Transform Calculus and it is Applications in Differential Equations

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Lecture - 42
Solution of Integral Equations using Fourier Transform

In the last lecture, we started with the application of Fourier transform in solving integral equations. If an integral equation is given to us, where a function $f(x)$ is inside the integral sign as an integrand, then by using Fourier transform, how to find out the solution of that equation, that we have studied.

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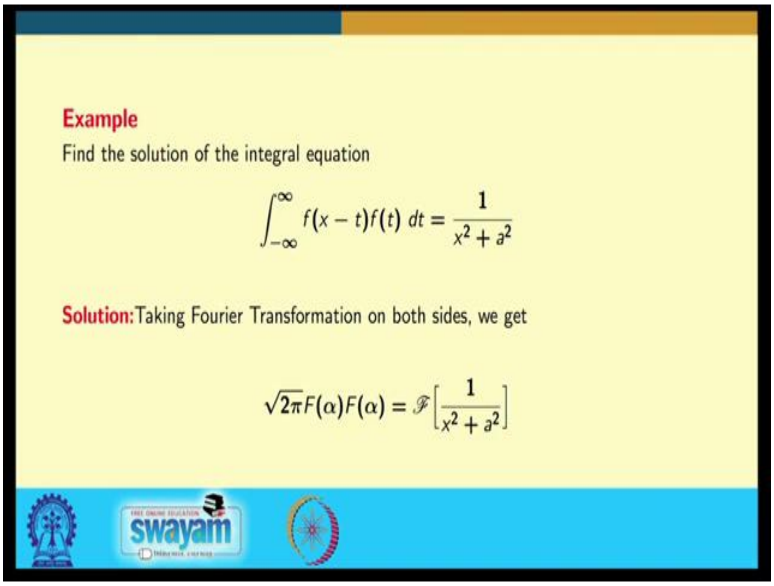

So, let us take another example.

$$\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{1}{x^2+a^2}$$

Please note that our aim is to find out the value of $f(x)$ from this integral equation.

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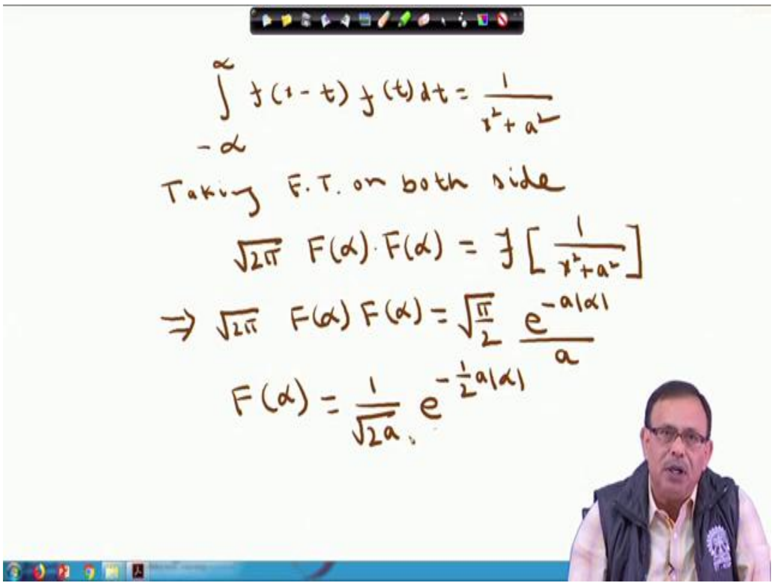

Now multiplying both sides of the given equation by $\frac{1}{\sqrt{2\pi}}$, we get,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{1}{\sqrt{2\pi}}.\frac{1}{x^2+a^2}$$

$$\Rightarrow f(x)*f(x) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{x^2+a^2}$$

Now taking Fourier transform on both sides and using the property of Convolution theorem, we obtain,

$$\sqrt{2\pi}\,F(\alpha)F(\alpha) = \mathcal{F}\left[\frac{1}{x^2+a^2}\right]$$

where $F(\alpha)$ is the Fourier transform of the function $f(x)$.

