Transform Calculus and it is Applications in Differential Equations

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Lecture - 43
Introduction to Partial Differential Equations

Till now, in the lectures, we have covered Laplace transform, the properties of Laplace transform, how to find out the Laplace transform of various functions, convolution etc. Similarly, after that we have gone through Fourier transform, Fourier sine transform, Fourier cosine transform and their properties. We have studied how to find out the Fourier transform or Fourier cosine or Fourier sine transform of various functions and how to use their properties.

Now, let us come to the third topic that is the application of Fourier and Laplace transform on partial differential equations. If we recall, we have seen how to find out the solution of an ODE using Laplace transform by converting it into a simple algebraic equation and by solving the algebraic equation, we are able to find the solution of the ODE.

So, now let us see how to find out the solution of partial differential equations using Laplace and Fourier transforms. So, before going to the application of this, let us very briefly talk about the partial differential equation first and then we will go to the other part.

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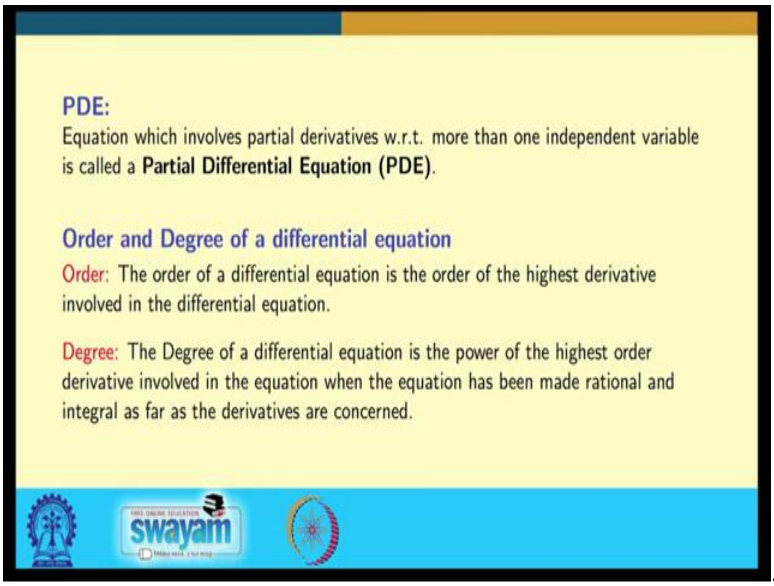
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An equation which involves the partial derivatives with respect to more than one independent variable, we call it as a partial differential equation. So, please note that we have an equation which involves the partial derivatives and the partial derivatives are with respect to more than one independent variable, then we call it as PDE or partial differential equation.

Whenever we talk about differential equation, we always try to talk about two things, one is the order, another one is the degree. The order of a differential equation is the order of the highest derivative involved in the differential equation. So, whatever highest derivative is there, that we call as the order of that differential equation. The degree of a differential equation is the power of the highest order derivative involved in the equation.

So, let us see some examples.

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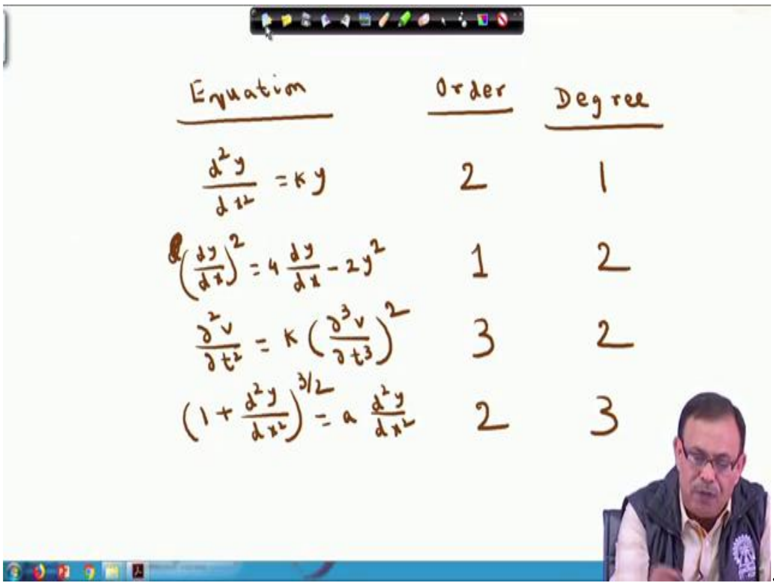

