Transform Calculus and its Applications in Differential Equations
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Lecture - 04
Laplace Transform of Derivative and Integration of a Function - I

In the earlier lecture, we have studied certain properties of Laplace transform. In this lecture, initially, we will go through some more examples, after which we will discuss certain other important properties of Laplace Transform.

The first one is an application of the change of scale property.

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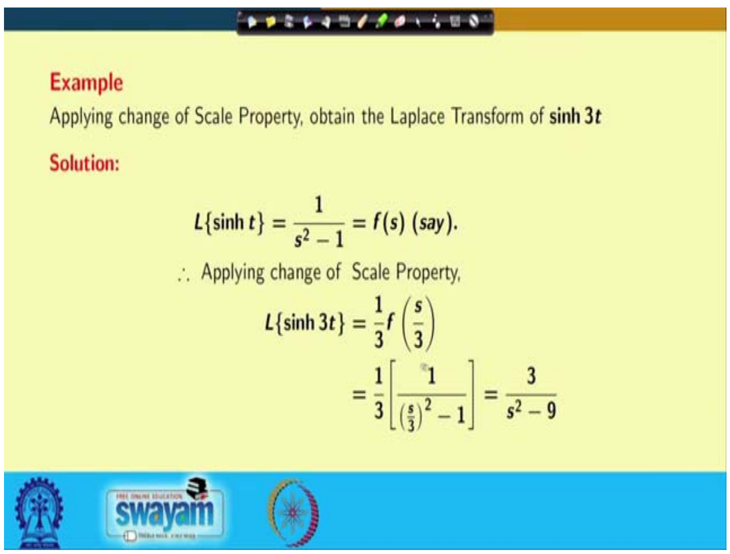

We need to find the Laplace transform of $sinh\, 3t$. So, as we know, $\mathrm{L}\{\sinh t\} = \frac{1}{s^2-1}$ (putting $a = 1$ in the formula for $\mathrm{L}\{\sinh at\}$).

$$\mathrm{L}\{\sinh t\} = \frac{1}{s^2-1} = f(s) \text{ (say).}$$

Therefore, in order to obtain the Laplace Transform of $sinh\, 3t$, we can apply change of scale property as

$$\mathrm{L}\{\sinh at\} = \frac{1}{a} f\left(\frac{s}{a}\right).$$

Applying the change of scale property for $a = 3$, we have

$$\text{L}\{\sinh 3t\} = \frac{1}{3} f\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3} \frac{9}{s^2 - 9}$$

$$= \frac{3}{s^2 - 9}.$$

So, we can see the advantage of applying the change of scale property. Once we know $\text{L}\{\sinh t\}$, then we can simply evaluate $\text{L}\{\sinh at\}$ for any given value of a.

The next example is on the similar line. Find the Laplace transform of $\cos 5t$.

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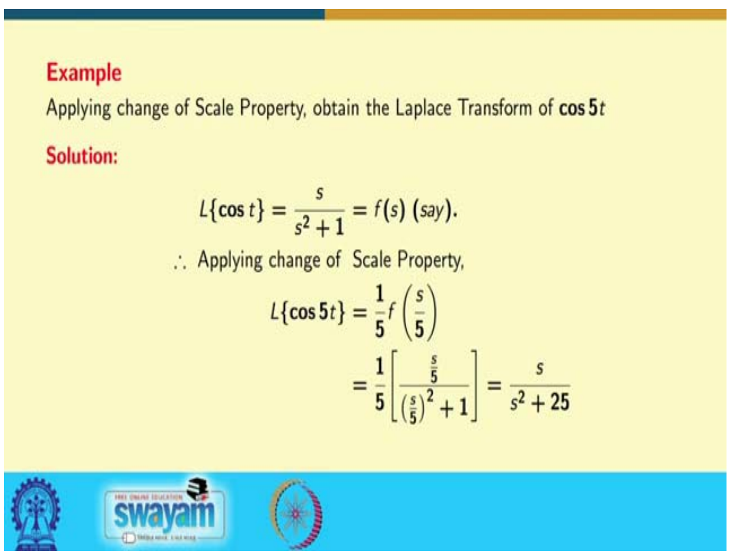

We know the Laplace transform of $\cos t$

$$L\{\cos t\} = \frac{s}{s^2+1} = f(s) \quad \text{(say)}.$$

Again applying the change of scale property, we can write down

$$L\{\cos 5t\} = \frac{1}{5} f\left(\frac{s}{5}\right)$$

$$\Rightarrow L\{\cos 5t\} = \frac{1}{5}\frac{\frac{s}{5}}{\left(\frac{s}{5}\right)^2 + 1}$$

$$= \frac{1}{5}\frac{5s}{s^2 + 25}$$

$$= \frac{s}{s^2 + 25}.$$

Let us see the next example.

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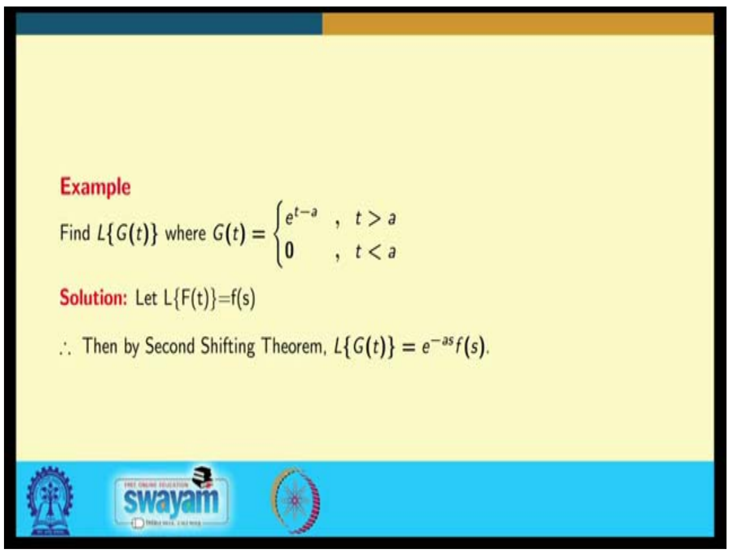

$G(t)$ is a given function defined as $G(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a \end{cases}$ and we need to find out its Laplace transform. So, initially we assume that Laplace transform of $F(t)$ is $f(s)$.