We have already derived the Fourier transform of $\frac{1}{x^2+a^2}$ earlier. So, using the result, we can write down,

$$\sqrt{2\pi}\,F(\alpha)F(\alpha) = \sqrt{\frac{\pi}{2}}\cdot\frac{e^{-a|\alpha|}}{a}$$

$$\Rightarrow F(\alpha) = \frac{1}{\sqrt{2a}}\cdot e^{-\frac{a|\alpha|}{2}}$$

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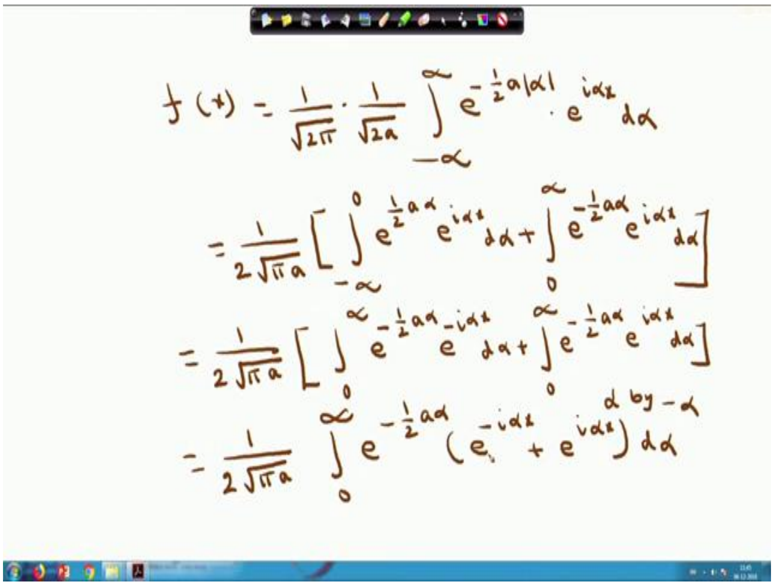

From here, we can write down by using inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2a}} \int_{-\infty}^{\infty} e^{-\frac{a|\alpha|}{2}} e^{i\alpha x} d\alpha$$

So, if we break it, we obtain,

$$f(x) = \frac{1}{2\sqrt{a\pi}} \left[\int_{-\infty}^{0} e^{\frac{a\alpha}{2}} e^{i\alpha x} d\alpha + \int_{0}^{\infty} e^{-\frac{a\alpha}{2}} e^{i\alpha x} d\alpha \right]$$

So, basically in the first integral, if we replace α by $-\alpha$, then we have,

$$f(x) = \frac{1}{2\sqrt{a\pi}} \left[\int_{0}^{\infty} e^{-\frac{a\alpha}{2}} e^{-i\alpha x} d\alpha + \int_{0}^{\infty} e^{-\frac{a\alpha}{2}} e^{i\alpha x} d\alpha \right]$$

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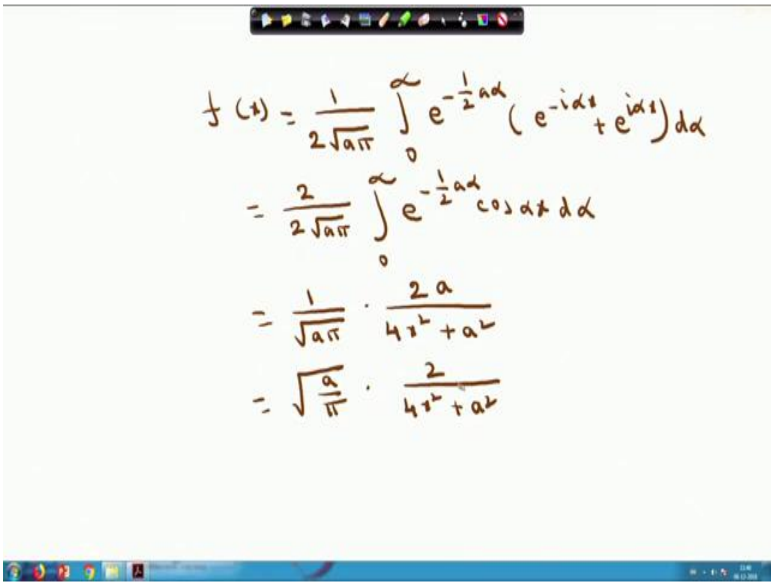

From the earlier one we will get,

$$f(x) = \frac{1}{2\sqrt{a\pi}} \int_0^\infty e^{-\frac{a\alpha}{2}} \left[e^{-i\alpha x} + e^{i\alpha x} \right] d\alpha$$

$$= \frac{1}{\sqrt{a\pi}} \int_0^\infty e^{-\frac{a\alpha}{2}} \cos \alpha x \, d\alpha$$

$$= \frac{1}{\sqrt{a\pi}} \cdot \frac{2a}{4x^2 + a^2} \text{ [use the integration formula for } e^{ax} \cos bx]$$

$$= \sqrt{\frac{a}{\pi}} \cdot \frac{2}{4x^2 + a^2}$$

Thus we can obtain the unknown function $f(x)$ from the given integral equation.

