Transform Calculus and its Applications in Differential Equations

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Lecture – 40
Applications of Fourier Transform to Ordinary Differential Equations – I

In the last few lectures, we have started the Fourier transform of a function where we have studied how to find out the Fourier transform, Fourier cosine transform or Fourier sine transform of a function. Then we have studied the properties of Fourier transform which include the scaling property, shaping property among others.

Besides these, we have also studied that if we know the Fourier transform of a function, then how to find out the Fourier transform of the derivative of the function or Fourier transform of integration of that function. Afterwards, we have discussed the convolution definition and the convolution theorem and then we have studied the Parseval’s theorem as well.

Now, we are going to study the applications of Fourier transform. The first one which we will study is the application of Fourier transform to ordinary differential equations. If we recall, we have done the same for Laplace transform also, that is we have discussed how to find the solution of an ordinary differential equation using Laplace transform. So, now, we will first study how to find out the solution of an ordinary differential equation, mostly the second order ODE, using the Fourier transform without using the concept of complementary function (CF) and particular integral (PI).

So, let us see the first example of this. The first topic is application of Fourier transform to ODE.

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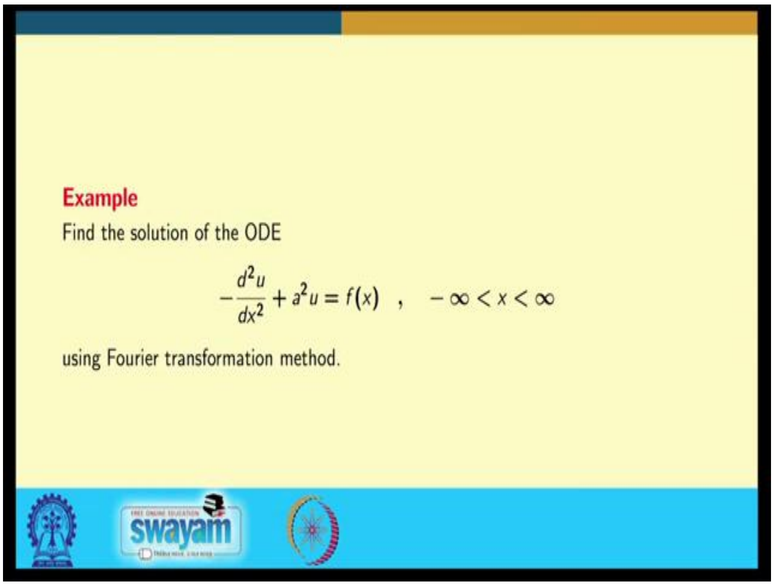

Here we have to find out the solution of the ODE given as,

$$-\frac{d^2u}{dx^2} + a^2u = f(x)\,, \qquad -\infty < x < \infty$$

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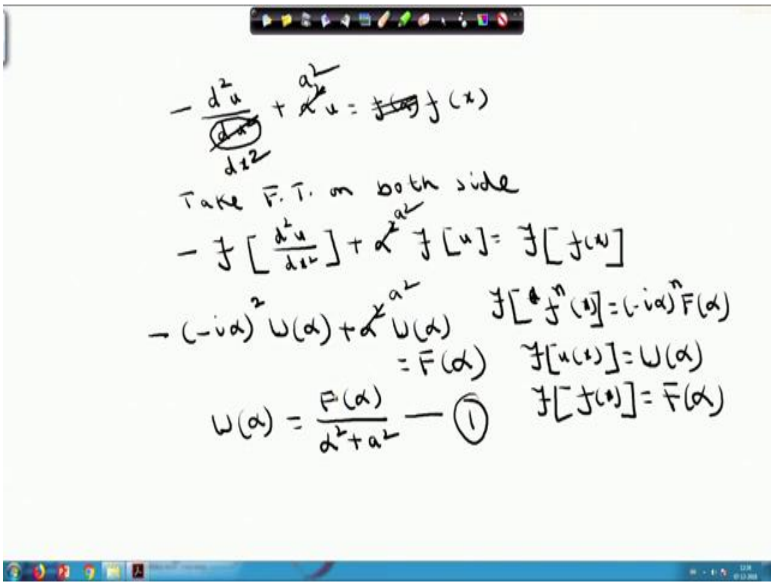