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So, suppose we have an equation,

$$\frac{d^2y}{dx^2} = ky$$

From the definition, order is the highest order derivative which appears in the equation. Here the highest order derivative is $\frac{d^2y}{dx^2}$. Therefore, the order of this equation will be 2 and since, there is no nonlinear term, so, degree will be equal to 1.

Let us consider another equation,

$$\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx} - 2y^2$$

Here, the highest order derivative present is $\frac{dy}{dx}$ and the highest power for this term is 2. Therefore, the order of the differential equation is 1 and degree is 2.

Let us take another equation,

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial t^3}\right)^2$$

So, here the highest order appearing is third order because we have $\frac{\partial^3 v}{\partial t^3}$. So, the order of this equation will be 3 whereas, the degree will be 2, because power of the highest order derivative is 2.

Similarly, if we take the equation

$$\left(1 + \frac{d^2 y}{dx^2}\right)^{3/2} = a\frac{d^2 y}{dx^2}$$

In this equation, the highest order derivative which appears is $\frac{d^2 y}{dx^2}$. So, the order is 2. Since, there is a fraction power in the equation, we try to remove it. For that, we square both sides of the equation to obtain integer power. In that case, highest power of the highest order derivative will be 3. So, the degree of the given equation is 3.

So, like this way, we find out the order and degree of a differential equation.

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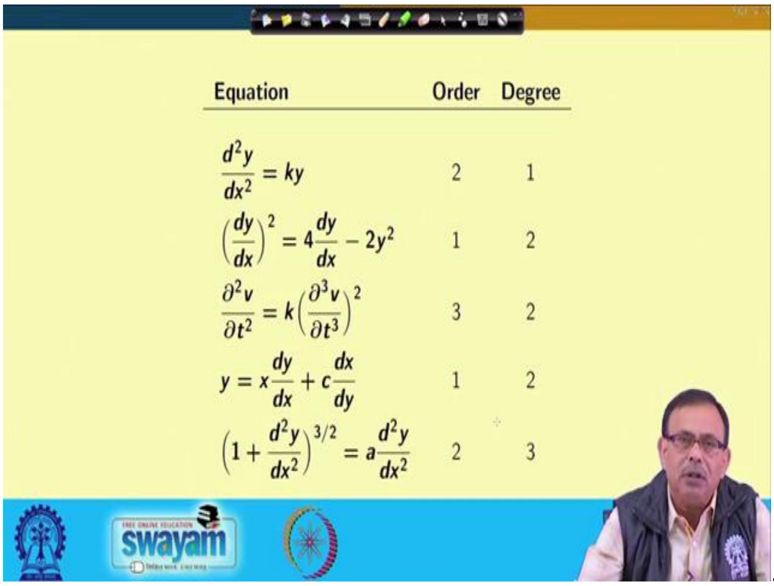

