Transform Calculus and it is Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 44
Solution of Partial Differential Equations using Laplace Transform

In this lecture, we are going to start the solution of partial differential equations. In the last lecture, we have given introduction to the partial differential equations, where we have seen the general form of a second order linear PDE and how to classify it into three types to generate the wave equation, Laplace equation and heat equation. So, in this lecture we are going to study at first how to find out the solution of those PDE using the Laplace transform.

Now, whenever a problem is given to us, what we have to decide at first is that whether we can use Laplace transform or Fourier transform to solve the problem.

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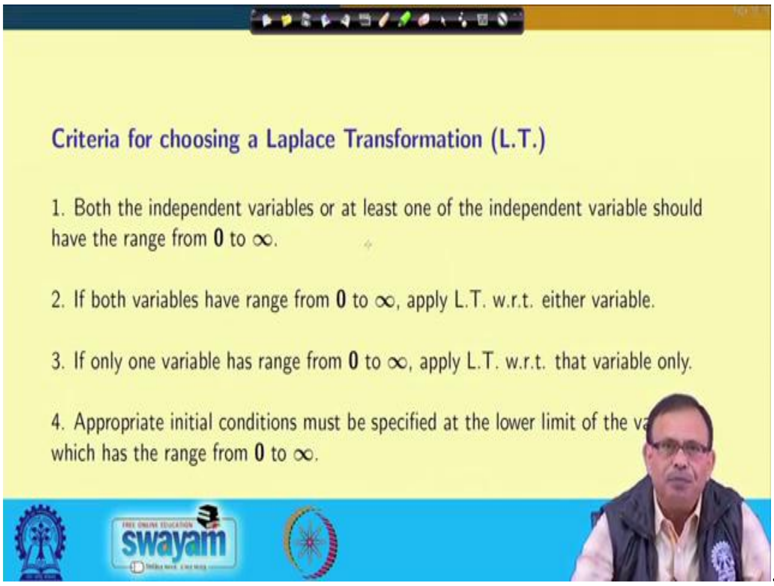

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In PDE, we will have more than one independent variable. But we take Laplace transform with respect to one variable only. So, with respect to which variable we will take the transform, that we have to decide. Since we are now going through the Laplace transform first, let us discuss the criteria for choosing the Laplace transform.

We know, Laplace transform has been defined in 0 to ∞ only. So, to apply Laplace transform, both the independent variables or at least one of the independent variables should have the range from 0 to ∞. So, the very first thing that we have to check is whether both the independent variables or at least one independent variable varies from 0 to ∞ or not.

If both the variables have range from 0 to ∞, we can apply Laplace transform with respect to any one of the variables. If only one variable has the range from 0 to ∞, then we will apply Laplace transform with respect to that variable only. Also, we need appropriate initial conditions at the lower limit of the variable which has the range from 0 to ∞. Because, whenever we are taking Laplace transform of second order derivative, whether it is partial or ordinary derivative, some constant values arrive and if we do not know the values of those constants, then it becomes difficult for us to solve the problem.

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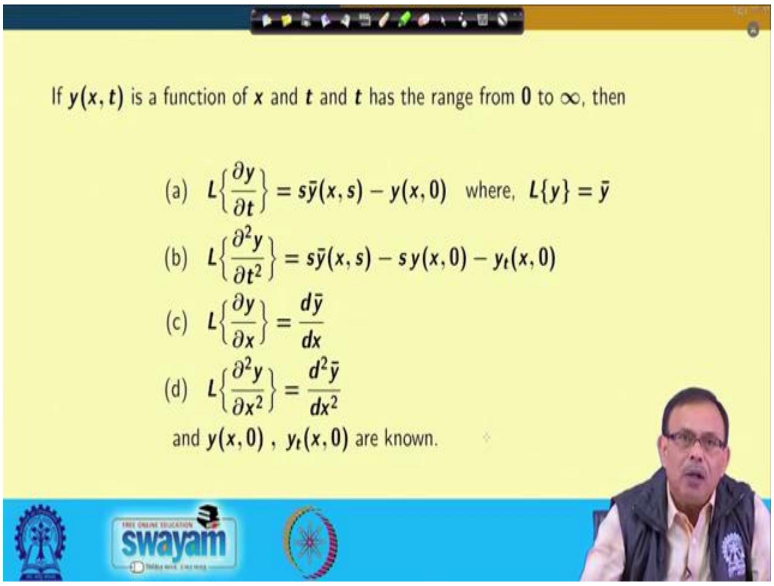