If we take the Fourier transform on both sides of the given equation, then we have,

$$-\mathcal{F}\left[\frac{d^2u}{dx^2}\right] + a^2\mathcal{F}[u] = \mathcal{F}[f(x)] \qquad (1)$$

We have already studied the Fourier transform of the derivative of a function and the formula is given as,

$$\mathcal{F}[f^n(x)] = (-i\alpha)^n \mathcal{F}[f(x)] = (-i\alpha)^n F(\alpha)$$

Suppose,

$$\mathcal{F}[u(x)] = U(\alpha) \;\; \text{and} \;\; \mathcal{F}[f(x)] = F(\alpha)$$

Therefore, the equation (1) can be written as,

$$-(-i\alpha)^2 U(\alpha) + a^2 U(\alpha) = F(\alpha)$$

After simplification, we will obtain,

$$U(\alpha) = \frac{F(\alpha)}{\alpha^2 + a^2} \tag{2}$$

So, whenever we want to solve a second order ODE, first we will take the Fourier transform on both sides of the given equation. Use the formula $\mathcal{F}[f^n(x)] = (-i\alpha)^n F(\alpha)$ and then simplify it.

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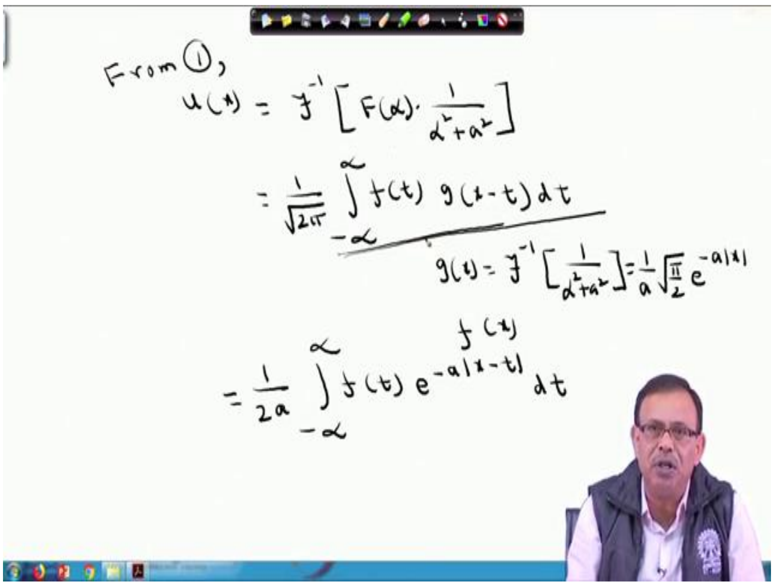

Now, from equation (2), if we take the inverse Fourier transform, then we will get $u(x)$ as,

$$u(x) = \mathcal{F}^{-1}\left[F(\alpha)\cdot\frac{1}{\alpha^2+a^2}\right]$$

From the convolution theorem, we have,

$$\mathcal{F}^{-1}[F(\alpha)\cdot G(\alpha)] = (f*g)(x) = \mathcal{F}^{-1}[F(\alpha)] * \mathcal{F}^{-1}[G(\alpha)]$$

Suppose,

$$\mathcal{F}[g(x)] = G(\alpha) = \frac{1}{\alpha^2+a^2}$$

$$\therefore g(x) = \mathcal{F}^{-1}\left[\frac{1}{\alpha^2+a^2}\right] = \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a|x|}$$

Using this result, we will get $u(x)$ as,

$$\begin{aligned} u(x) &= \mathcal{F}^{-1}\left[F(\alpha)\cdot\frac{1}{\alpha^2+a^2}\right] \\ &= (f*g)(x) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt \\ &= \frac{1}{2a}\int_{-\infty}^{\infty} f(t)\, e^{-a|x-t|}\, dt \end{aligned}$$

Once $f(x)$ is known to us, then we can evaluate the integral and hence calculate $u(x)$.