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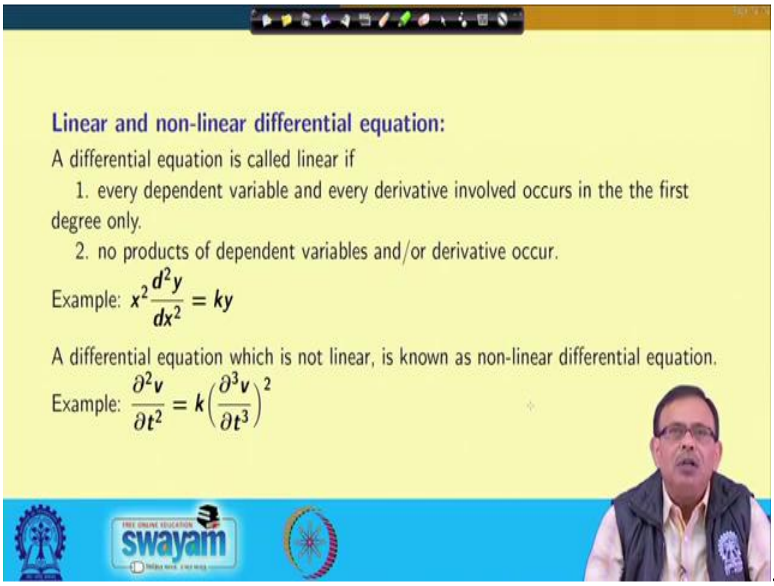

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A differential equation is known as linear if every dependent variable and every derivative involved occurs in the first degree only and no products of dependent variables and/or derivatives occur. Consider the following example:

$$x^2\frac{d^2y}{dx^2} = ky$$

A non-linear differential equation is defined as a differential equation which is not linear. So, we can give an example as

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial t^3}\right)^2$$

Therefore, if the dependent variable and its partial derivatives are occurring in the first degree only and there is no term containing their products, then we call it as linear, otherwise it is non-linear. So, to check whether it is linear or non-linear, it is always desirable that the concept on the order and degree is totally clear, otherwise it will be difficult to judge whether an equation is linear or non-linear.

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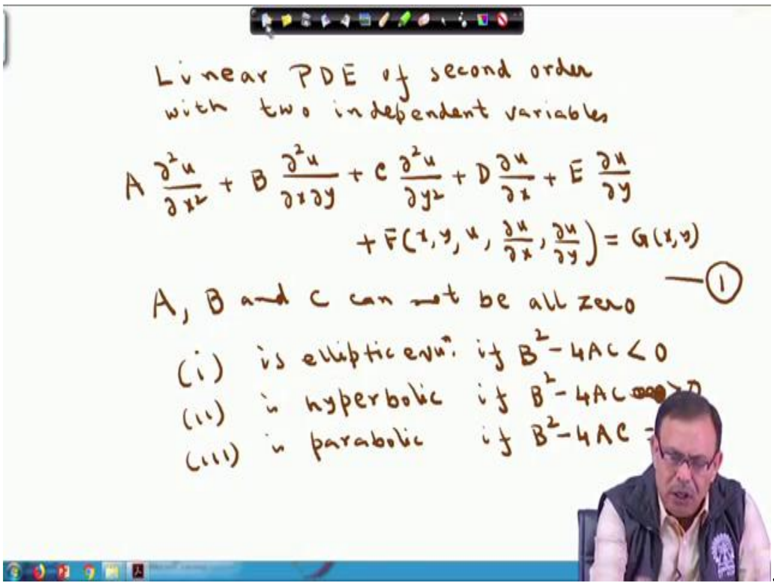

The general form of a linear PDE of second order in two independent variables is given as,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where, A, B, C cannot be all zero and $u = u(x, y)$ and A, B, C, D, E, F, G are function of x and y only.

Whenever we have this general equation, then we can make the classification of this general equation depending upon the values of A, B and C as follows:

(1) If $B^2 - 4AC < 0$, then the equation is elliptic ($Laplace\ Equation$).

(2) If $B^2 - 4AC > 0$, then the equation is hyperbolic ($Wave\ Equation$).

(3) If $B^2 - 4AC = 0$, then the equation is parabolic ($Heat\ Equation$).

