

Transform Calculus and its Applications in Differential Equations
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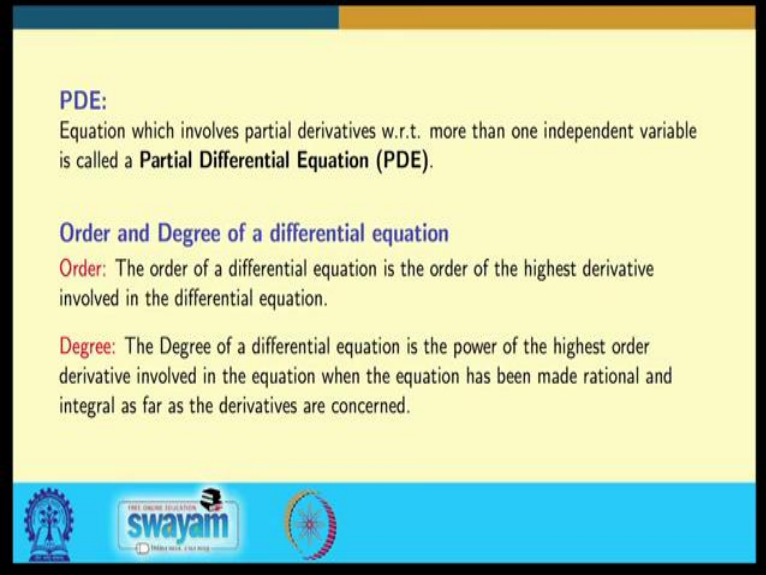
Lecture - 43
Introduction to Partial Differential Equations

Till now, in the lectures, we have covered Laplace transform, the properties of Laplace transform, how to find out the Laplace transform of various functions, convolution etc. Similarly, after that we have gone through Fourier transform, Fourier sine transform, Fourier cosine transform and their properties. We have studied how to find out the Fourier transform or Fourier cosine or Fourier sine transform of various functions and how to use their properties.

Now, let us come to the third topic that is the application of Fourier and Laplace transform on partial differential equations. If we recall, we have seen how to find out the solution of an ODE using Laplace transform by converting it into a simple algebraic equation and by solving the algebraic equation, we are able to find the solution of the ODE.

So, now let us see how to find out the solution of partial differential equations using Laplace and Fourier transforms. So, before going to the application of this, let us very briefly talk about the partial differential equation first and then we will go to the other part.

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PDE:
Equation which involves partial derivatives w.r.t. more than one independent variable is called a **Partial Differential Equation (PDE)**.

Order and Degree of a differential equation

Order: The order of a differential equation is the order of the highest derivative involved in the differential equation.

Degree: The Degree of a differential equation is the power of the highest order derivative involved in the equation when the equation has been made rational and integral as far as the derivatives are concerned.

An equation which involves the partial derivatives with respect to more than one independent variable, we call it as a partial differential equation. So, please note that we have an equation which involves the partial derivatives and the partial derivatives are with respect to more than one independent variable, then we call it as PDE or partial differential equation.

Whenever we talk about differential equation, we always try to talk about two things, one is the order, another one is the degree. The order of a differential equation is the order of the highest derivative involved in the differential equation. So, whatever highest derivative is there, that we call as the order of that differential equation. The degree of a differential equation is the power of the highest order derivative involved in the equation.

So, let us see some examples.

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Equation	Order	Degree
$\frac{d^2y}{dx^2} = ky$	2	1
$\left(\frac{dy}{dx}\right)^2 = 4 \frac{dx}{dt} - 2y^2$	1	2
$\frac{d^2v}{dt^2} = k \left(\frac{dv}{dt}\right)^2$	3	2
$\left(1 + \frac{d^2y}{dx^2}\right)^{3/2} = a \frac{d^2y}{dx^2}$	2	3

So, suppose we have an equation,

$$\frac{d^2y}{dx^2} = ky$$

From the definition, order is the highest order derivative which appears in the equation.

Here the highest order derivative is $\frac{d^2y}{dx^2}$. Therefore, the order of this equation will be 2 and since, there is no nonlinear term, so, degree will be equal to 1.

Let us consider another equation,

$$\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx} - 2y^2$$

Here, the highest order derivative present is $\frac{dy}{dx}$ and the highest power for this term is 2. Therefore, the order of the differential equation is 1 and degree is 2.

Let us take another equation,

$$\frac{\partial^2 v}{\partial t^2} = k \left(\frac{\partial^3 v}{\partial t^3}\right)^2$$

So, here the highest order appearing is third order because we have $\frac{\partial^3 v}{\partial t^3}$. So, the order of this equation will be 3 whereas, the degree will be 2, because power of the highest order derivative is 2.