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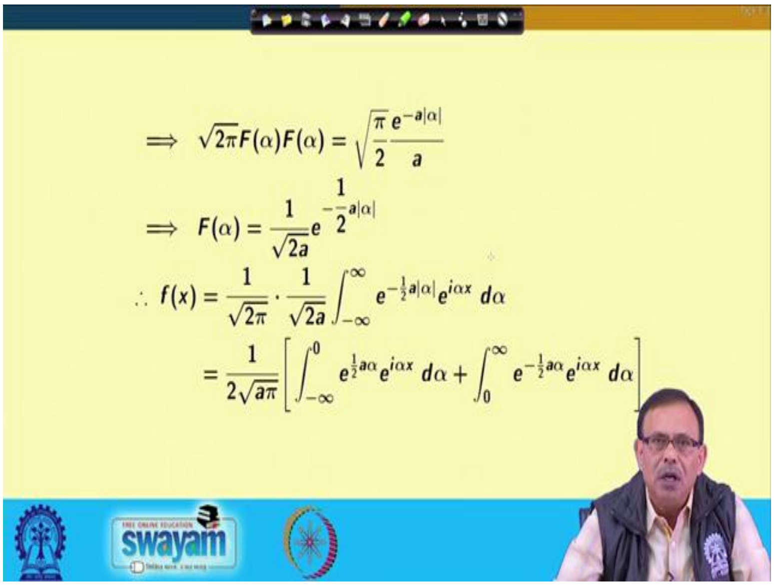

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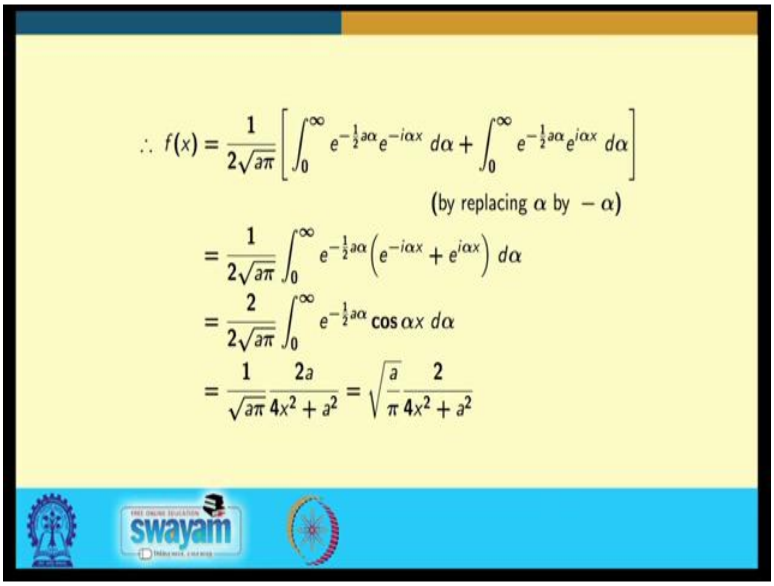

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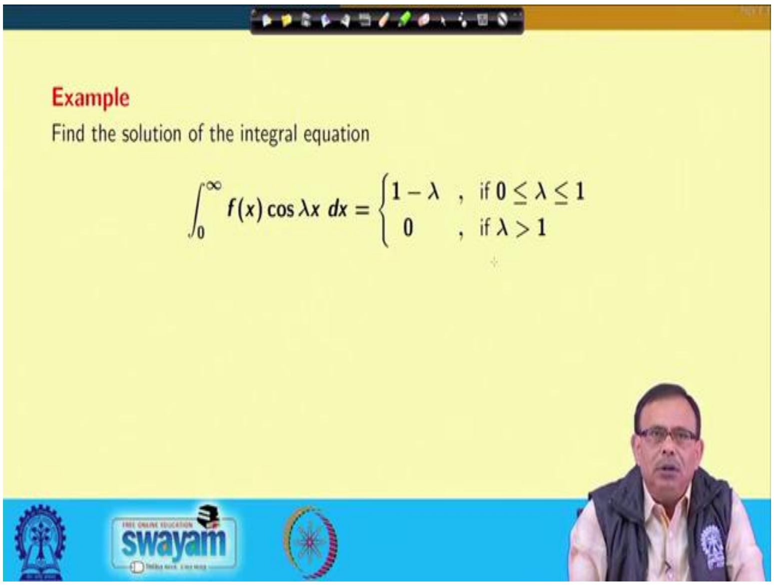