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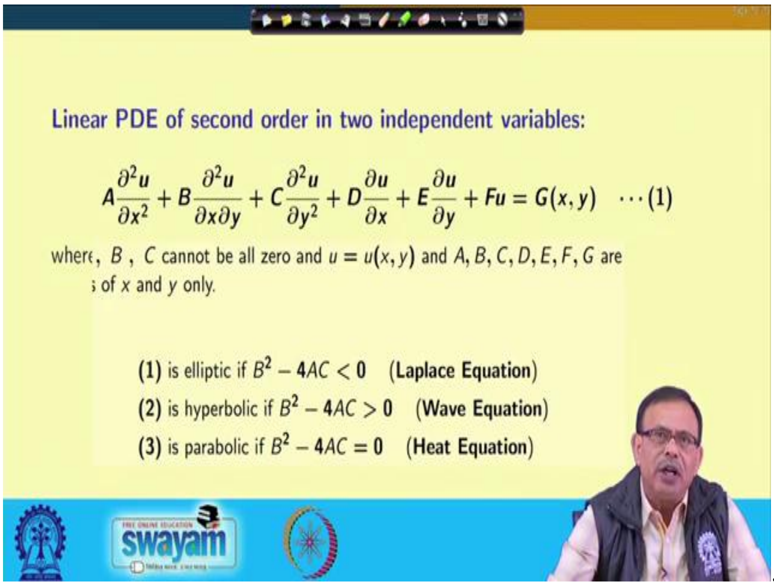

Let us discuss about wave equation first.

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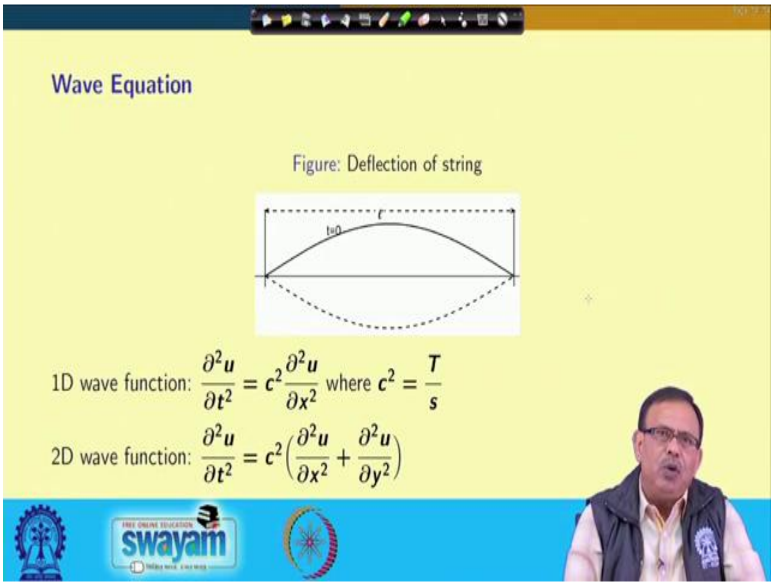

Let us have an elastic string which is fixed at two end points. At time $t = 0$, suppose we allow the string to vibrate.

So, if the vibration is $u(x, t)$ at any time $t > 0$ and at any point x, then the wave function can be defined by an equation which we call as wave equation.