We have done Laplace transform of derivatives only, not partial derivatives. So, now we will give some formulae for Laplace transform of partial derivatives.

If $y(x,t)$ is a function of x and t and t has range from 0 to ∞, then,

$$L\left\{\frac{\partial y}{\partial t}\right\} = s\,\bar{y}(x,s) - y(x,0) \quad \text{where}, L\{y\} = \bar{y}$$

The initial condition at $t = 0$ i.e. $y(x,0)$ must be given otherwise we cannot evaluate Laplace transform of the partial derivative. Similarly, other formulae are

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = s^2\, \bar{y}(x,s) - s\, y(x,0) - y_t(x,0)$$

$$L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d\bar{y}}{dx}$$

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2\bar{y}}{dx^2}$$

where, $y(x,0)$ and $y_t(x,0)$ are given.

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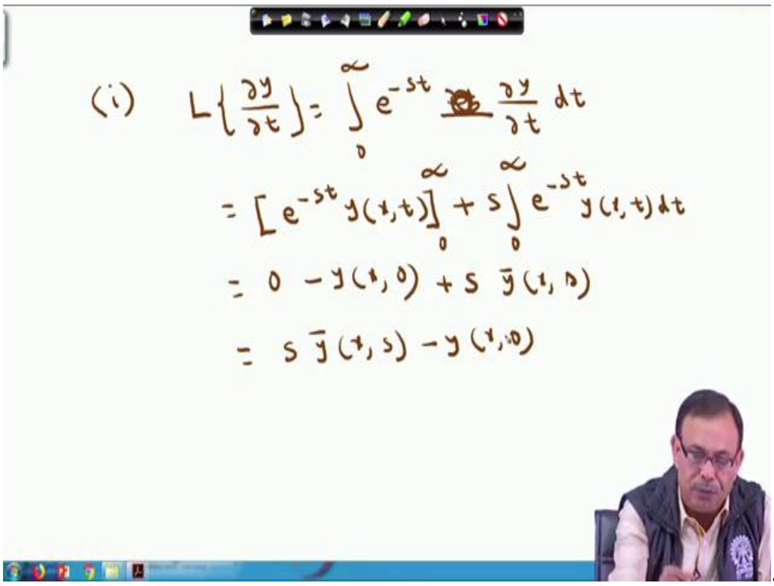

For proof of the first one, if we use the definition of Laplace transform, then we can write down,

$$L\left\{\frac{\partial y}{\partial t}\right\} = \int_0^\infty e^{-st}\,\frac{\partial y}{\partial t}\,dt$$

So, if we use the integration by parts on this particular equation, we will obtain,

$$L\left\{\frac{\partial y}{\partial t}\right\} = [e^{-st}\, y(x,t)]_0^\infty + s\int_0^\infty e^{-st} y(x,t)dt$$

$$= 0 - y(x,0) + s\,\bar{y}(x,s)$$

$$= s\,\bar{y}(x,s) - y(x,0)$$

This completes the proof of the first part.