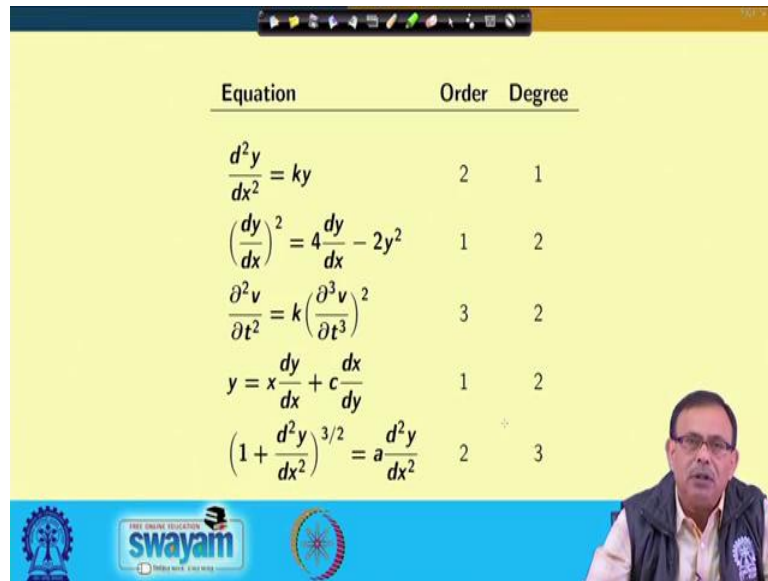
Similarly, if we take the equation

$$\left(1 + \frac{d^2 y}{dx^2}\right)^{3/2} = a \frac{d^2 y}{dx^2}$$

In this equation, the highest order derivative which appears is $\frac{d^2 y}{dx^2}$. So, the order is 2. Since, there is a fraction power in the equation, we try to remove it. For that, we square both sides of the equation to obtain integer power. In that case, highest power of the highest order derivative will be 3. So, the degree of the given equation is 3.

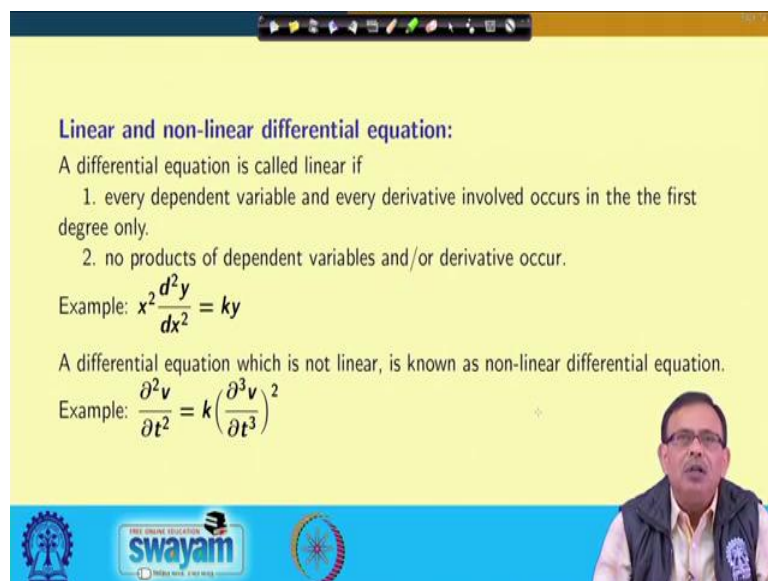
So, like this way, we find out the order and degree of a differential equation.

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Equation	Order	Degree
$\frac{d^2y}{dx^2} = ky$	2	1
$\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx} - 2y^2$	1	2
$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial t^3}\right)^2$	3	2
$y = x\frac{dy}{dx} + c\frac{dx}{dy}$	1	2
$\left(1 + \frac{d^2y}{dx^2}\right)^{3/2} = a\frac{d^2y}{dx^2}$	2	3

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Linear and non-linear differential equation:

A differential equation is called linear if

1. every dependent variable and every derivative involved occurs in the the first degree only.
2. no products of dependent variables and/or derivative occur.

Example: $x^2 \frac{d^2y}{dx^2} = ky$

A differential equation which is not linear, is known as non-linear differential equation.