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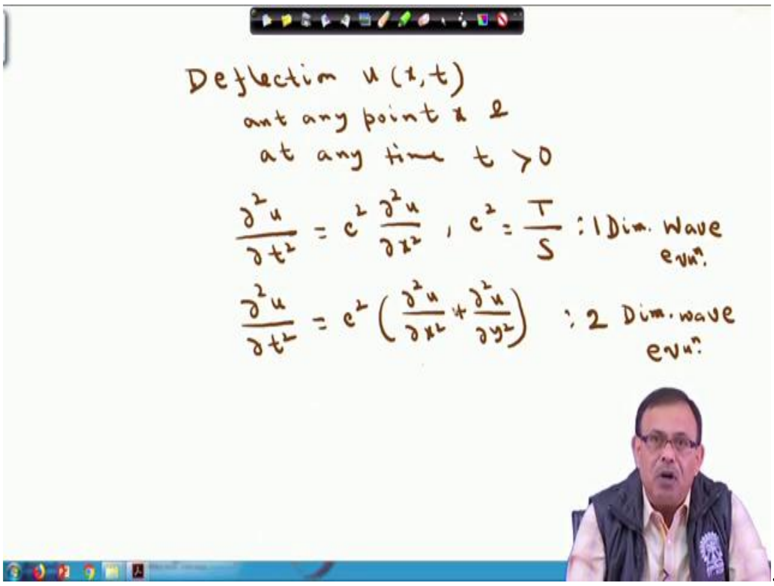

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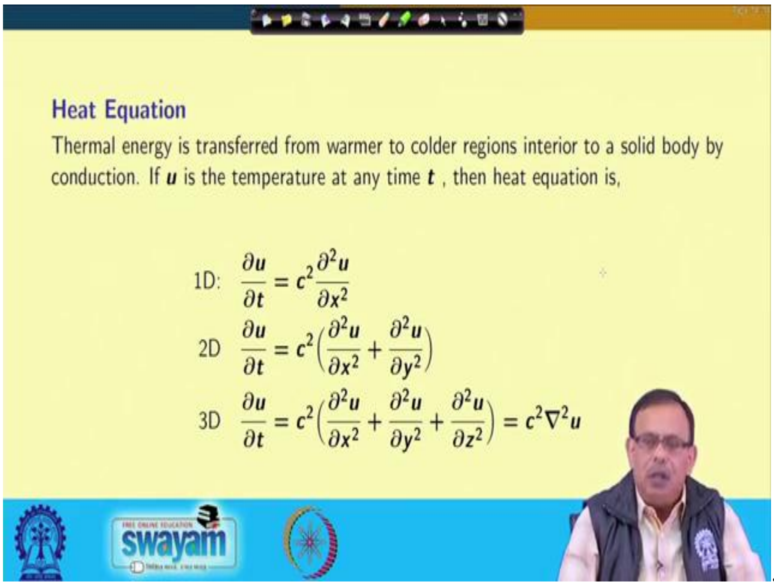

One dimensional wave equation is given as,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where, } c^2 = \frac{T}{s}$$

Two dimensional wave equation is given as,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

The next one is the heat equation. Thermal energy is transferred from warmer to colder regions. If $u(x,t)$ is the temperature at any time t, then the one dimensional heat equation can be written as

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Similarly, two dimensional heat equation can be written as

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

And three dimensional heat equation is given as,

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = c^2 \nabla^2 u$$

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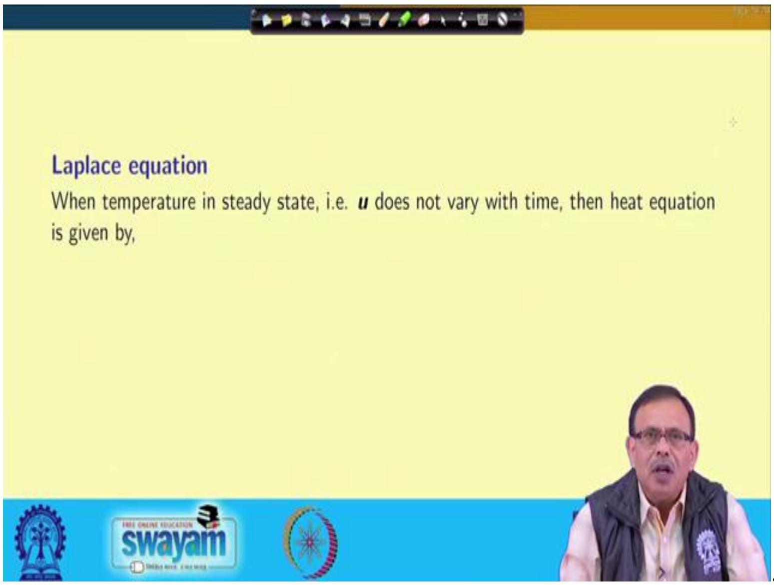

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And the third one is the Laplace equation. When the temperature is in steady state, that is u does not vary with time, then the heat equation is known as Laplace equation.