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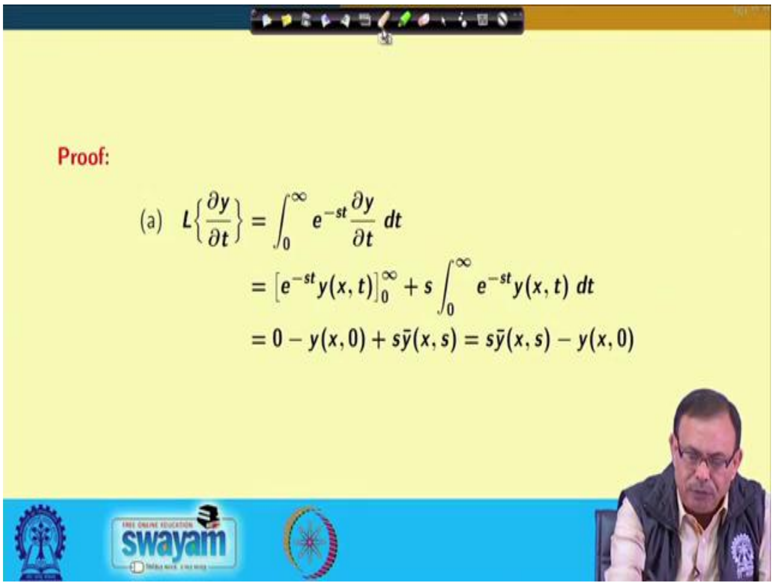

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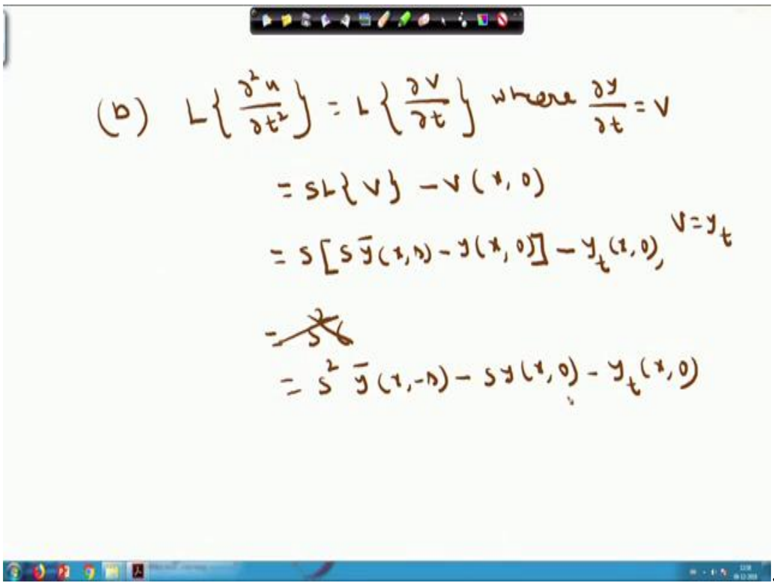

Let us see the proof of the Laplace transform of $\frac{\partial^2 y}{\partial t^2}$. Let $\frac{\partial y}{\partial t} = v$. Therefore,

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\}$$

From the previous formula, we have,

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\}$$
$$= s\,L\{v\} - v(x,0)$$
$$= s[s\,\bar{y}(x,s) - y(x,0)] - y_t(x,0) \;\; (\because v = y_t)$$
$$= s^2\,\bar{y}(x,s) - s\,y(x,0) - y_t(x,0)$$

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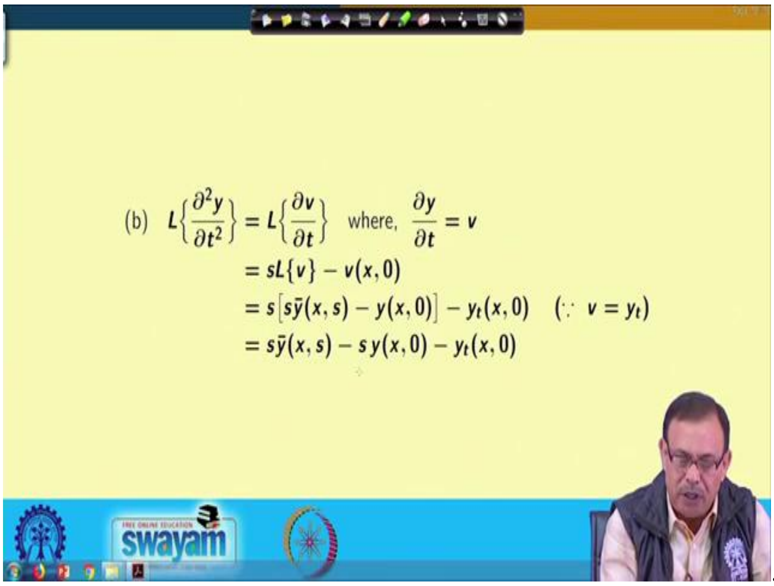