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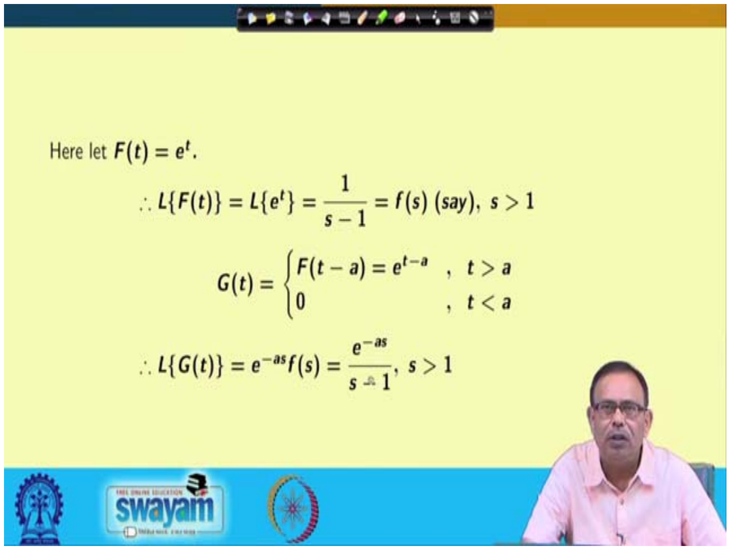

Next we assume $F(t) = e^t$ (say).

$$\therefore L\{F(t)\} = L\{e^t\} = \frac{1}{s-1} = f(s), \qquad s > 1.$$

Then, $e^{t-a} = F(t-a)$ and by the given definition of $G(t)$, we have

$$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

Then, by second shifting property, Laplace transform of $G(t)$ would be $e^{-as} f(s)$.

$$\Rightarrow L\{G(t)\} = e^{-as} f(s)$$

$$= \frac{e^{-as}}{s-1}, \qquad s > 1.$$

This gives us the desired result for $L\{G(t)\}$ as $\frac{e^{-as}}{s-1}$.

Let us now move to the next example.

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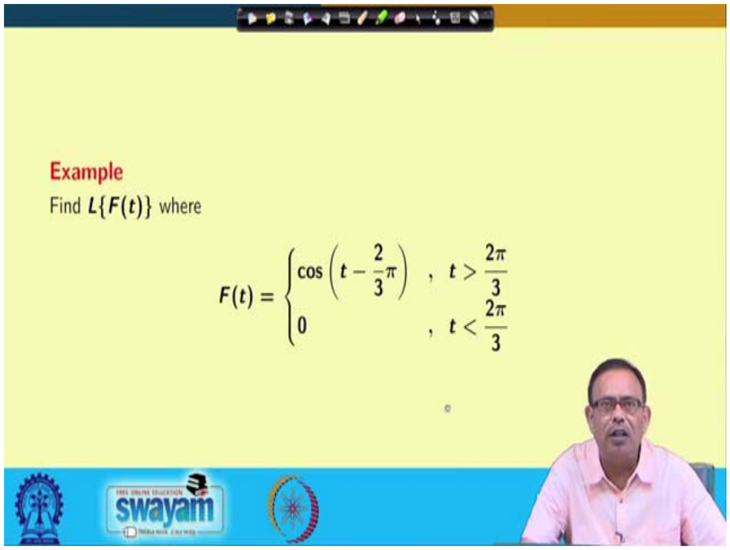

We are given a function $F(t)$ as

$$F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$$

whose Laplace transform we need to evaluate.

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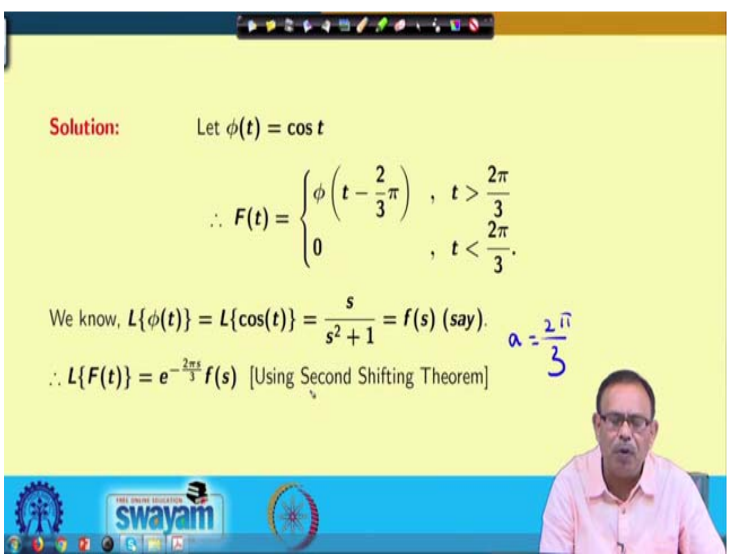

So, at first we assume $\phi(t) = \cos t$ so that $\phi\left(t - \frac{2\pi}{3}\right) = \cos\left(t - \frac{2\pi}{3}\right)$. Therefore, we have

$$L\{\phi(t)\} = L\{\cos t\} = \frac{s}{s^2 + 1} = f(s) \quad \text{(say).}$$

Then, $F(t)$ becomes

$$F(t) = \begin{cases} \phi\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$$

so that we can apply the second shifting property to evaluate $L\{F(t)\}$ as

$$L\{F(t)\} = e^{-\frac{2\pi}{3}s} f(s)$$

$$= e^{-\frac{2\pi s}{3}} \frac{s}{s^2 + 1}, \qquad s > 0.$$

So, whenever we are using this second shifting property, it becomes very easy for us to find out the Laplace transform of some unknown functions knowing the Laplace transform of certain known functions.