Let us take another example.

$$\int_0^\infty f(x)\cos\lambda x\ dx = \begin{cases} 1-\lambda & , \text{ if } 0 \le \lambda \le 1 \\ 0 & , \text{ if } \lambda > 1 \end{cases}$$

We have to find out the unknown function $f(x)$.

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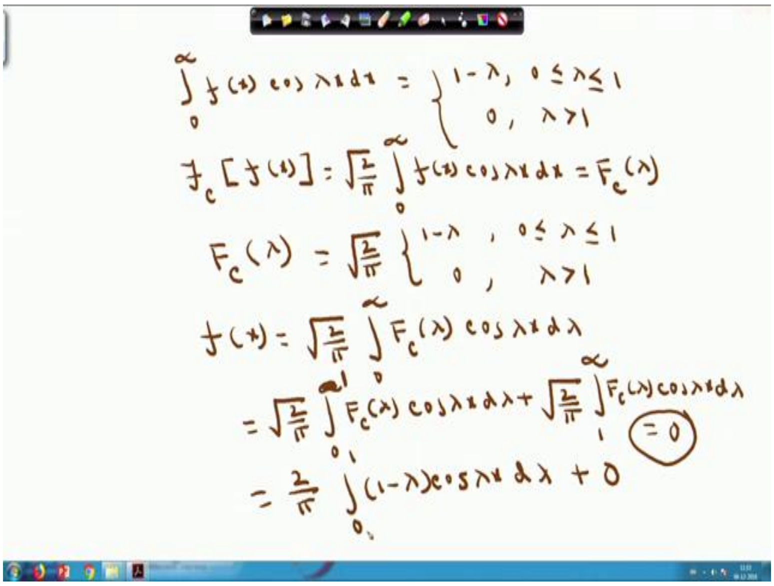

From the definition of Fourier cosine transform, we have,

$$\mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x \ dx = F_c(\lambda)$$

Therefore, from the given equation, we can say that,

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \begin{cases} 1 - \lambda & , \ \text{if } 0 \le \lambda \le 1 \\ 0 & , \ \text{if } \lambda > 1 \end{cases}$$

And once we know the Fourier cosine transform of the function in terms of lambda, then we can always tell what would be the function $f(x)$ by using the inverse Fourier cosine transform. So, now if we use the inverse Fourier cosine transform on this particular function, we can write down,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\lambda) \cos \lambda x \ d\lambda$$

Now putting the value of the function $F_c(\lambda)$ in the above relation, we get,

$$f(x) = \frac{2}{\pi} \int_0^1 (1 - \lambda) \cos \lambda x \ d\lambda + \frac{2}{\pi} \int_1^\infty 0 \cdot \cos \lambda x \ d\lambda$$

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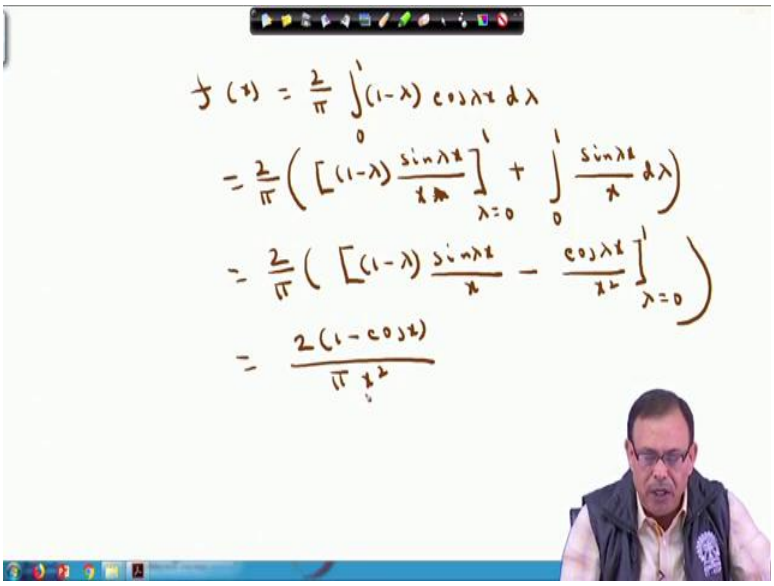