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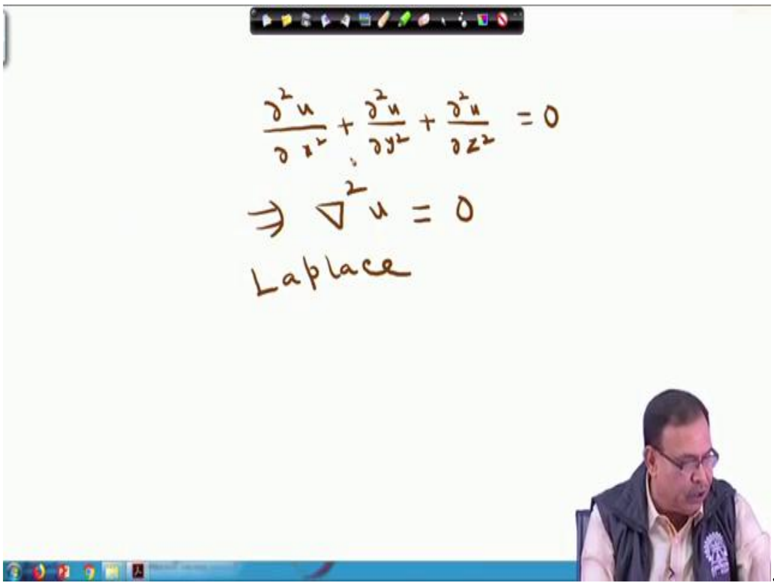

Therefore, Laplace equation is given by,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$i.e.\ \nabla^2 u = 0$$

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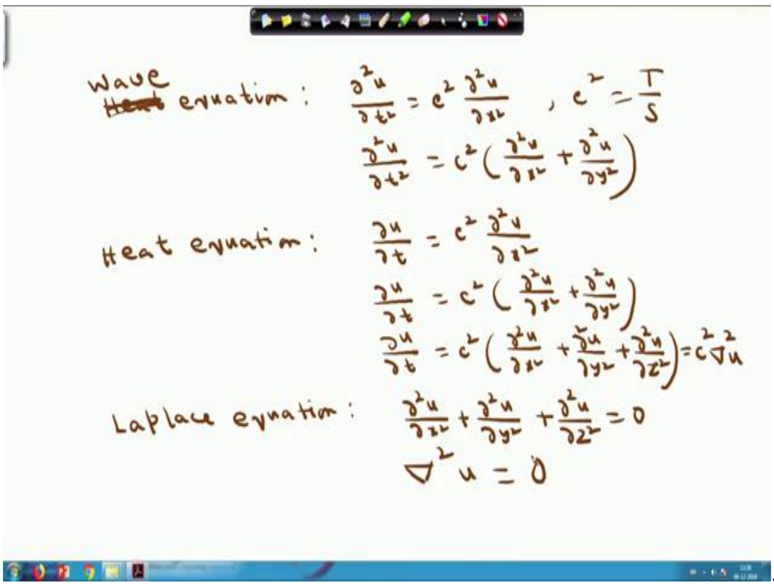

So, basically, whenever any second order linear PDE is there, we can classify it into either elliptic or hyperbolic or parabolic and these three are corresponding to the Laplace equation, wave equation and heat equation and our basic aim will be to find out the solution of this type of equation using Laplace transform and Fourier transform. In the subsequent lectures, we will see how we can find out the solution of this. Thank you.