Example: $\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial t^3}\right)^2$

A differential equation is known as linear if every dependent variable and every derivative involved occurs in the first degree only and no products of dependent variables and/or derivatives occur. Consider the following example:

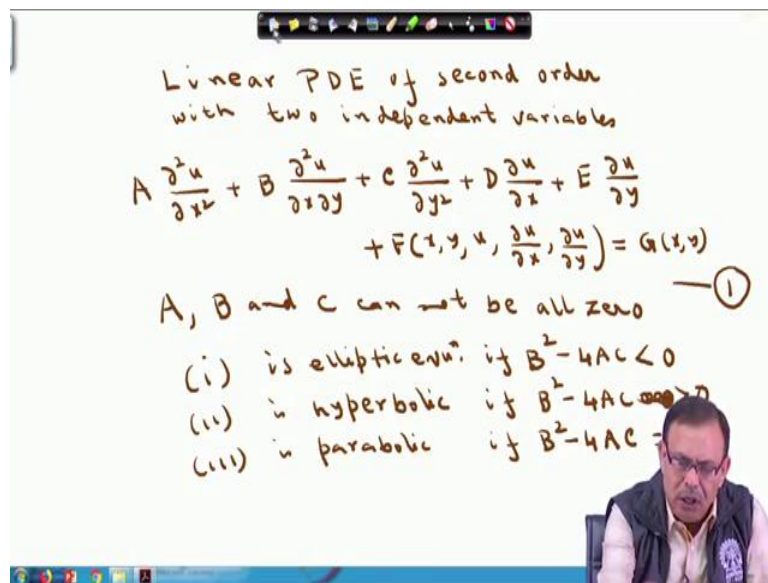
$$x^2 \frac{d^2y}{dx^2} = ky$$

A non-linear differential equation is defined as a differential equation which is not linear. So, we can give an example as

$$\frac{\partial^2 v}{\partial t^2} = k \left(\frac{\partial^3 v}{\partial t^3} \right)^2$$

Therefore, if the dependent variable and its partial derivatives are occurring in the first degree only and there is no term containing their products, then we call it as linear, otherwise it is non-linear. So, to check whether it is linear or non-linear, it is always desirable that the concept on the order and degree is totally clear, otherwise it will be difficult to judge whether an equation is linear or non-linear.

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The general form of a linear PDE of second order in two independent variables is given as,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

where, A, B, C cannot be all zero and $u = u(x, y)$ and A, B, C, D, E, F, G are function of x and y only.

Whenever we have this general equation, then we can make the classification of this general equation depending upon the values of A, B and C as follows:

- (1) If $B^2 - 4AC < 0$, then the equation is elliptic (*Laplace Equation*).
- (2) If $B^2 - 4AC > 0$, then the equation is hyperbolic (*Wave Equation*).

(3) If $B^2 - 4AC = 0$, then the equation is parabolic (*Heat Equation*).

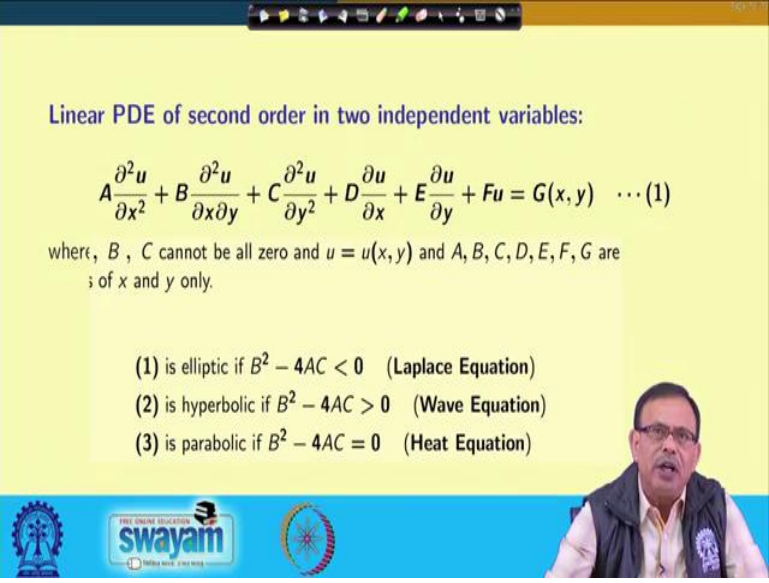
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Linear PDE of second order in two independent variables:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G(x, y) \quad \dots (1)$$

where, B, C cannot be all zero and $u = u(x, y)$ and A, B, C, D, E, F, G are functions of x and y only.

(1) is elliptic if $B^2 - 4AC < 0$ (Laplace Equation)
 (2) is hyperbolic if $B^2 - 4AC > 0$ (Wave Equation)
 (3) is parabolic if $B^2 - 4AC = 0$ (Heat Equation)

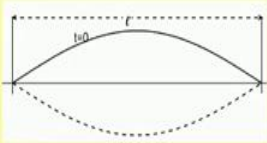


Let us discuss about wave equation first.

