Transform Calculus and it is Applications in Differential Equations
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Lecture - 41
Application of Fourier Transform to Ordinary Differential Equation – II

Continuing with the earlier lecture, where we have just started the solution of ODE using Fourier Transform, let us do one or two more problems so that it is more clearly understandable how to find out the solution of an ODE using Fourier transform and its properties. Initially, we are taking Fourier transform on both sides. Using the properties, we are getting some equation in terms of $y(\alpha)$, then we are taking inverse Fourier transform. There, we may have to use the convolution theorem. Then, we will get the solution.

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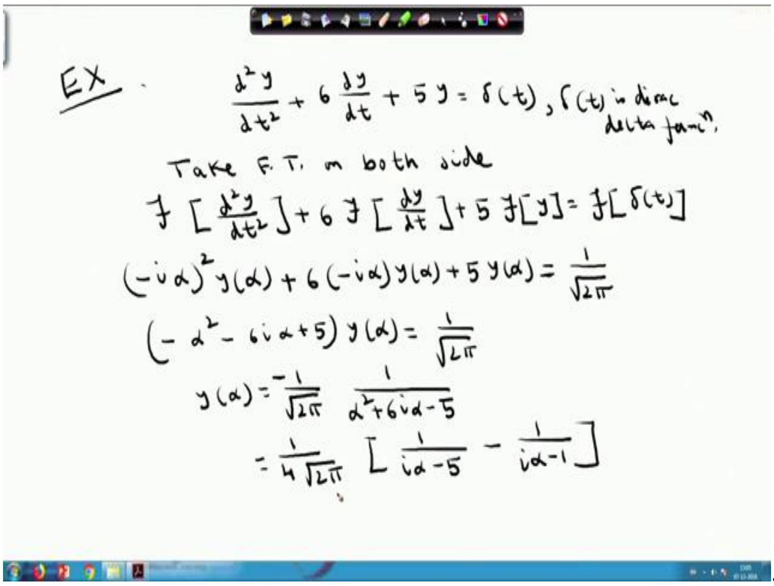

So, now let us consider this example,

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = \delta(t)$$

where $\delta(t)$ is Dirac delta function.

Now, if we take the Fourier transform on both sides, we will obtain,

$$\mathcal{F}\left[\frac{d^2y}{dt^2}\right] + 6\,\mathcal{F}\left[\frac{dy}{dt}\right] + 5\,\mathcal{F}[y] = \mathcal{F}[\delta(t)]$$

We have already done that,

$$\mathcal{F}\left[\frac{d^ny}{dt^n}\right] = (-i\alpha)^n\mathcal{F}[y] = (-i\alpha)^nY(\alpha)$$

$$\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}}$$

So, if we use these formulae, then we can write down,

$$(-i\alpha)^2Y(\alpha) + 6(-i\alpha)Y(\alpha) + 5Y(\alpha) = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow (-\alpha^2 - 6i\alpha + 5)\,Y(\alpha) = \frac{1}{\sqrt{2\pi}}$$

So that we can write down,

$$Y(\alpha) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(-\alpha^2 - 6i\alpha + 5)}$$

If we break this, we will obtain,

$$\begin{aligned} Y(\alpha) &= -\,\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(\alpha + i)(\alpha + 5i)} \\ &= -\,\frac{1}{4i\,\sqrt{2\pi}} \cdot \left[\frac{1}{(\alpha + i)} - \frac{1}{(\alpha + 5i)}\right] \\ &= \frac{1}{4i\,\sqrt{2\pi}} \cdot \left[\frac{1}{(\alpha + 5i)} - \frac{1}{(\alpha + i)}\right] \\ &= \frac{1}{4\sqrt{2\pi}} \cdot \left[\frac{1}{(i\alpha - 5)} - \frac{1}{(i\alpha - 1)}\right] \end{aligned}$$