So, please note that to find out the solution of an ODE using Fourier transform, we will take the Fourier transform on both sides of the given ODE, then we will find an equation in $U(\alpha)$ where $U(\alpha)$ equals some function of α and $U(\alpha)$ is nothing but the Fourier transform of $u(x)$. Then we take the inverse Fourier transform to find out $u(x)$. Sometimes, we may have to use the convolution theorem just like we have used over here.

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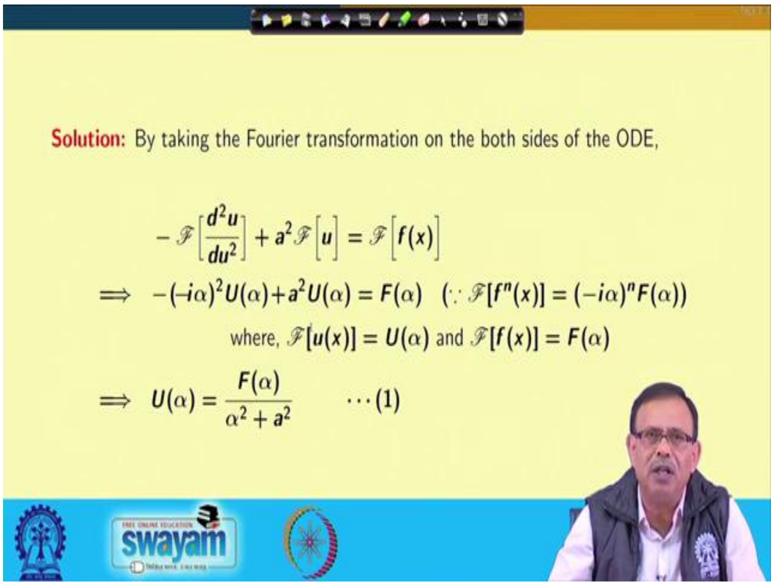

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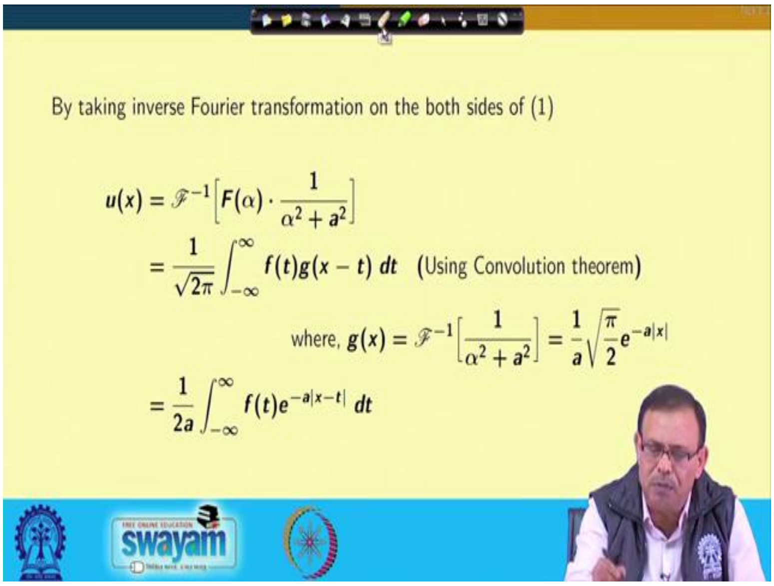

Now, let us take some more examples.