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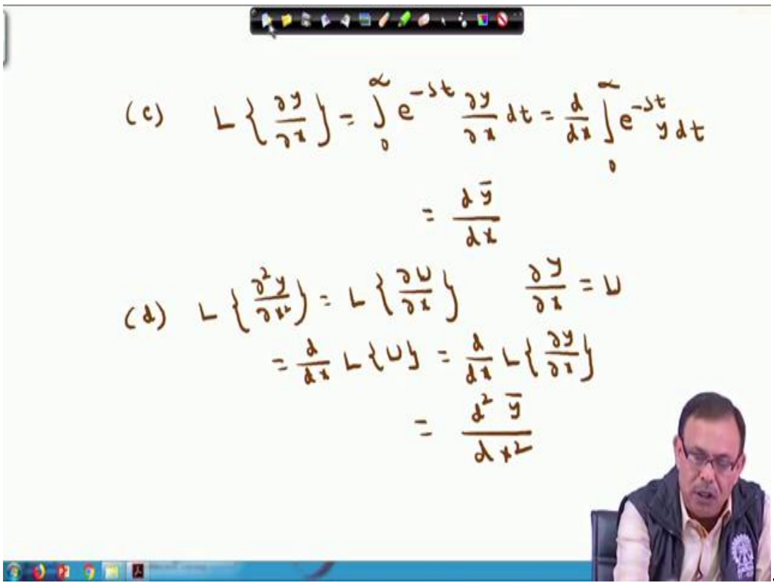

For the next proof, using definition of Laplace transform, we have,

$$L\left\{\frac{\partial y}{\partial x}\right\} = \int_0^{\infty} e^{-st} \frac{\partial y}{\partial x}\, dt$$

Since x and t are independent, we can take $\frac{d}{dx}$ outside the integration. Therefore,

$$L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d}{dx}\int_0^{\infty} e^{-st} y\, dt = \frac{d\bar{y}}{dx}$$

Similarly, for the last one, let $\frac{\partial y}{\partial x} = u$. Therefore,

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d}{dx} L\{u\}$$
$$= \frac{d}{dx} L\left\{\frac{\partial y}{\partial x}\right\}$$
$$= \frac{d^2\bar{y}}{dx^2}$$

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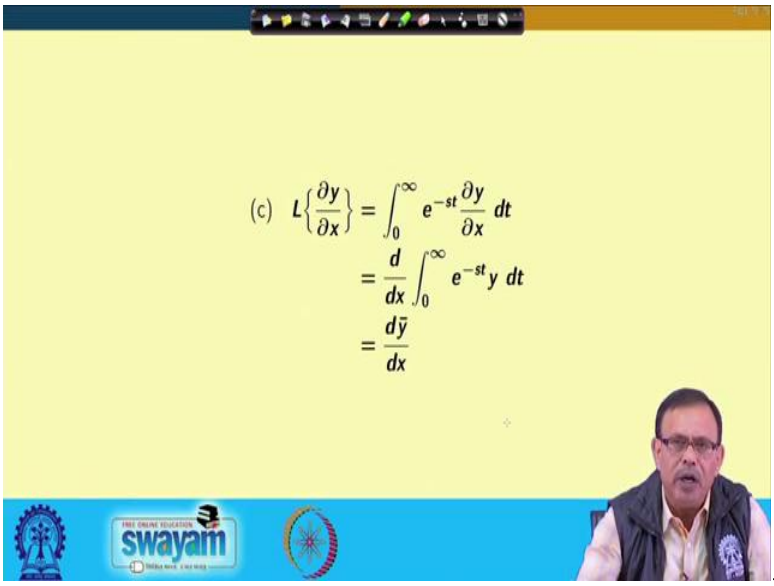