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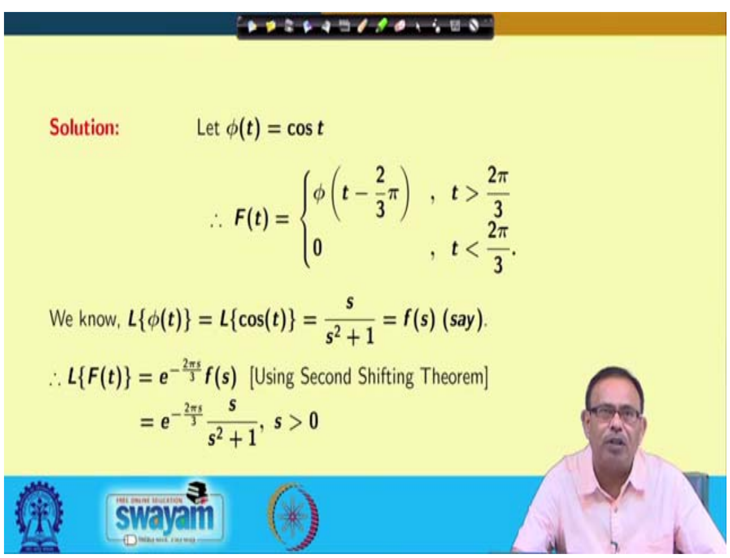

We can use an alternative method as well to solve the previous problem.

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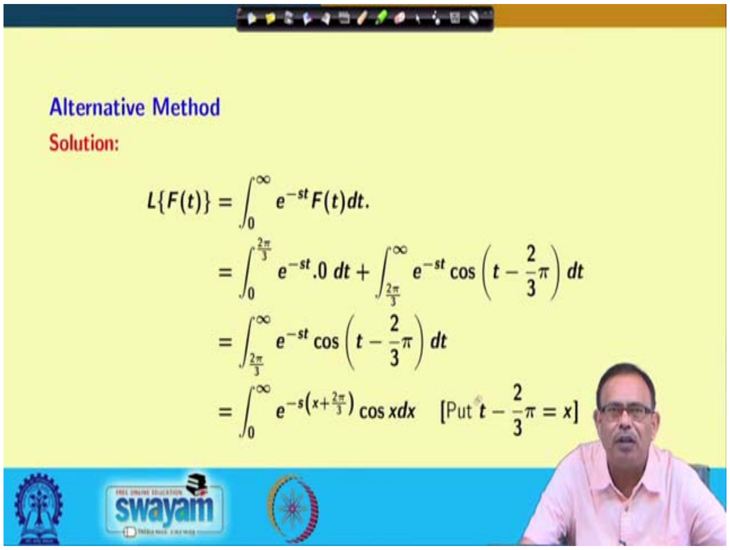

Using the definition of Laplace transform, we can directly write it as

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt.$$

We can now break this integral into two parts according to the definition of $F(t)$ as follows:

$$L\{F(t)\} = \int_0^{\frac{2\pi}{3}} e^{-st}.0\,dt + \int_{\frac{2\pi}{3}}^\infty e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$$

$$= \int_{\frac{2\pi}{3}}^\infty e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt.$$

In order to evaluate the integral, we put $t-\frac{2\pi}{3} = x$ so that $dt = dx$ and the limits of the integration are changed from $\left[\frac{2\pi}{3},\infty\right)$ to $[0,\infty)$.

$$L\{F(t)\} = \int_0^\infty e^{-s\left(x+\frac{2\pi}{3}\right)}\cos x\,dx$$

$$\Rightarrow L\{F(t)\} = e^{-\frac{2\pi s}{3}} \int_0^\infty e^{-sx} \cos x \, dx$$

$$= e^{-\frac{2\pi s}{3}} \int_0^\infty e^{-st} \cos t \, dt$$

$$= e^{-\frac{2\pi s}{3}} L\{\cos t\}$$

$$= e^{-\frac{2\pi s}{3}} \frac{s}{s^2 + 1}, \qquad s > 0.$$

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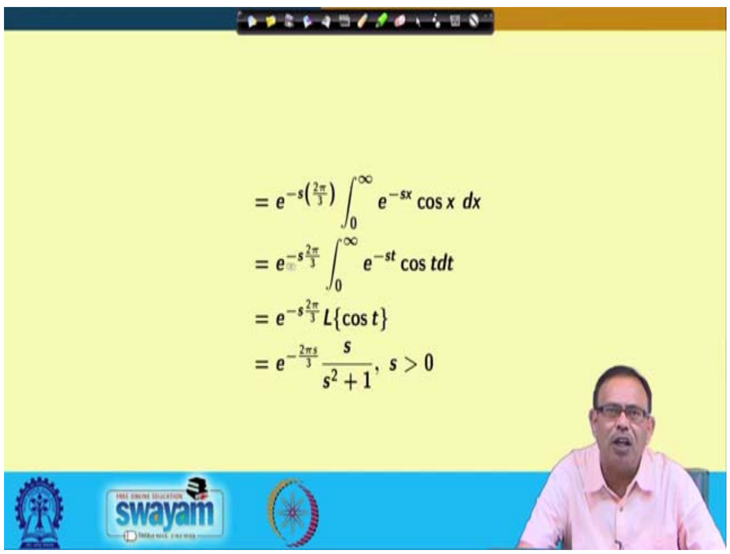

So, from both the methods, we observe that if we know the second shifting property, we can directly evaluate the Laplace transform of given $F(t)$ and we do not need to evaluate the integral.

