Transform Calculus and its Applications in Differential Equations
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Lecture - 39
Representation of a function as Fourier Integral

In the last lecture, we have covered how to find out the Fourier transform of some functions as well as what is the Fourier transform of Dirac delta function and some useful values of some given integrals. Now let us take some other types of problems.

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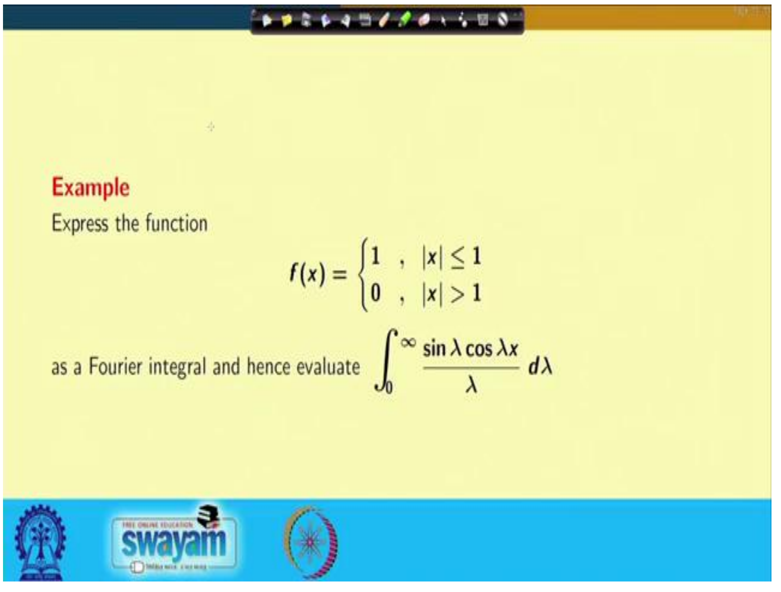

We want to express $f(x)$ as a Fourier integral, that means, in terms of Fourier integral representation, where $f(x)$ is defined as,

$$f(x) = \begin{cases} 1 & , \quad |x| \le 1 \\ 0 & , \quad |x| > 1 \end{cases}$$

And then, whatever result we will obtain, using that result we will try to find out the value of

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

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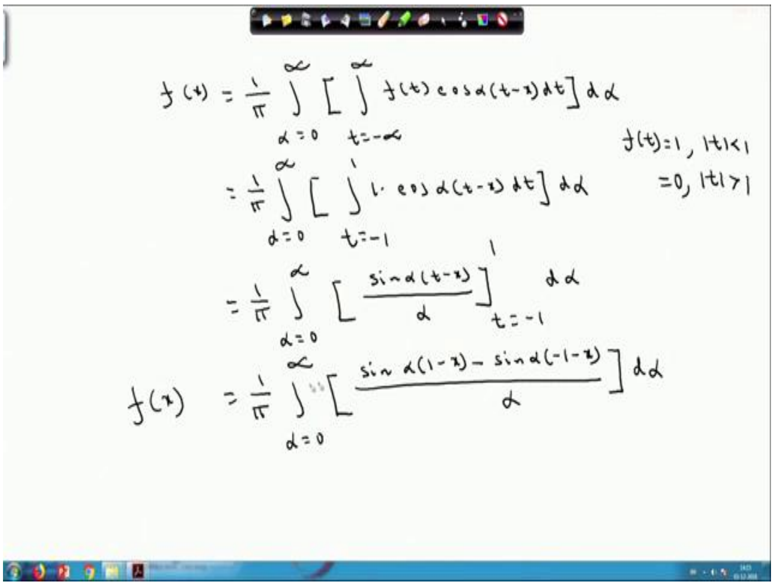

Given function $f(x)$ can be rewritten as,

$$f(x) = \begin{cases} 1 \ , & -1 \leq x \leq 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

We had discussed about the Fourier integral representation (FIR) in the introductory lectures of the Fourier transform. FIR of a function $f(x)$ is given as,

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-\infty}^{\infty} f(t) \cos \alpha(t-x) \ dt \right] d\alpha$$

Since, the function takes the value 1 in the interval $[-1,1]$ and 0 otherwise, therefore,

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-1}^{1} 1 \cdot \cos \alpha(t-x) \ dt \right] d\alpha$$

If we evaluate the bracketed integral, we will obtain,

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(t-x)}{\alpha} \right]_{t=-1}^{1} d\alpha$$

$$= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(1-x) - \sin \alpha(-1-x)}{\alpha} \right] d\alpha$$