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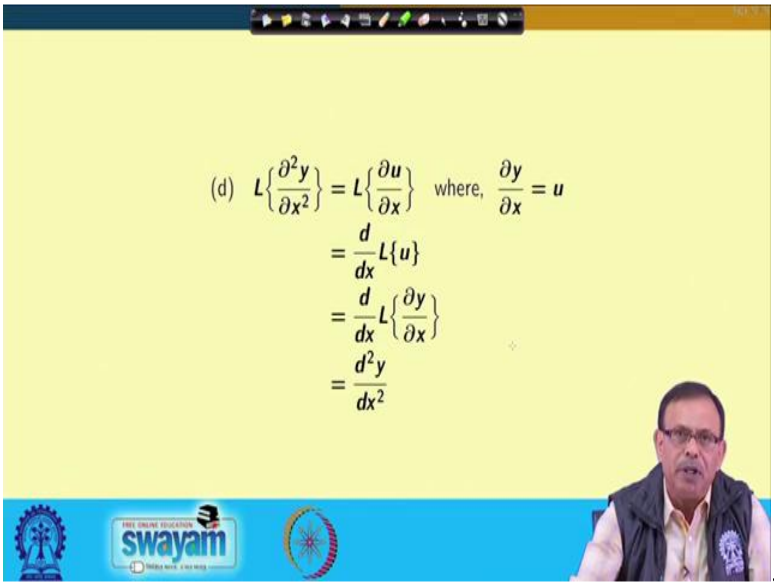

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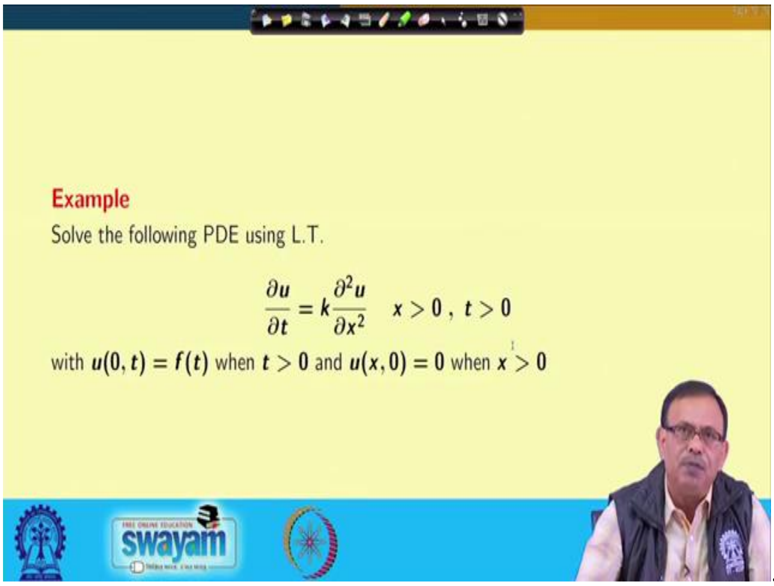

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Now, let us see the first application. We want to solve the following

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad x > 0, \ \ t > 0$$

with, $u(0,t) = f(t)$ when $t > 0$ and $u(x,0) = 0$ when $x > 0$

Let us see the solution process of this. At first we have to decide with respect to which variable we should take the Laplace transform.