Next example is of the similar type as the previous one.

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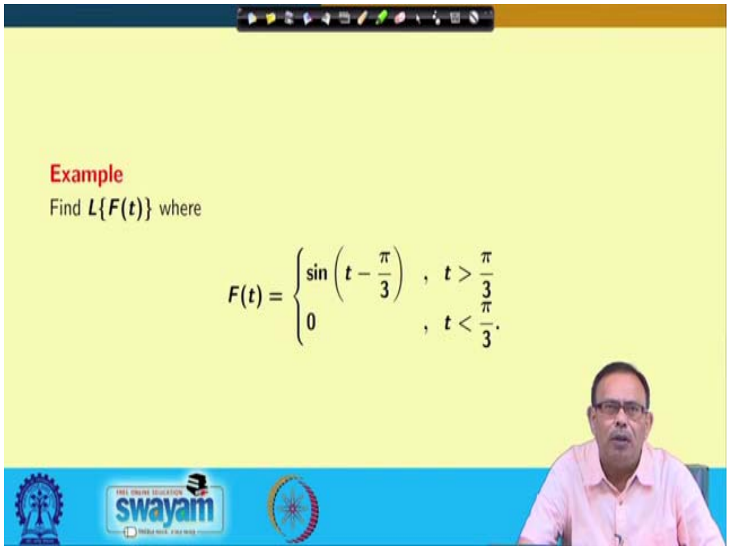

A function is given as

$$F(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases}$$

We need to evaluate $L\{F(t)\}$.

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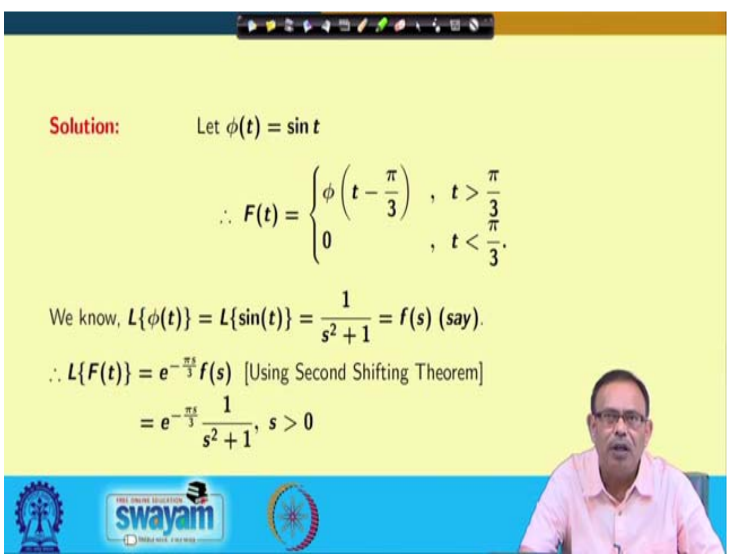

Here, we assume $\phi(t) = \sin t$ so that $\phi\left(t - \frac{\pi}{3}\right) = \sin\left(t - \frac{\pi}{3}\right)$. Therefore, we have

$$L\{\phi(t)\} = L\{\sin t\} = \frac{1}{s^2+1} = f(s) \quad \text{(say).}$$

Then, $F(t)$ becomes

$$F(t) = \begin{cases} \phi\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases}$$

so that we can apply the second shifting property to evaluate $L\{F(t)\}$ as

$$L\{F(t)\} = e^{-\frac{\pi}{3}s} f(s)$$

$$= \frac{e^{-\frac{\pi s}{3}}}{s^2+1}, \qquad s > 0.$$

Now, we come to another important property that is Laplace transform of derivatives of $F(t)$.

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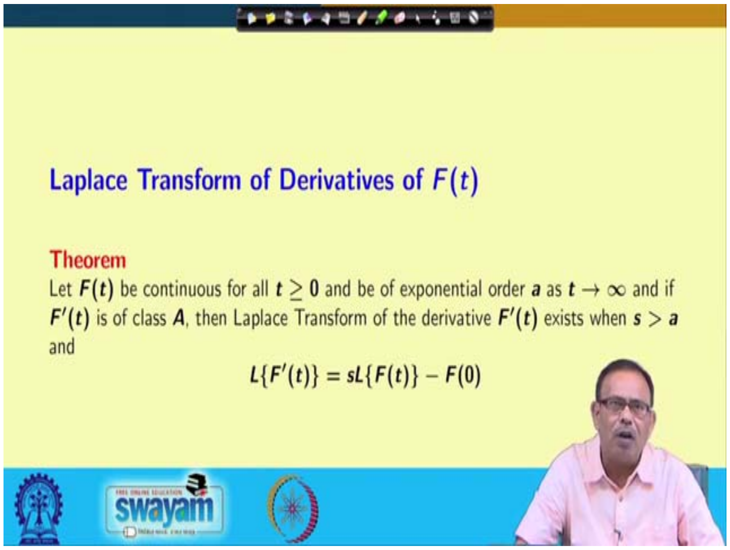

Let $F(t)$ be a continuous function for all $t \geq 0$ and be of exponential order a as $t \to \infty$. And if $F'(t)$ is of class A, i.e., $F'(t)$ is piecewise continuous and is of exponential order as $t \to \infty$, then Laplace transform of $F'(t)$exists, when $s > a$.

And we can say that $L\{F'(t)\}$ equals $sL\{F(t)\} - F(0)$. Let us see the proof of this property.