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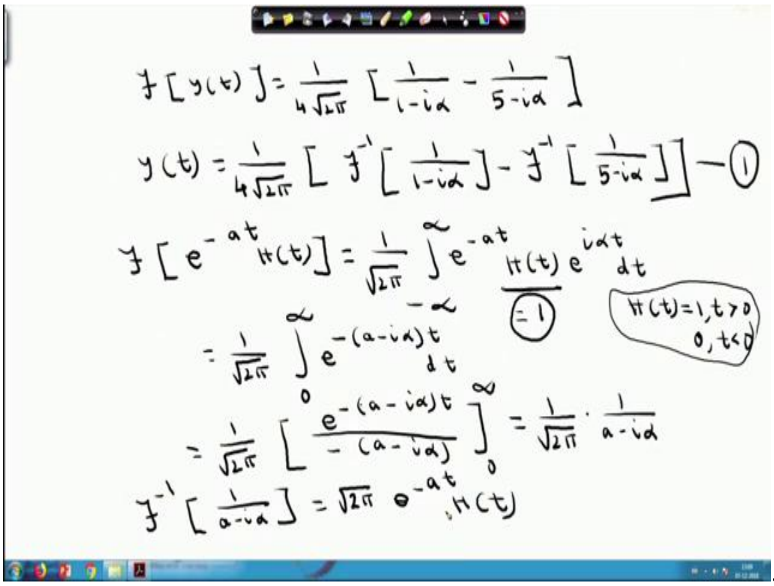

Therefore $Y(\alpha)$ is given as,

$$Y(\alpha) = \frac{1}{4\sqrt{2\pi}} \cdot \left[\frac{1}{(1-i\alpha)} - \frac{1}{(5-i\alpha)}\right]$$

Now we can obtain $y(t)$ by taking the inverse Fourier transform on both sides. Therefore,

$$y(t) = \frac{1}{4\sqrt{2\pi}}\left(\mathcal{F}^{-1}\left[\frac{1}{(1-i\alpha)}\right] - \mathcal{F}^{-1}\left[\frac{1}{(5-i\alpha)}\right]\right)$$

So, to find out the value of above Fourier inverse, let us start with the Fourier transform of $e^{-at}H(t)$, where $H(t)$ is the Heaviside unit step function. So, from the definition of Fourier transform, we can write down,

$$\mathcal{F}[e^{-at}H(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-at}H(t)\,e^{i\alpha t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-(a-i\alpha)t}\,dt \qquad \left[\because H(t) = \begin{cases}1 & , \quad t > 0 \\ 0 & , \quad t < 0\end{cases}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(a-i\alpha)t}}{-(a-i\alpha)}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(a-i\alpha)}$$

$$\therefore \mathcal{F}^{-1}\left[\frac{1}{a-i\alpha}\right] = \sqrt{2\pi}\,e^{-at}H(t)$$

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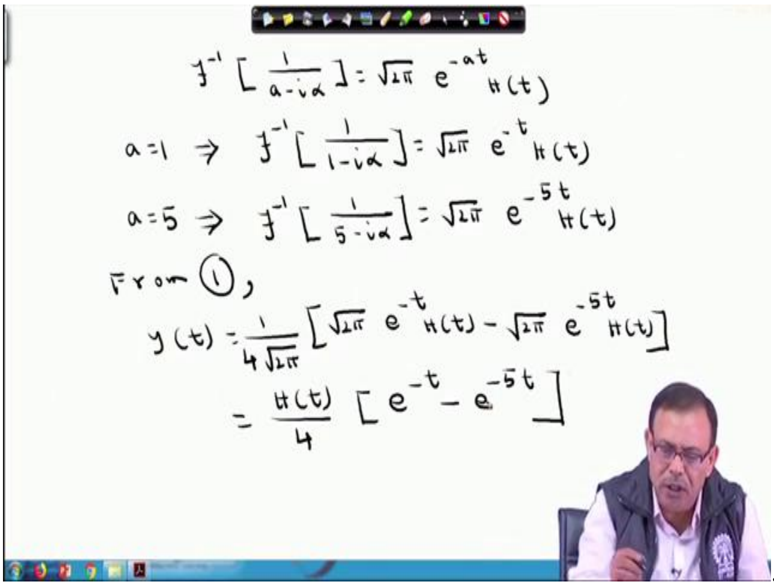