Therefore,

$$
\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^1 (1-\lambda)\cos\lambda x \ d\lambda \\
&= \frac{2}{\pi}\left(\left[\frac{(1-\lambda)\sin\lambda x}{x}\right]_{\lambda=0}^{1} + \int_0^1 \frac{\sin\lambda x}{x}\ d\lambda\right) \\
&= \frac{2}{\pi}\left[-\frac{\cos\lambda x}{x^2}\right]_{\lambda=0}^{1} \\
&= \frac{2(1-\cos x)}{\pi x^2}
\end{aligned}
$$

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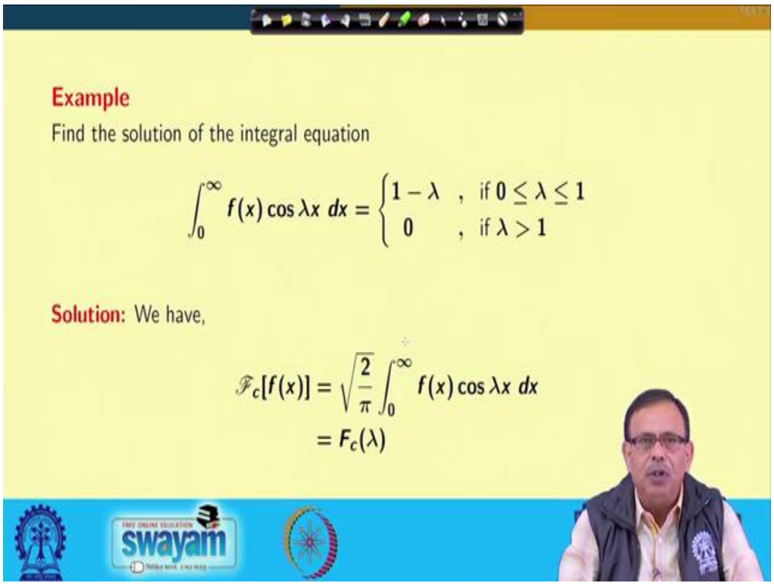

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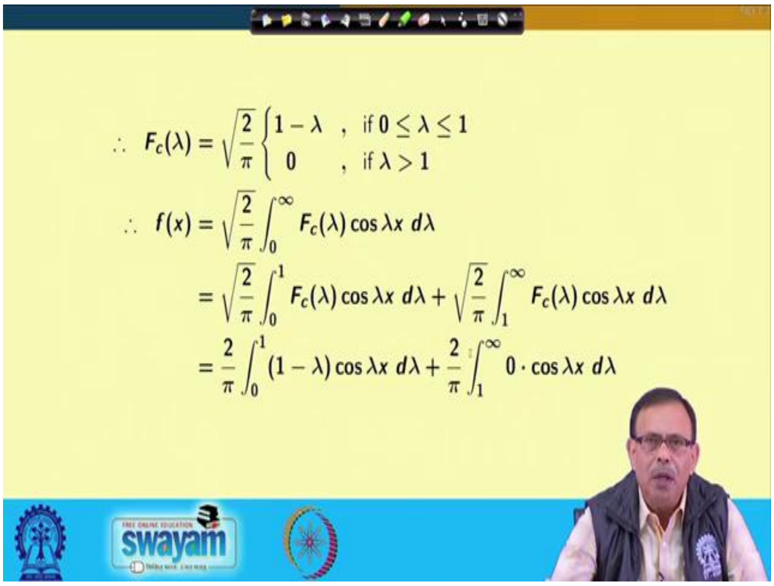