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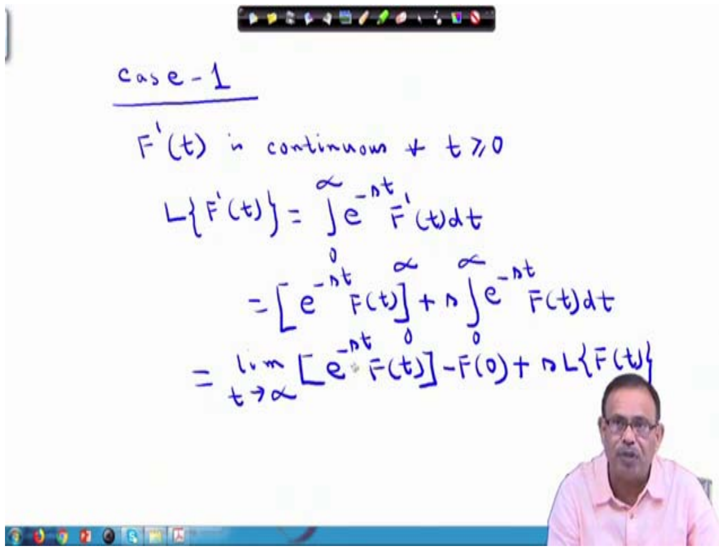

Since $F'(t)$ is of class A, so we know that $F'(t)$ is a piecewise continuous function. However, $F'(t)$ may be continuous $\forall t \geq 0$ as well. Thus, two cases may arise as follows:

Case 1: $F'(t)$ is continuous $\forall t \geq 0$

Case 2: $F'(t)$ is piecewise continuous

We start with Case 1.

In this case, we are assuming that $F'(t)$ is continuous $\forall t \geq 0$. We can write down from the definition of Laplace transform,

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t) dt.$$

We will use integration by parts to evaluate this.

$$\therefore L\{F'(t)\} = [e^{-st} F(t)]_{t=0}^\infty + s\int_0^\infty e^{-st} F(t) dt$$

$$= \lim_{t\to\infty} [e^{-st} F(t)] - F(0) + sL\{F(t)\}. \qquad (1)$$

We know the $F(0)$ has a finite value and $\int_0^\infty e^{-st}F(t)dt = L\{F(t)\}$. So, we only have to check whether $\lim_{t\to\infty}[e^{-st}F(t)]$ is finite or not. If this limiting value exists, then we can say that Laplace transform of $F'(t)$ exists, and it will have some finite value.

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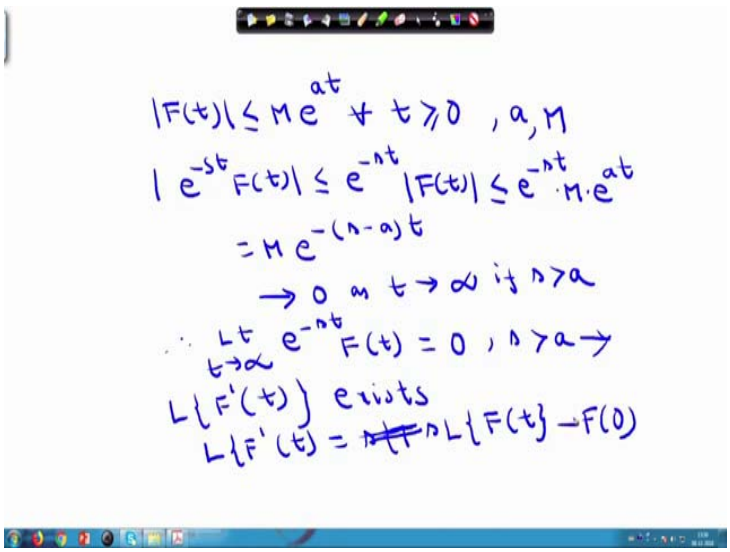

Since $F(t)$ is of exponential order a as $t \to \infty$, so there exists a positive real number M and a number $a > 0$ and a finite number t_0 such that

$$|F(t)| \le Me^{at} \quad \forall\, t \ge t_0$$

$$\therefore\ |e^{-st}F(t)| \le e^{-st}|F(t)|$$

$$\le e^{-st}Me^{at}$$

$$= Me^{-(s-a)t}$$

$$\to 0 \text{ as } t \to \infty \text{ if } s > a.$$

$$\therefore \lim_{t\to\infty}[e^{-st}F(t)] = 0, \qquad s > a.$$

Therefore, $L\{F'(t)\}$ exists.

From (1), $$L\{F'(t)\} = \lim_{t\to\infty}[e^{-st}F(t)] - F(0) + sL\{F(t)\}$$

$$= sL\{F(t)\} - F(0).$$

This completes the proof for Case 1.

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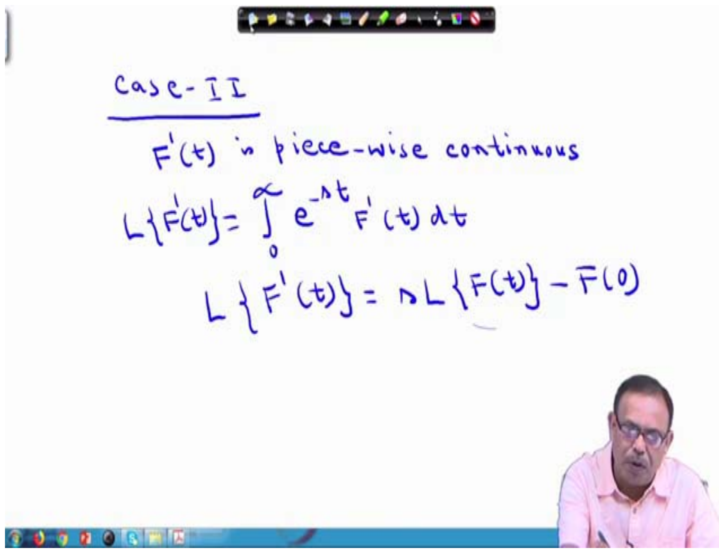