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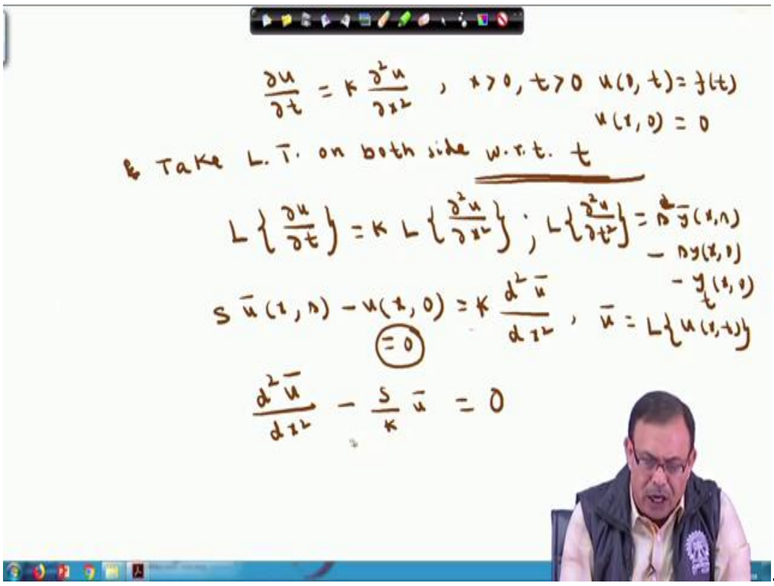

Here, both the variables have range from 0 to ∞. Now, if we take Laplace transform w.r.t. x, then we need $u_x(0,t)$ because of $\frac{\partial^2 u}{\partial x^2}$ but this value is not given. So, we will take Laplace transform on both sides with respect to t. Now using the formulae for the Laplace transform for partial derivatives, we will obtain,

$$L\left\{\frac{\partial u}{\partial t}\right\} = kL\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow s\,\bar{u}(x,s) - u(x,0) = k\frac{d^2\bar{u}}{dx^2}$$

$$\Rightarrow \frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u} = 0 \quad (\because u(x,0) = 0\,)$$

Therefore, from the given PDE, we obtained an ODE.

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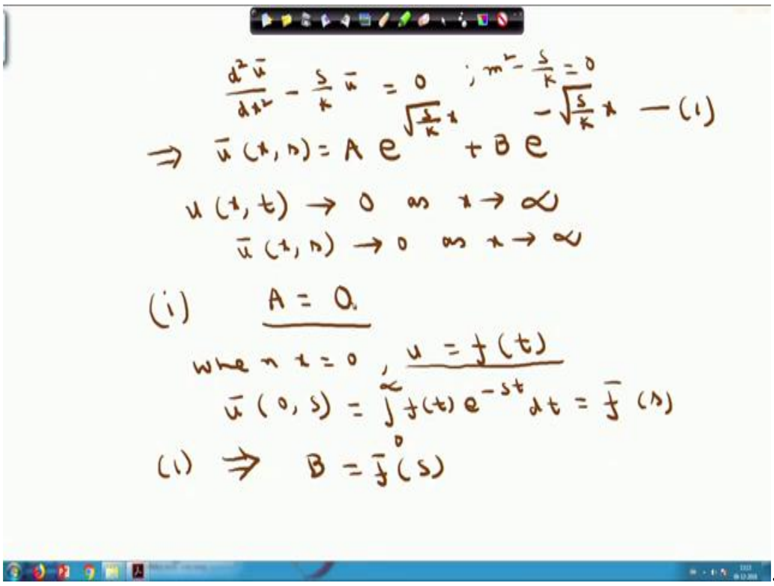

Auxiliary equation for the obtained ODE is,

$$m^2 - \frac{s}{k} = 0 \Rightarrow m = \pm\sqrt{\frac{s}{k}}$$

Therefore, the general solution of the ODE is given as,

$$\bar{u}(x, s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x} \qquad (1)$$

From the given initial condition, we have, $u(x, 0) = 0$ when $x > 0$. So, we can say that, $u(x, t) \to 0$ as $x \to \infty$.

$$\therefore \bar{u}(x, s) \to 0 \text{ as } x \to \infty$$

$$\therefore (1) \Rightarrow A = 0$$

Again, $u = f(t)$ at $x = 0$.

