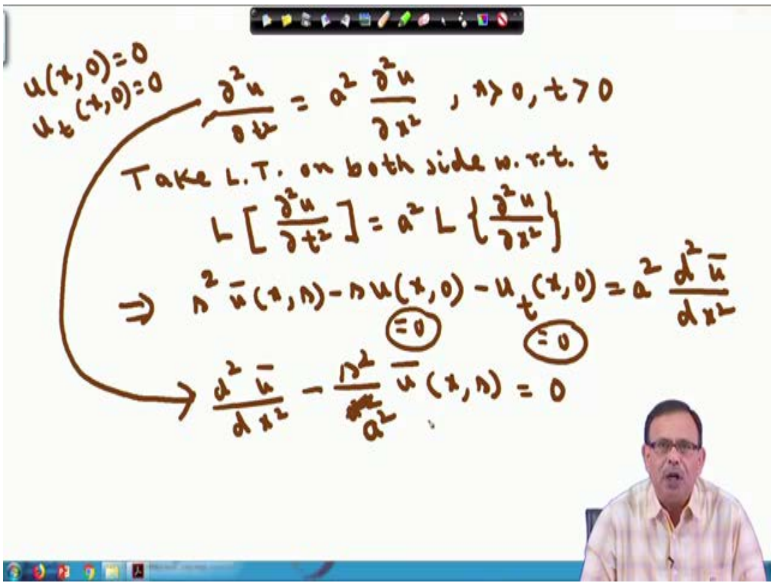

Although both the variables x and t are greater than 0, but depending upon the supplied initial conditions, we have to decide that we will take Laplace transform with respect to which independent variable. Since t has the range from 0 to ∞ and $u(x,0)$ and $u_t(x,0)$ are given, we will take the Laplace transform with respect to t. Therefore from the given equation, we will get,

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = a^2 L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow s^2\,\bar{u}(x,s) - s\,u(x,0) - u_t(x,0) = a^2 \frac{d^2\bar{u}}{dx^2}$$

where, $\bar{u}(x,s) = L\{u(x,t)\}$. Since $u(x,0) = 0, u_t(x,0) = 0$, above equation is reduced to the following ODE:

$$\Rightarrow \frac{d^2\bar{u}}{dx^2} - \frac{s^2}{a^2}\bar{u} = 0$$

Auxiliary equation for the obtained ODE is,

$$m^2 - \frac{s^2}{a^2} = 0 \Rightarrow m = \pm\frac{s}{a}$$

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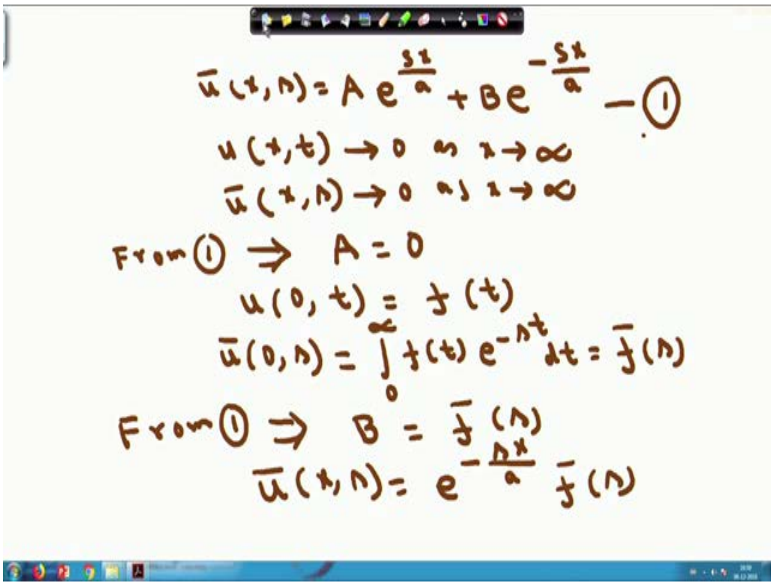