Now, we come to the second case. In case 2, we assume that $F'(t)$ is piecewise continuous, which means, in each particular sub-domain of $[0, \infty)$, the function $F'(t)$ will be continuous. We can write from the definition of Laplace transform,

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t) dt.$$

This integral can be broken down into n number of finite sub-intervals, say $[0, a_1]$, $[a_1, a_2]$, $[a_2, a_3]$ … $[a_n, \infty)$ such that in each of them, the function $F'(t)$ is continuous. Therefore, similar to Case 1, we can prove that for each of these sub-intervals, the integral exists and it has a finite value and if we calculate it, we will get the same result as in Case 1 i.e.,

$$L\{F'(t)\} = sL\{F(t)\} - F(0).$$

So, once we know the Laplace transform of $F(t)$, then using this theorem, we can easily evaluate the Laplace transform of its derivative also.

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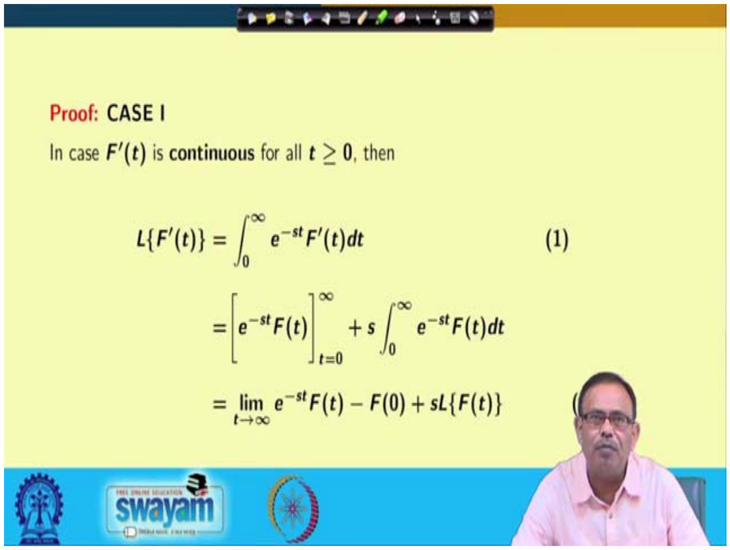

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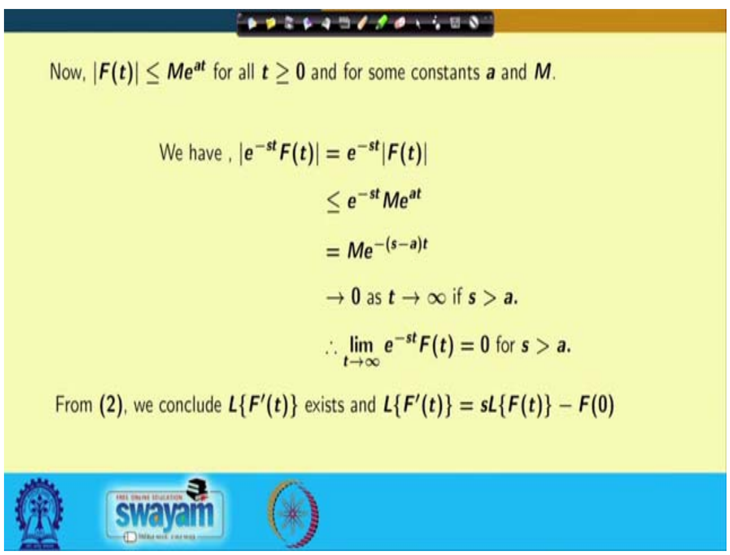

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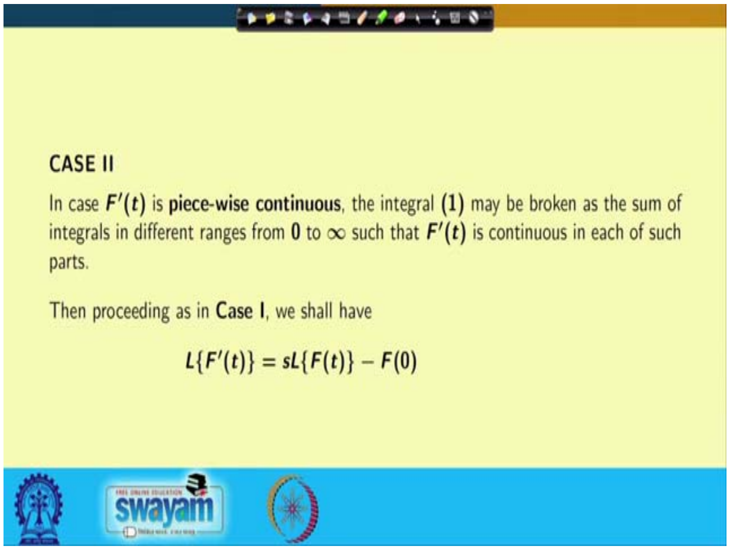

Next, let us consider the Laplace transform of n^{th} derivative of $F(t)$.