$$\therefore \bar{u}(0, s) = \int_0^\infty f(t)e^{-st}dt = \bar{f}(s)$$

$$\therefore (1) \Rightarrow B = \bar{f}(s)$$

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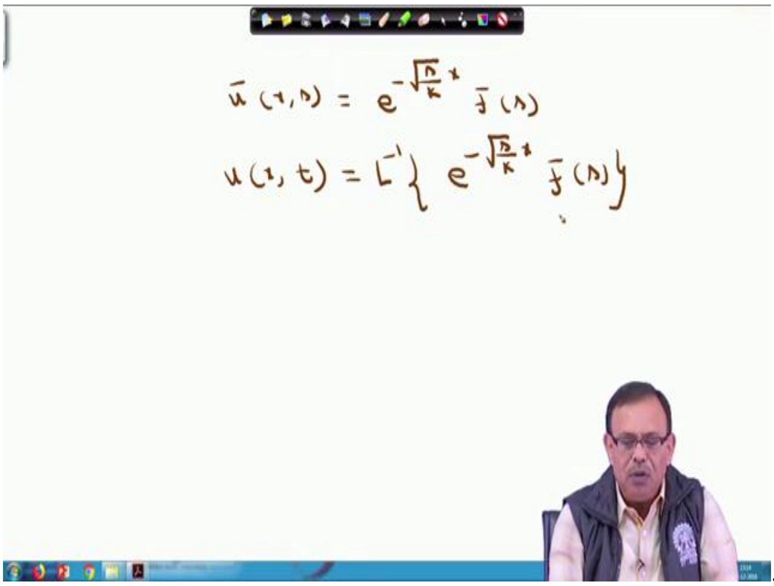

Now we have both the constants A and B.

$$\therefore \bar{u}(x,s) = e^{-\sqrt{\frac{s}{k}}x}\,\bar{f}(s)$$

Now taking inverse Laplace transform, we can obtain $u(x,t)$ as,

$$u(x,t) = L^{-1}\left\{e^{-\sqrt{\frac{s}{k}}x}\,\bar{f}(s)\right\}$$

We can evaluate it if the function $f(t)$ is known to us. We have done it for general form. So, by this way, we can find the solution of the equation.

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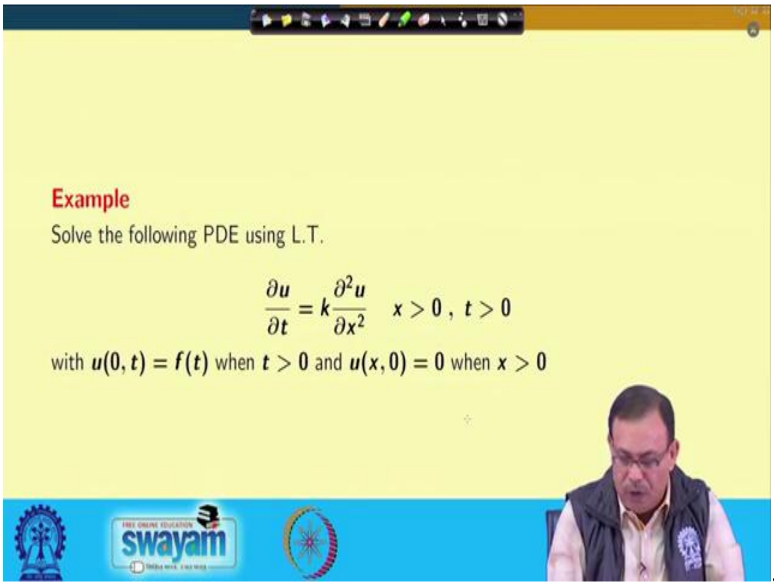