Therefore, the general solution of the ODE is given as,

$$\bar{u}(x,s) = Ae^{\frac{sx}{a}} + Be^{-\frac{sx}{a}} \qquad (2)$$

where, A and B are the constants of integration. From the initial condition, we have $u(x,0) = 0$ when $x > 0$. So, we can say that, $u(x,t) \to 0$ as $x \to \infty$.

$$\therefore \bar{u}(x,s) \to 0 \text{ as } x \to \infty$$

$$\therefore (2) \Rightarrow A = 0$$

Again we have, $u(0,t) = f(t)$.

$$\therefore \bar{u}(0,s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \bar{f}(s)$$

$$\therefore (2) \Rightarrow B = \bar{f}(s)$$

$$\therefore \bar{u}(x,s) = e^{-\frac{sx}{a}}\bar{f}(s)$$

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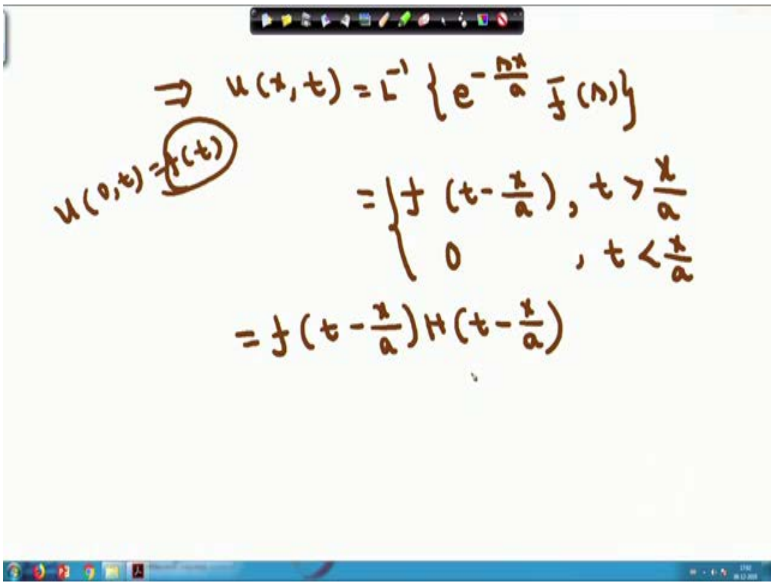

To obtain the solution, we will use the inverse Laplace transform. Therefore, we will get,

$$u(x,t) = L^{-1}\left\{e^{-\frac{sx}{a}}\bar{f}(s)\right\}$$

Again we know that,

$$L^{-1}\{e^{-as}\bar{f}(s)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

Therefore, $u(x,t)$ is given by,

$$u(x,t) = \begin{cases} f\left(t-\frac{x}{a}\right), & t > \frac{x}{a} \\ 0, & t < \frac{x}{a} \end{cases}$$

$$= f\left(t-\frac{x}{a}\right)H\left(t-\frac{x}{a}\right)$$

where, $H(t)$ is Heaviside unit step function.

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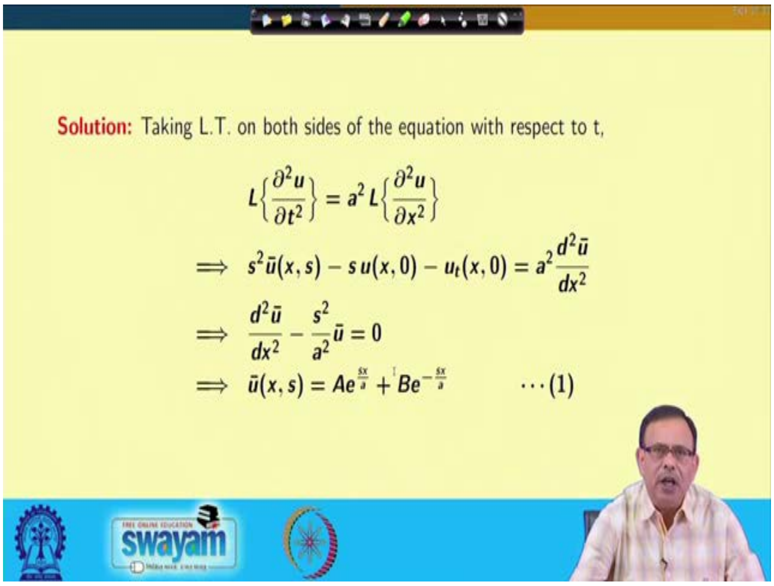