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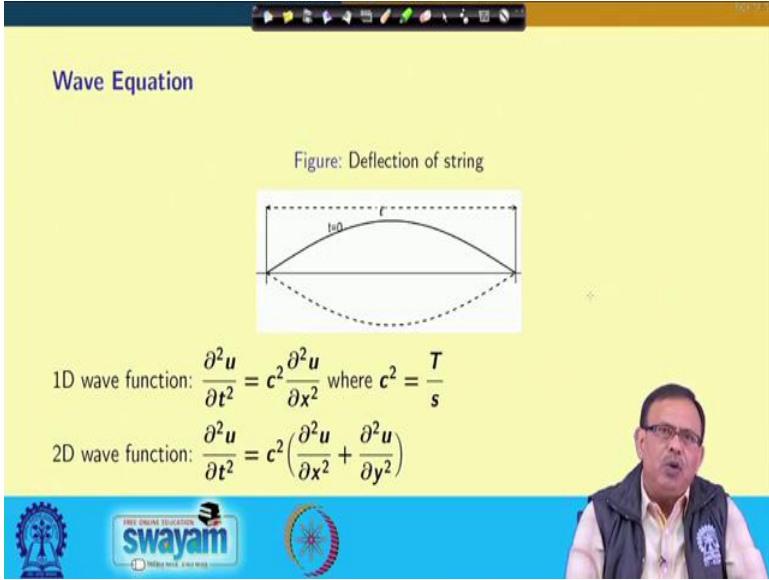
Wave Equation

Figure: Deflection of string



1D wave function: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{s}$

2D wave function: $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$



Let us have an elastic string which is fixed at two end points. At time $t = 0$, suppose we allow the string to vibrate.

So, if the vibration is $u(x, t)$ at any time $t > 0$ and at any point x , then the wave function can be defined by an equation which we call as wave equation.

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Deflection $u(x,t)$
at any point x &
at any time $t > 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{S} \quad \text{: 1 Dim. Wave eqn.}$$
$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{: 2 Dim. wave eqn.}$$

The image shows a whiteboard with handwritten mathematical definitions and equations. A person is visible in the bottom right corner of the frame.

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Heat Equation

Thermal energy is transferred from warmer to colder regions interior to a solid body by conduction. If u is the temperature at any time t , then heat equation is,

$$\text{1D: } \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$
$$\text{2D } \frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$
$$\text{3D } \frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \nabla^2 u$$

The slide includes the Swayam logo and a person in the bottom right corner.

One dimensional wave equation is given as,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where, } c^2 = \frac{T}{S}$$

Two dimensional wave equation is given as,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

The next one is the heat equation. Thermal energy is transferred from warmer to colder regions. If $u(x, t)$ is the temperature at any time t , then the one dimensional heat equation can be written as

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Similarly, two dimensional heat equation can be written as

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

And three dimensional heat equation is given as,

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \nabla^2 u$$

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The image shows a screenshot of a video lecture slide. The slide has a yellow background and a blue footer. The title is "Laplace equation" in blue text. Below the title, it says "When temperature in steady state, i.e. u does not vary with time, then heat equation is given by,". In the bottom right corner, there is a small video inset of a man speaking. The footer contains logos for "swayam" and "Digital India" along with the text "Free Online Education" and "Digital India".

And the third one is the Laplace equation. When the temperature is in steady state, that is u does not vary with time, then the heat equation is known as Laplace equation.

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The image shows a whiteboard with handwritten mathematical equations. At the top, the equation is written as $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. Below this, it is simplified to $\Rightarrow \nabla^2 u = 0$. The word "Laplace" is written at the bottom. A person's head and shoulders are visible in the bottom right corner of the whiteboard frame.

Therefore, Laplace equation is given by,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

i. e. $\nabla^2 u = 0$

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The image shows a whiteboard with handwritten text and equations. It lists three types of second-order linear PDEs: "Wave equation" with $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ and $c^2 = \frac{T}{S}$; "Heat equation" with $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$; and "Laplace equation" with $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ and $\nabla^2 u = 0$. The word "Heat" is crossed out under "Wave equation".

So, basically, whenever any second order linear PDE is there, we can classify it into either elliptic or hyperbolic or parabolic and these three are corresponding to the Laplace

equation, wave equation and heat equation and our basic aim will be to find out the solution of this type of equation using Laplace transform and Fourier transform. In the subsequent lectures, we will see how we can find out the solution of this. Thank you.