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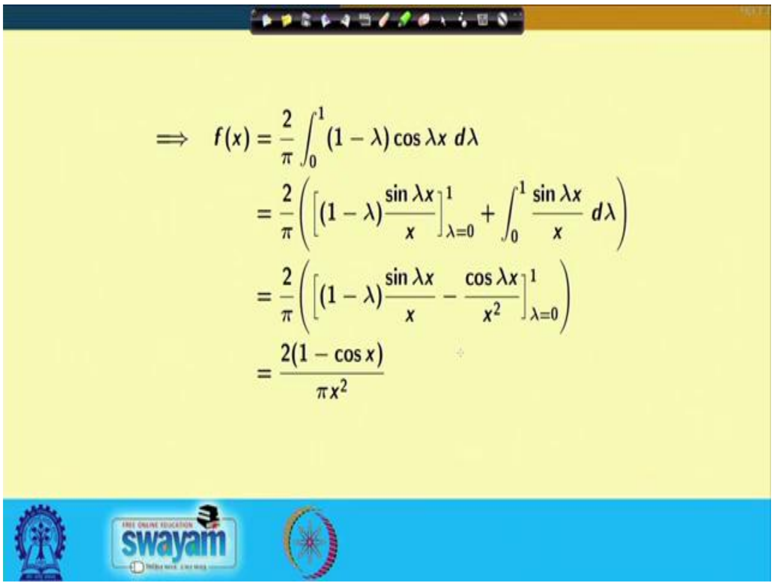

So, like this way, whenever we have the integral equations, to find out the solution of the integral equation that is to find the value of the function $f(x)$, we can use these steps.

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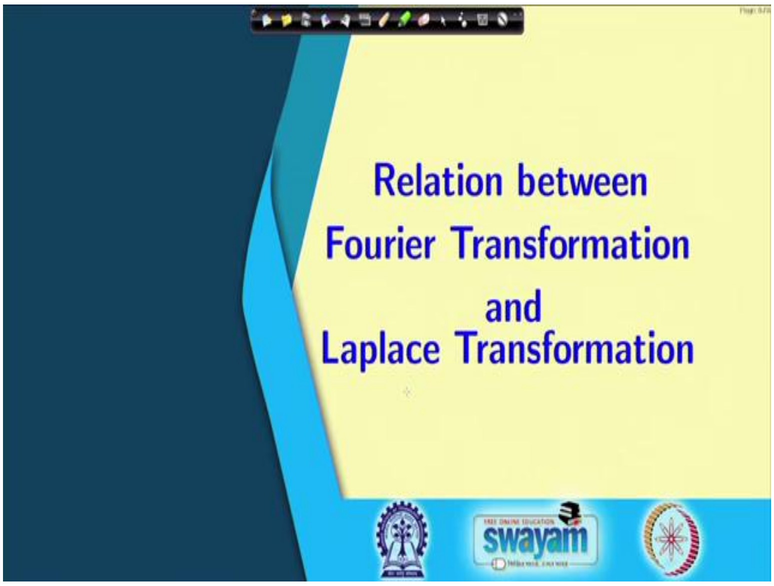

Now, let us see a very small part that is relation between Fourier transform and Laplace transform that is, we will check if any relationship between Fourier transform and Laplace transform exists or not for a particular function. That is under what condition will the Laplace transform of a function be equal to the Fourier transform of some transformed function.

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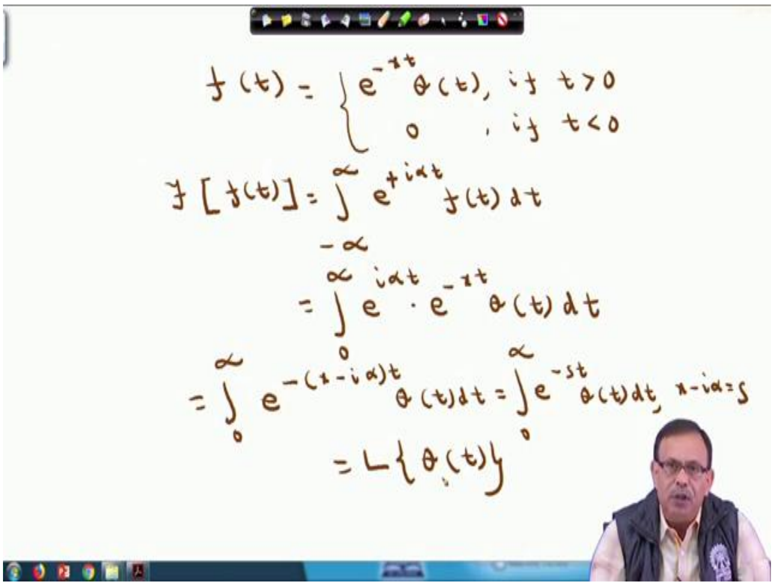

Let us define a function as,

$$f(t) = \begin{cases} e^{-xt}\, Q(t) \;, & \text{if } t > 0 \\ 0 \;, & \text{if } t < 0 \end{cases}$$