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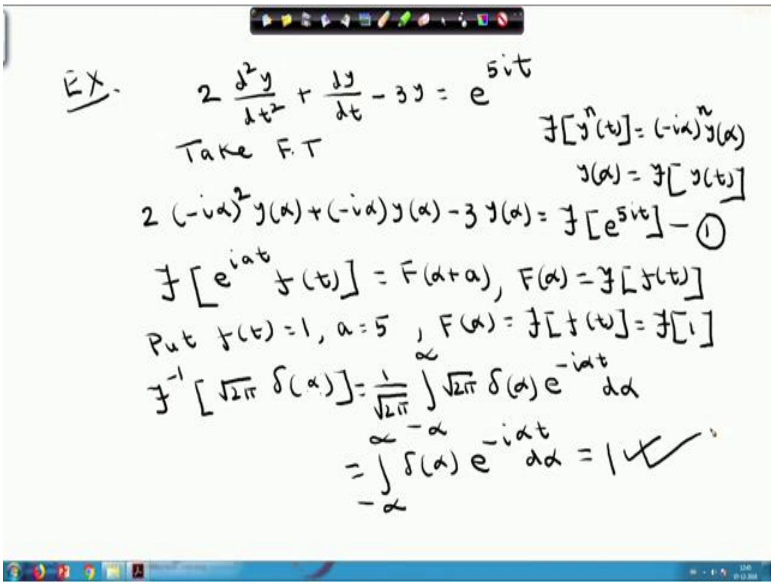

The earlier one was a generic one. Now, let us take another example, say,

$$2\frac{d^2y}{dt^2}+\frac{dy}{dt}-3y=e^{5it}$$

To solve this, we will use Fourier transform on both sides of the given equation. From the formula for Fourier transform of derivatives, we have $\mathcal{F}[y^n(x)]=(-i\alpha)^n\mathcal{F}[y(x)]=(-i\alpha)^nY(\alpha)$. Therefore, taking Fourier transform on both sides of the given equation, we will obtain,

$$2(-i\alpha)^2Y(\alpha)+(-i\alpha)Y(\alpha)-3Y(\alpha)=\mathcal{F}\left[e^{5it}\right]$$

$$\Rightarrow -(2\alpha^2+i\alpha+3)Y(\alpha)=\mathcal{F}\left[e^{5it}\right] \qquad (3)$$

Now, very first thing what we have to find out is the Fourier transform of e^{5it} otherwise we cannot proceed here. Now, we know that,

$$\mathcal{F}\left[e^{iax}f(x)\right]=F(\alpha+a)$$

Therefore, if we take $f(x)=1$, $\mathcal{F}[f(x)]=F(\alpha)$ and $a=5$, then we will have,

$$\mathcal{F}\left[e^{5it}\right]=F(\alpha+5)$$

Now, from the definition, Fourier inverse of $\sqrt{2\pi}\,\delta(\alpha)$ is

$$\mathcal{F}^{-1}\left[\sqrt{2\pi}\,\delta(\alpha)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{2\pi}\,\delta(\alpha)\,e^{-i\alpha x}\,d\alpha$$

$$= \int_{-\infty}^{\infty} \delta(\alpha)\,e^{-i\alpha x}\,d\alpha$$

Using the property of Dirac Delta function, we get,

$$\mathcal{F}^{-1}\left[\sqrt{2\pi}\,\delta(x)\right] = \left[e^{-i\alpha x}\right]_{\alpha=0} = 1$$