Therefore,

$$\text{for } a = 1, \quad \mathcal{F}^{-1}\left[\frac{1}{1-i\alpha}\right] = \sqrt{2\pi}\, e^{-t} H(t)$$

$$\text{for } a = 5, \quad \mathcal{F}^{-1}\left[\frac{1}{5-i\alpha}\right] = \sqrt{2\pi}\, e^{-5t} H(t)$$

So, therefore, if we substitute these values, we will obtain $y(t)$ as,

$$y(t) = \frac{1}{4\sqrt{2\pi}}\left(\sqrt{2\pi}\, e^{-t} H(t) - \sqrt{2\pi}\, e^{-5t} H(t)\right)$$

$$= \frac{H(t)}{4}\left(e^{-t} - e^{-5t}\right)$$

So, like this way, we can solve a given ODE.

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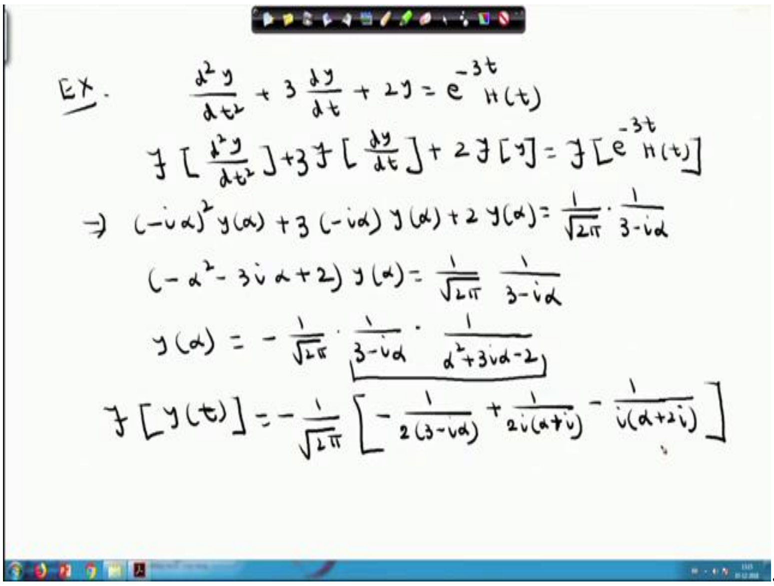

Now, let us take one more example and then we will go to the next topic. So, next example states:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-3t}H(t)$$

where $H(t)$ is the Heaviside unit step function.

If we take the Fourier transform on both sides of the given equation, following the similar process, we will obtain,

$$\mathcal{F}\left[\frac{d^2y}{dt^2}\right] + 3\mathcal{F}\left[\frac{dy}{dt}\right] + 2\mathcal{F}[y] = \mathcal{F}[e^{-3t}H(t)]$$

We know that,

$$\mathcal{F}[e^{-at}H(t)] = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(a - i\alpha)}$$

Therefore,

$$(-\alpha^2 - 3i\alpha + 2)\, Y(\alpha) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(3 - i\alpha)}$$

So, directly we can write,

$$
\begin{aligned}
Y(\alpha) &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(3 - i\alpha)(-\alpha^2 - 3i\alpha + 2)} \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2(3 - i\alpha)} - \frac{1}{2i(\alpha + i)} + \frac{1}{i(\alpha + 2i)} \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2(3 - i\alpha)} + \frac{1}{2(1 - i\alpha)} - \frac{1}{(2 - i\alpha)} \right]
\end{aligned}
$$

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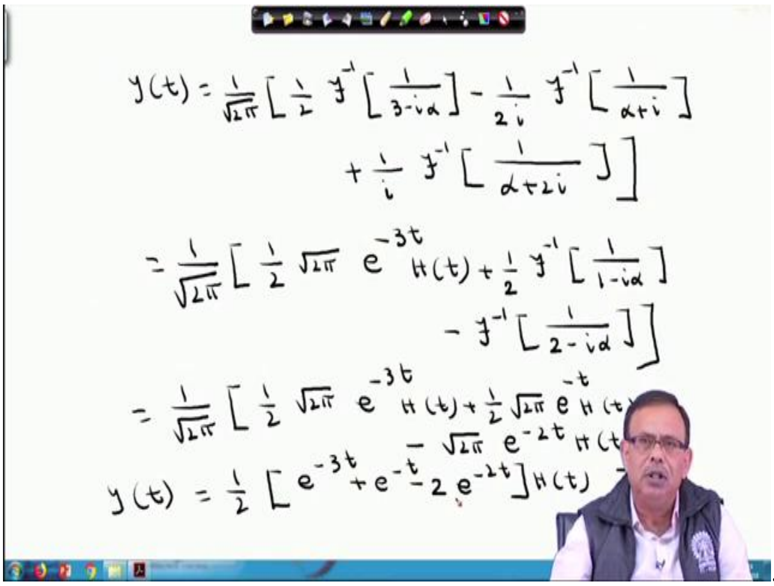