Now Fourier transform of $f(t)$ is given as,

$$\begin{aligned} \mathcal{F}[f(t)] &= \int_{-\infty}^{\infty} e^{i\alpha t}\, f(t)dt \\ &= \int_{0}^{\infty} e^{i\alpha t}\, e^{-xt}\, Q(t)dt \\ &= \int_{0}^{\infty} e^{-(x-i\alpha)t}\, Q(t)dt \\ &= \int_{0}^{\infty} e^{-st}\, Q(t)dt \quad \text{where,} \;\; (x - i\alpha) = s \\ &= L\{Q(t)\} \end{aligned}$$

[Please note that we are concerned here with the integration part only, any constant multiplication is ignored]

So, this is the relationship between the Fourier transform and Laplace transform of a function.

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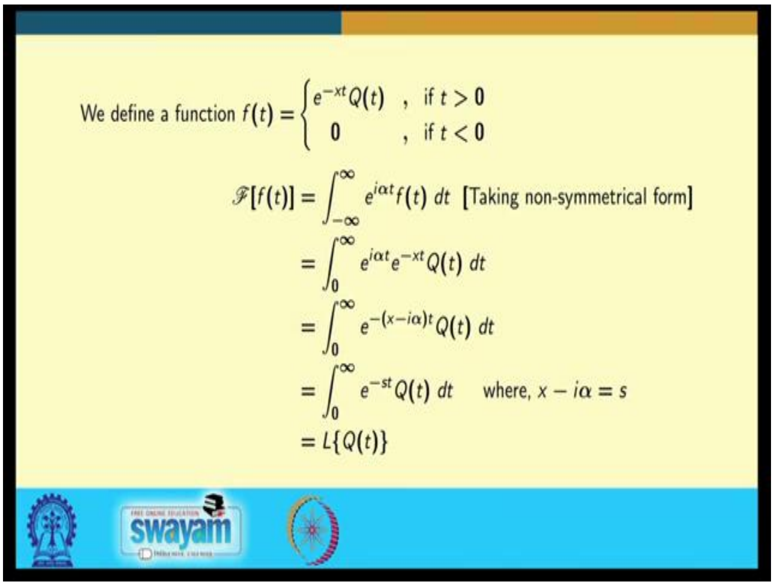

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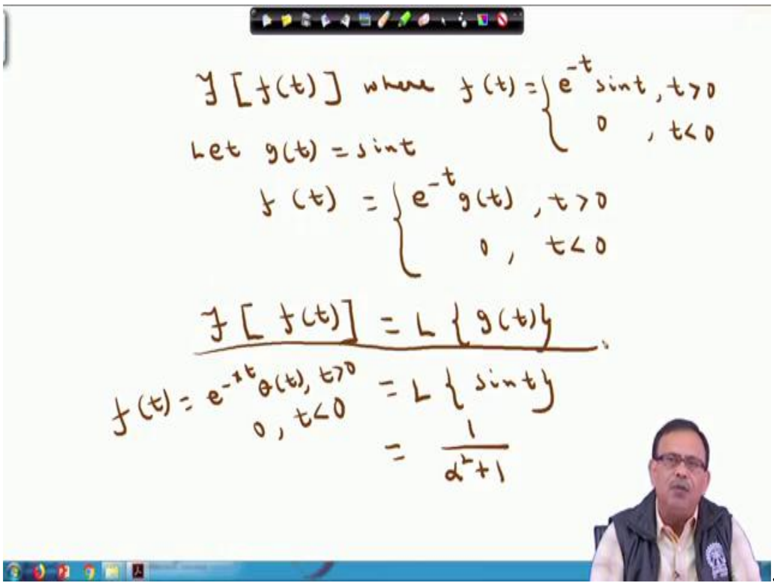

Let us see one example over here. We want to find out the Fourier transform of $f(t)$, where $f(t)$ is given by

$$f(t) = \begin{cases} e^{-t}\sin t & , \text{ if } t > 0 \\ 0 & , \text{ if } t < 0 \end{cases}$$

So, this is in a similar form. Let, $g(t) = \sin t$

So, using the relationship between Laplace and Fourier Transforms, we have,

$$\begin{aligned} \mathcal{F}[f(t)] &= L\{g(t)\} \\ &= L\{\sin t\} \\ &= \frac{1}{1+\alpha^2} \end{aligned}$$

Thank you.