$$\Rightarrow \mathcal{F}[1] = \sqrt{2\pi}\,\delta(\alpha) = F(\alpha)$$

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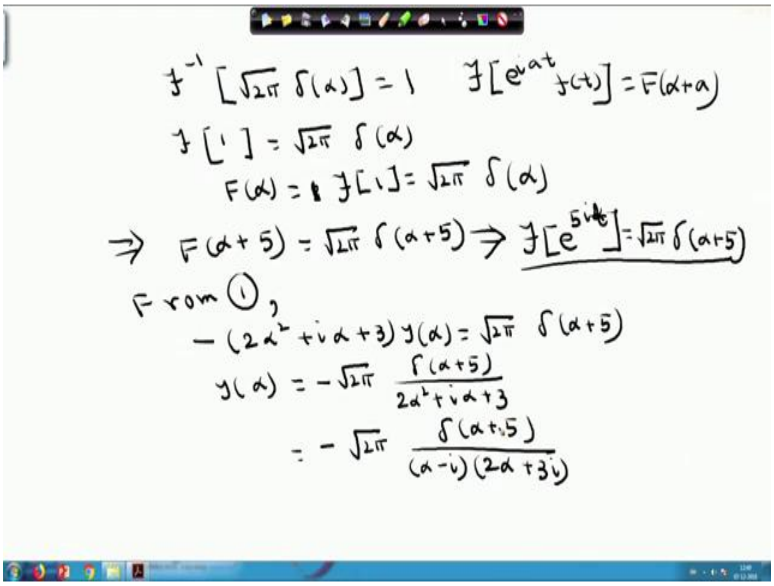

Therefore,

$$\mathcal{F}\left[e^{5it}\right] = F(\alpha+5) = \sqrt{2\pi}\,\delta(\alpha+5)$$

So, once we have obtained the Fourier transform of e^{5it}, from equation (3) we will get,

$$-(2\alpha^2 + i\alpha + 3)Y(\alpha) = \sqrt{2\pi}\,\delta(\alpha+5)$$

After simplification, we can write the above equation as,

$$Y(\alpha) = -\sqrt{2\pi}\,\frac{\delta(\alpha+5)}{(2\alpha^2 + i\alpha + 3)}$$

$$= -\sqrt{\frac{\pi}{2}}\,\frac{\delta(\alpha+5)}{(\alpha - i)\left(\alpha + \frac{3}{2}i\right)}$$

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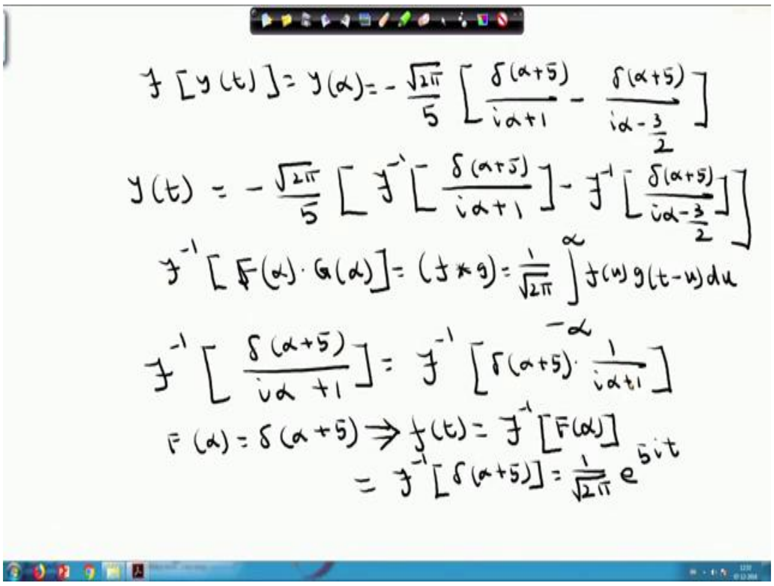

Now, from the last equation we have,

$$Y(\alpha) = -\sqrt{\frac{\pi}{2}}\,\frac{\delta(\alpha+5)}{(\alpha - i)\left(\alpha + \frac{3}{2}i\right)}$$

$$= -\sqrt{\frac{\pi}{2}}\,\delta(\alpha+5)\cdot\frac{2}{5i}\left[\frac{\left(\alpha + \frac{3}{2}i\right) - (\alpha - i)}{(\alpha - i)\left(\alpha + \frac{3}{2}i\right)}\right]$$

$$= -\sqrt{2\pi}\left[\frac{\delta(\alpha+5)}{5i(\alpha - i)} - \frac{\delta(\alpha+5)}{5i\left(\alpha + \frac{3}{2}i\right)}\right]$$

$$= -\frac{\sqrt{2\pi}}{5}\left[\frac{\delta(\alpha+5)}{(i\alpha+1)} - \frac{\delta(\alpha+5)}{\left(i\alpha - \frac{3}{2}\right)}\right]$$