So, now, using inverse Fourier transform, we get,

$$
y(t) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2} \mathcal{F}^{-1} \left[\frac{1}{(3 - i\alpha)} \right] + \frac{1}{2} \mathcal{F}^{-1} \left[\frac{1}{(1 - i\alpha)} \right] - \mathcal{F}^{-1} \left[\frac{1}{(2 - i\alpha)} \right] \right)
$$

We have already obtained $\mathcal{F}^{-1} \left[\frac{1}{(a - i\alpha)} \right]$ in the last example. Using that result, we will get $y(t)$ as,

$$
\begin{aligned}
y(t) &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2} \sqrt{2\pi}\, e^{-3t} H(t) + \frac{1}{2} \sqrt{2\pi}\, e^{-t} H(t) - \sqrt{2\pi}\, e^{-2t} H(t) \right) \\
&= \frac{1}{2} H(t) \left(e^{-3t} + e^{-t} - 2\, e^{-2t} \right)
\end{aligned}
$$

So, we have studied how to find out the solution of an ODE using Fourier transform.

Now let us go to the next topic, that is application of Fourier transform to integral equations. Whenever a function satisfies an integral equation, we aim at finding out the solution of that integral equation using the Fourier transform.

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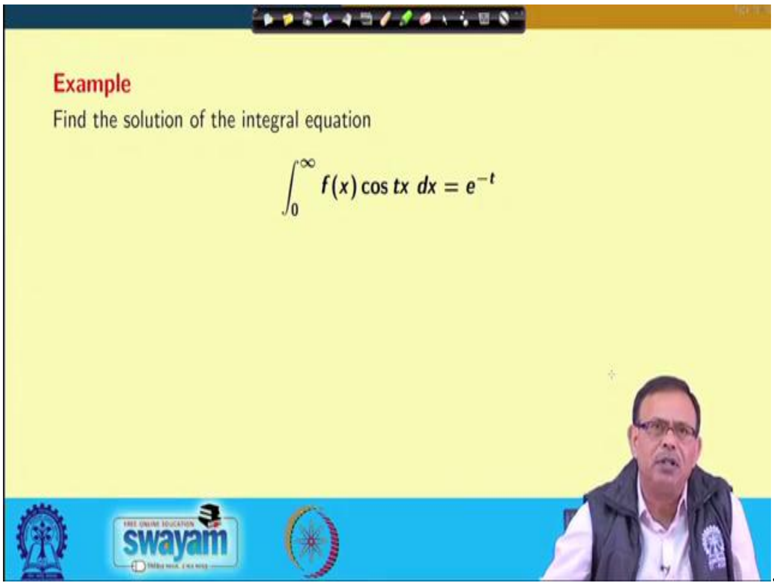

Let us take an integral equation where we would like to find the unknown function $f(x)$.