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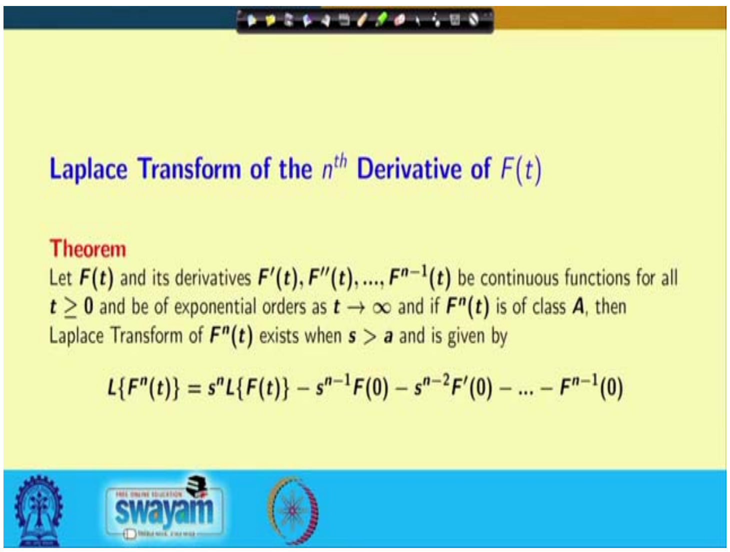

As the result shows,

$$L\{F^n(t)\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{n-1}(0).$$

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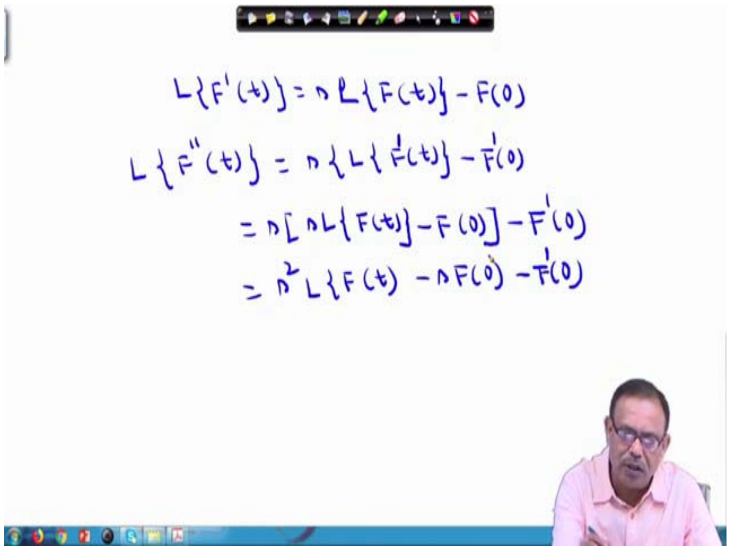

So, let us see how we can prove this theorem.

We have already proved

$$L\{F'(t)\} = sL\{F(t)\} - F(0). \qquad (2)$$

Using (2), we try to evaluate $L\{F''(t)\}$. So we obtain

$$L\{F''(t)\} = sL\{F'(t)\} - F'(0).$$

Here, again we substitute $L\{F'(t)\} = sL\{F(t)\} - F(0)$ from (2) to get

$$L\{F''(t)\} = sL\{F'(t)\} - F'(0)$$

$$= s[sL\{F(t)\} - F(0)] - F'(0)$$

$$= s^2L\{F(t)\} - sF(0) - F'(0). \qquad (3)$$

So, now we know the Laplace transform of $F''(t)$.

Now, when we try to find out Laplace transform of $F'''(t)$ that is third derivative of the function $F(t)$, we can similarly write

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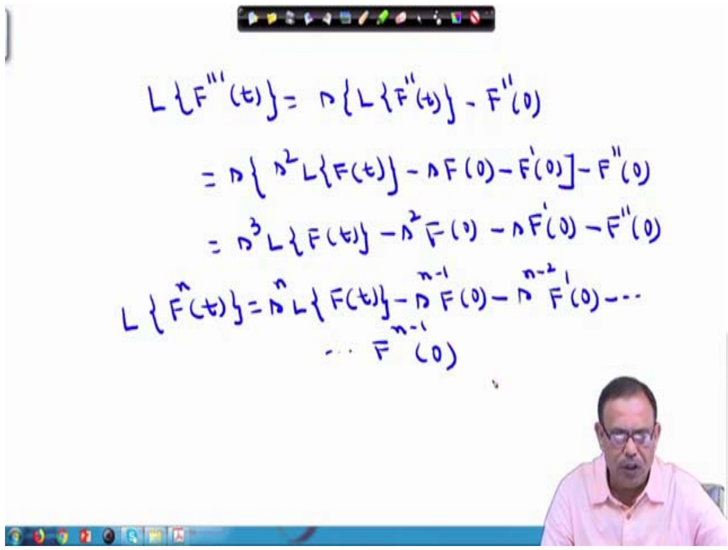

$$L\{F'''(t)\} = sL\{F''(t)\} - F''(0).$$

We replace $L\{F''(t)\}$ from (3), so we have

$$L\{F'''(t)\} = s[s^2L\{F(t)\} - sF(0) - F'(0)] - F''(0)$$

$$= s^3L\{F(t)\} - s^2F(0) - sF'(0) - F''(0).$$

If we proceed similarly, we can conclude that

$$L\{F^n(t)\} = s^nL\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{n-1}(0).$$

So, when the Laplace transform of a function is known to us, we can easily evaluate the Laplace transform of n^{th} derivative of the function as well.

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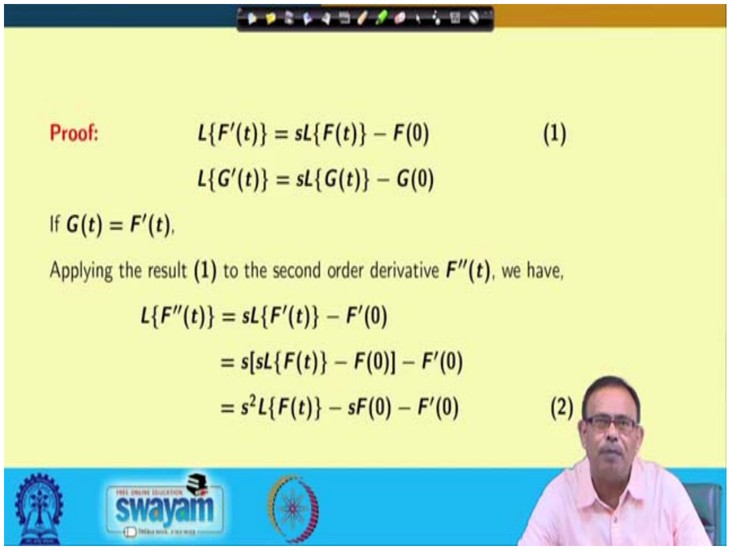