Now taking the inverse Fourier transform, we will get $y(t)$ as,

$$y(t) = -\frac{\sqrt{2\pi}}{5}\left(\mathcal{F}^{-1}\left[\frac{\delta(\alpha+5)}{(i\alpha+1)}\right] - \mathcal{F}^{-1}\left[\frac{\delta(\alpha+5)}{\left(i\alpha-\frac{3}{2}\right)}\right]\right)$$

Suppose,

$$\begin{aligned} F(\alpha) &= \delta(\alpha+5) \\ \Rightarrow f(t) &= \mathcal{F}^{-1}[F(\alpha)] \\ &= \mathcal{F}^{-1}[\delta(\alpha+5)] \\ &= \frac{1}{\sqrt{2\pi}}e^{5it} \end{aligned}$$

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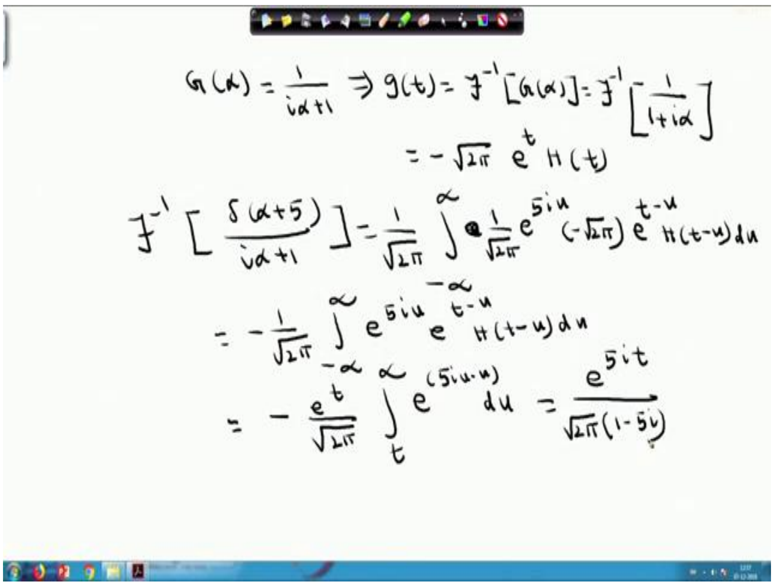

Suppose,

$$G(\alpha) = \frac{1}{(i\alpha+1)}$$

$$\therefore \mathcal{F}^{-1}\left[\frac{\delta(\alpha+5)}{(i\alpha+1)}\right] = \mathcal{F}^{-1}[F(\alpha)\cdot G(\alpha)]$$

If we take the inverse Fourier transform, then,

$$\begin{aligned} g(t) &= \mathcal{F}^{-1}[G(\alpha)] \\ &= \mathcal{F}^{-1}\left[\frac{1}{(i\alpha+1)}\right] \end{aligned}$$

If we evaluate this, then we have,

$$g(t) = \sqrt{2\pi}\, e^{t} H(-t)$$

where $H(t)$ is Heaviside unit step function. Therefore,

$$H(-t) = \begin{cases} 1 & , \quad t < 0 \\ 0 & , \quad t > 0 \end{cases}$$

Now, we have,

$$\mathcal{F}^{-1}\left[\frac{\delta(\alpha+5)}{(i\alpha+1)}\right] = \mathcal{F}^{-1}[F(\alpha)\cdot G(\alpha)]$$