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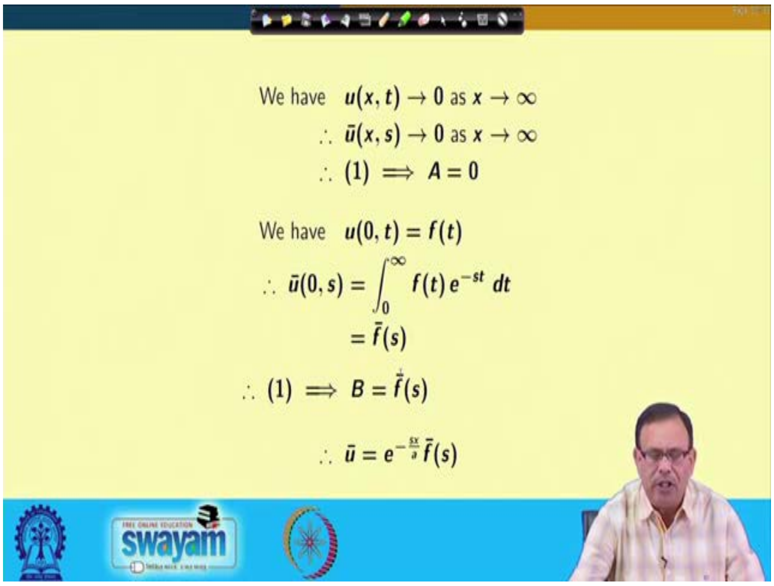

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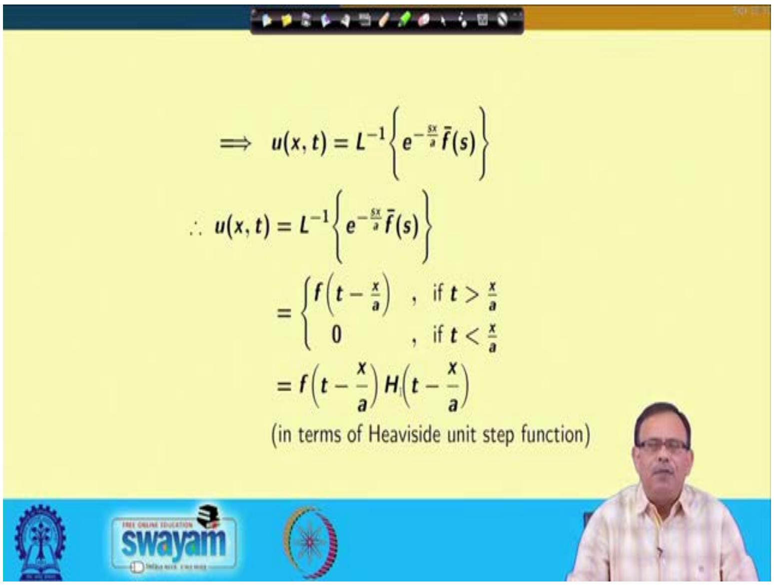

So, in this lecture, we have solved the problems of heat equation and the wave equation applying Laplace transform. We hope it is clear how to use the Laplace transform to find out the solution of different kinds of partial differential equations.

So, it consists of 3 steps. In the first step, a PDE is converted into an ODE using Laplace transform, then in the second step, we will find the solution of that ODE and in the third step, using inverse Laplace transform, we will find $u(x, t)$. Thank you.