$$\int_0^\infty f(x) \cos tx \ dx = e^{-t}$$

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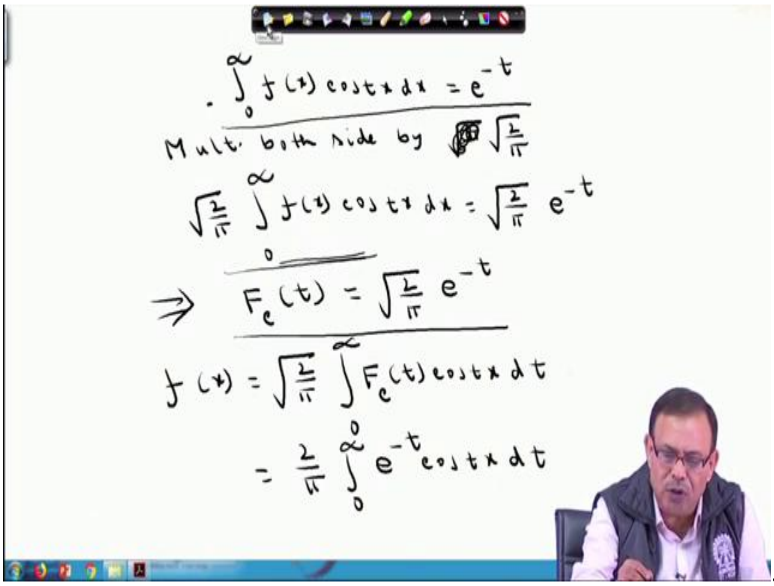

Now, multiplying both sides by $\sqrt{\frac{2}{\pi}}$, we obtain,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos tx \;\, dx = \sqrt{\frac{2}{\pi}}e^{-t}$$

If you remember, integral on the LHS is nothing but the Fourier cosine transform of $f(x)$. So, we can write it as,

$$F_c(t) = \sqrt{\frac{2}{\pi}}\; e^{-t}$$

So, once we are obtaining Fourier cosine transform of a function, then we can use the inverse formula to derive what is $f(x)$. From definition, we have,

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(t)\cos tx \;\, dt \\ &= \frac{2}{\pi}\int_0^\infty e^{-t}\cos tx \;\, dt \\ &= \frac{2}{\pi}\; I \;\; \text{(say)} \end{aligned}$$

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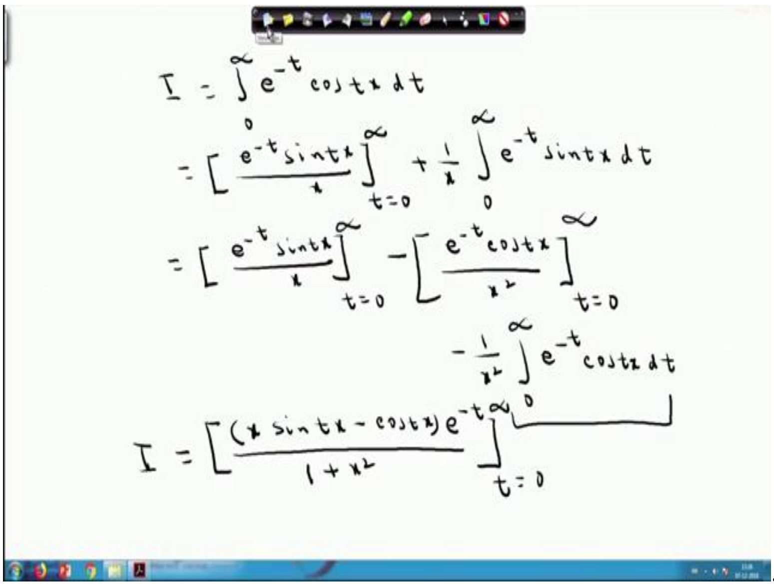
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$$
\begin{aligned}
I &= \int_0^\infty e^{-t}\cos tx \ dt \\
&= \left[\frac{e^{-t}\sin tx}{x}\right]_{t=0}^{\infty} + \frac{1}{x}\int_0^\infty e^{-t}\sin tx \ dt \\
&= [0-0] - \left[\frac{e^{-t}\cos tx}{x^2}\right]_{t=0}^{\infty} - \frac{1}{x^2}\int_0^\infty e^{-t}\cos tx \ dt \\
&= -\left[0-\frac{1}{x^2}\right] - \frac{1}{x^2}\, I
\end{aligned}
$$