Using the convolution theorem, we will obtain,

$$\begin{aligned} \mathcal{F}^{-1}\left[\frac{\delta(\alpha+5)}{(i\alpha+1)}\right] &= \mathcal{F}^{-1}[F(\alpha)] * \mathcal{F}^{-1}[G(\alpha)] \\ &= f(t) * g(t) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u) g(t-u) du \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} e^{5iu}\right]\left[\sqrt{2\pi}\, e^{t-u} H(u-t)\right] du \\ &= \frac{e^{t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{5iu}\, e^{-u} H(u-t)\, du \end{aligned}$$

Now we have,

$$H(u-t) = \begin{cases} 1 & , \quad u > t \\ 0 & , \quad u < t \end{cases}$$

Therefore,

$$\mathcal{F}^{-1}\left[\frac{\delta(\alpha+5)}{(i\alpha+1)}\right] = \frac{e^{t}}{\sqrt{2\pi}}\int_{t}^{\infty} e^{-u+5iu}\,du$$
$$= \frac{e^{t}}{\sqrt{2\pi}}\left[\frac{e^{-u+5iu}}{(-1+5i)}\right]_{t}^{\infty}$$
$$= \frac{e^{t}}{\sqrt{2\pi}}\cdot\frac{e^{-t+5it}}{(1-5i)}$$
$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{e^{5it}}{(1-5i)}$$

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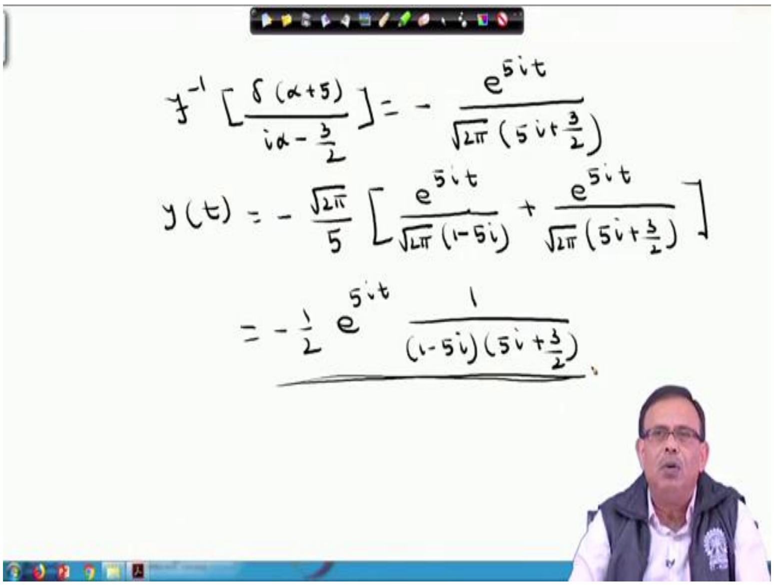

Using the same procedure, we can obtain,

$$\mathcal{F}^{-1}\left[\frac{\delta(\alpha+5)}{\left(i\alpha-\frac{3}{2}\right)}\right] = -\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{5it}}{\left(\frac{3}{2}+5i\right)}$$

Finally, we will obtain $y(t)$ as,

$$y(t) = -\frac{\sqrt{2\pi}}{5}\left(\frac{1}{\sqrt{2\pi}} \cdot \frac{e^{5it}}{(1-5i)} + \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{5it}}{\left(\frac{3}{2}+5i\right)}\right)$$

$$= -\frac{e^{5it}}{5} \cdot \frac{\frac{3}{2}+5i+1-5i}{(1-5i)\left(\frac{3}{2}+5i\right)}$$

$$= -\frac{e^{5it}}{2} \cdot \frac{1}{(1-5i)\left(\frac{3}{2}+5i\right)}$$

In the next lecture also, we will solve one or more problems, so that we can understand how to find out the solution of ODE using Fourier transform. Thank you.