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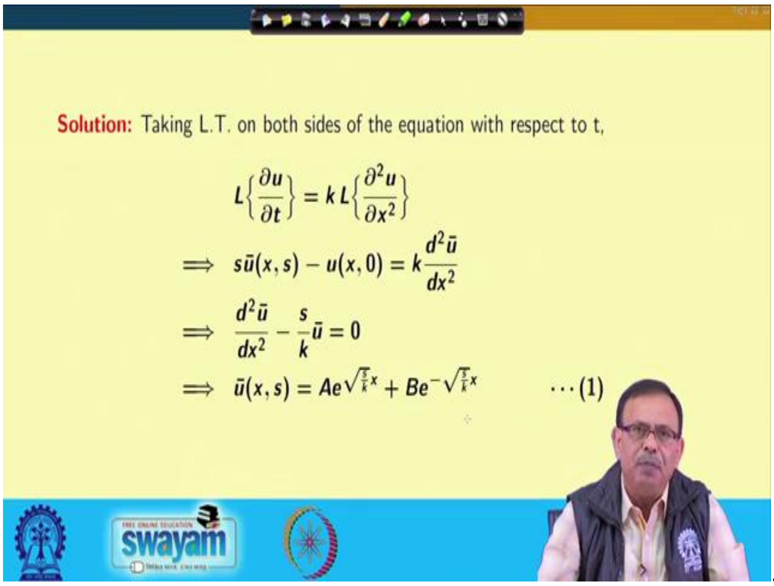

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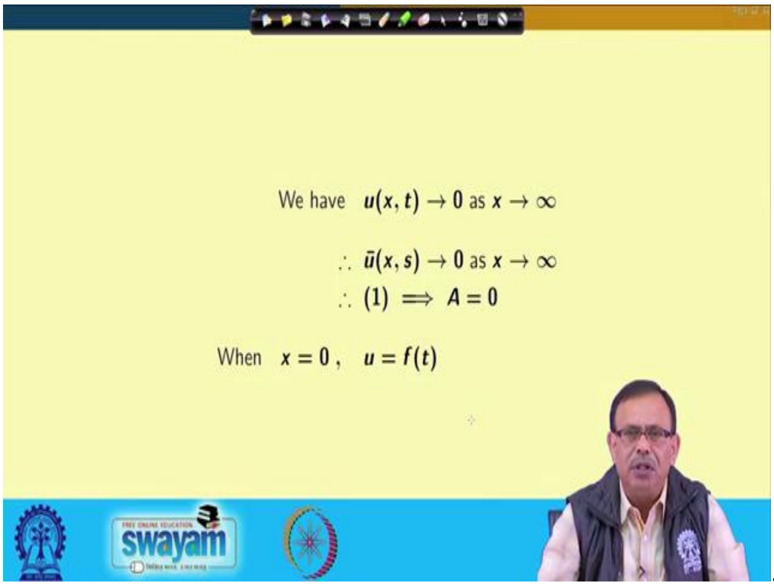

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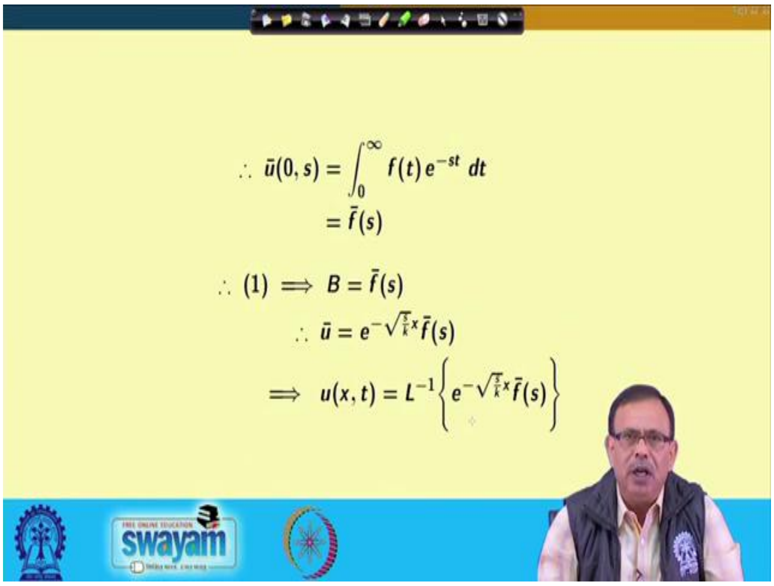

So, by this process, we can solve the equations. In the next lectures also, we will solve some other PDEs of other types. Thank you.