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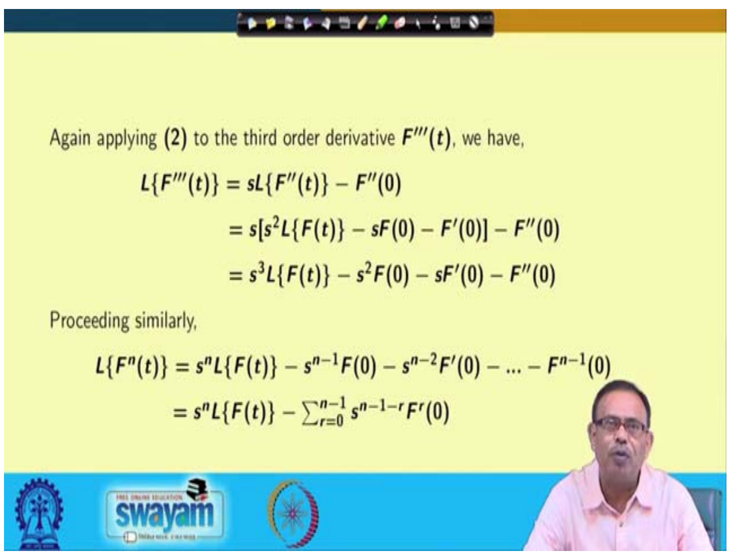

Next we come to Laplace transform of integrals of a function.

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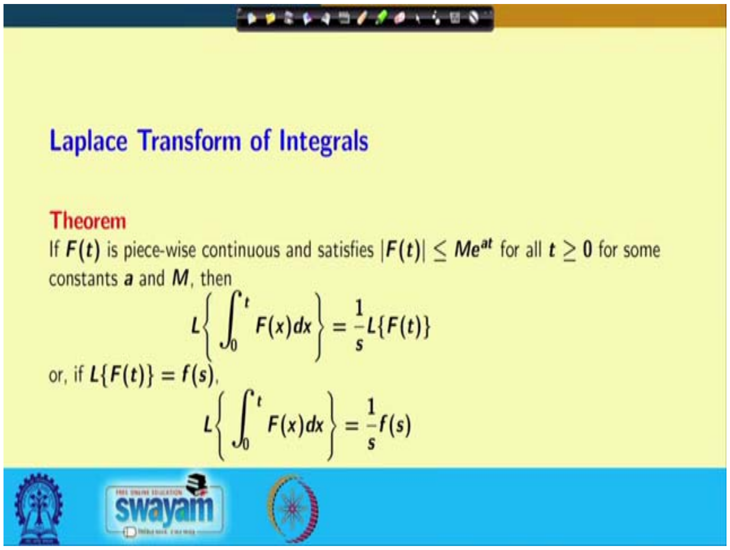

Let $F(t)$ be a piece-wise continuous function which satisfies

$$|F(t)| \leq Me^{at} \quad \forall\, t \geq 0$$

for some constants a and M i.e., in other sense, we can say that $F(t)$ is of exponential order a as $t \to \infty$. Then

$$L\left\{\int_0^t F(x)dx\right\} = \frac{1}{s}L\{F(t)\}$$

$$= \frac{1}{s}f(s)$$

where $L\{F(t)\} = f(s)$.

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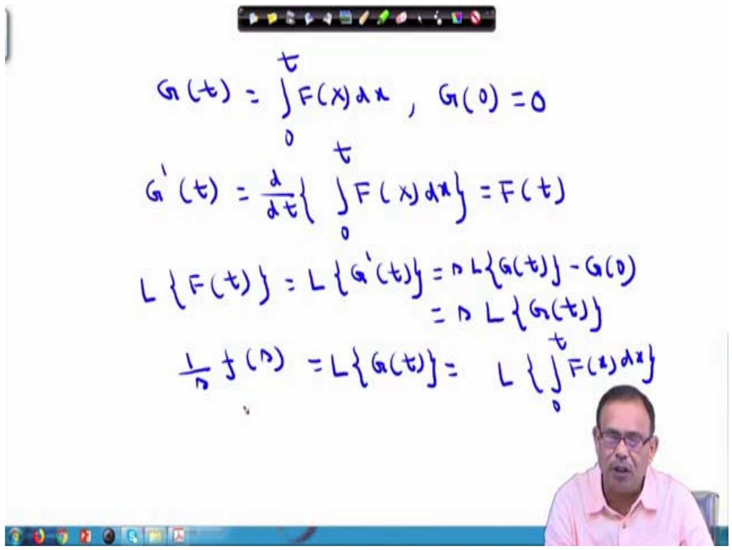

Let us see the proof. First of all, let us assume

$$G(t) = \int_0^t F(x)dx.$$

So that clearly, $G(0) = 0$. Again,

$$G'(t) = \frac{d}{dt}\{G(t)\} = \frac{d}{dt}\left\{\int_0^t F(x)dx\right\} = F(t).$$

Now, from the Laplace transform of derivatives of a function, we know,

$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$

$$\Rightarrow L\{F(t)\} = sL\left\{\int_0^t F(x)dx\right\} - 0$$

$$\Rightarrow f(s) = sL\left\{\int_0^t F(x)dx\right\}$$

$$\Rightarrow L\left\{\int_0^t F(x)dx\right\} = \frac{1}{s}f(s).$$

This completes the proof. Thank you.