$$
\therefore I = \frac{1}{1+x^2}
$$

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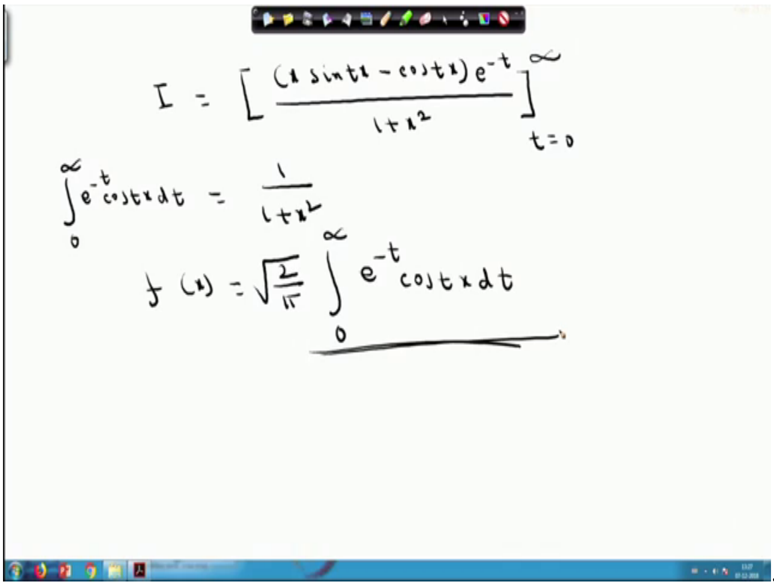

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Now, putting the value of I, we obtain $f(x)$ as,

$$
f(x) = \frac{2}{\pi(1+x^2)}
$$

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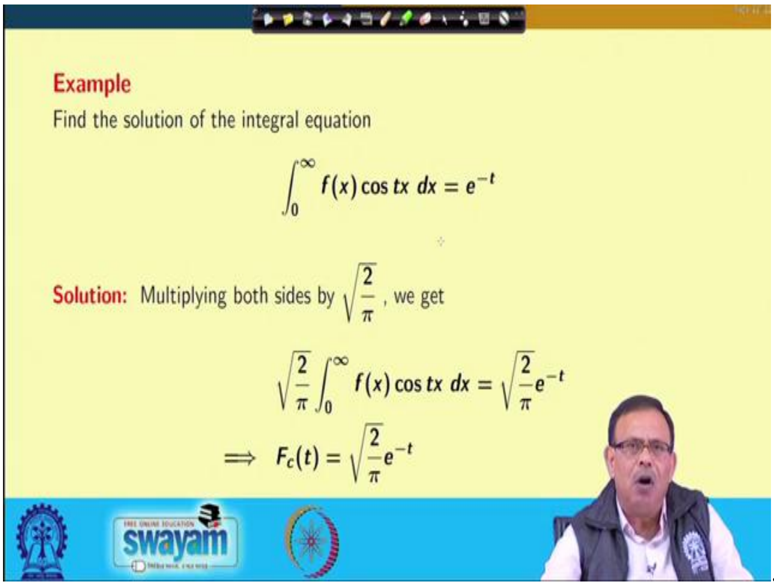

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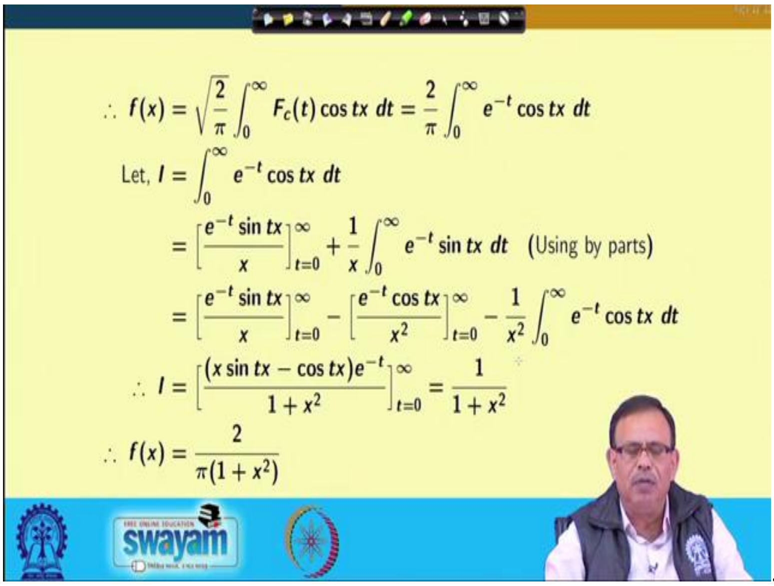

Thank you.