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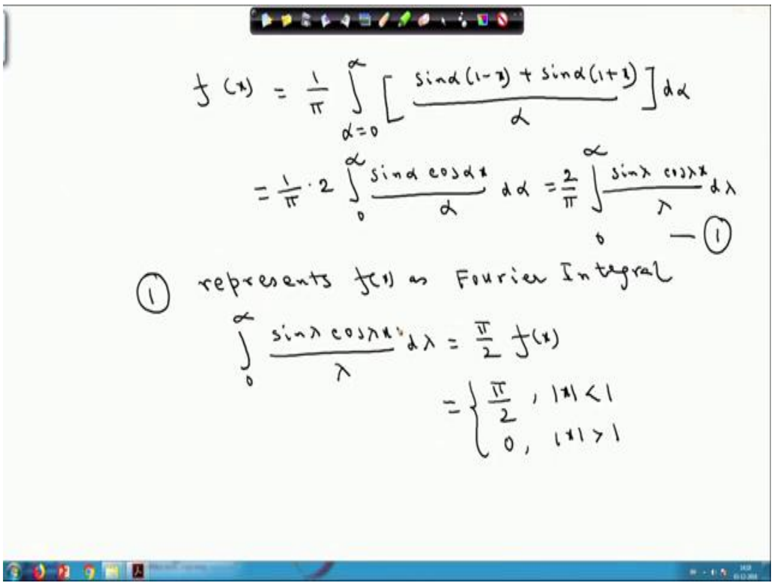

So, the last integral can be written as

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\left[\frac{\sin\alpha(1-x)+\sin\alpha(1+x)}{\alpha}\right]d\alpha$$

If we use the trigonometric identity, $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$, then,

$$f(x) = \frac{2}{\pi}\int_{\alpha=0}^{\infty}\frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha$$

If we replace α by λ in the above relation, then we have,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda \qquad (1)$$

This is the Fourier integral representation for the given function $f(x)$. From (1), we have

$$\int_{0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{2}f(x)$$

$$= \begin{cases}\frac{\pi}{2}, & |x| \leq 1\\ 0, & |x| > 1\end{cases}$$

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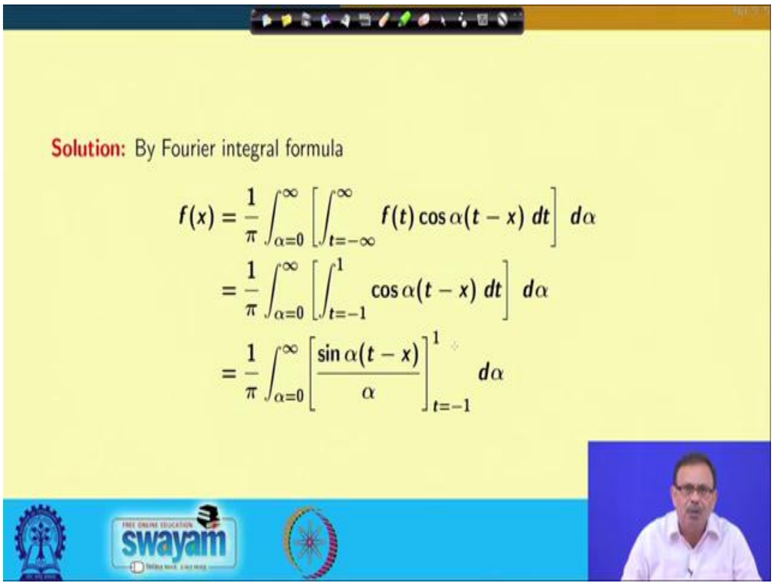

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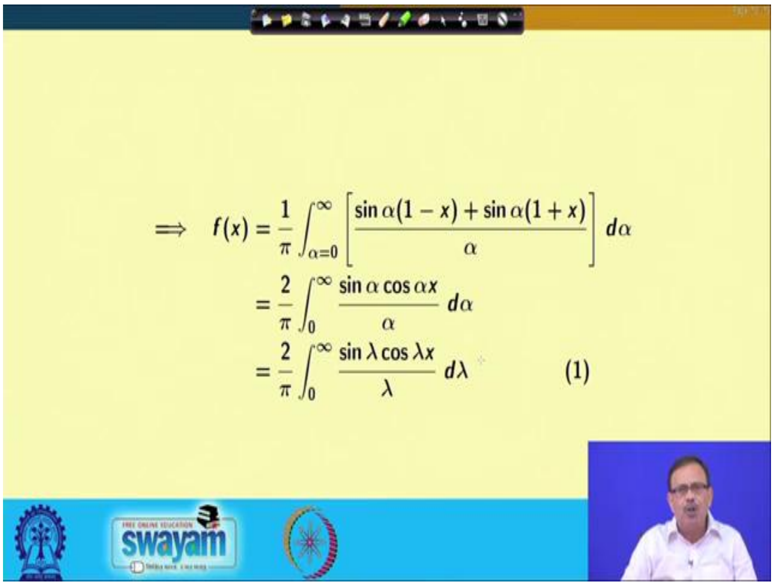

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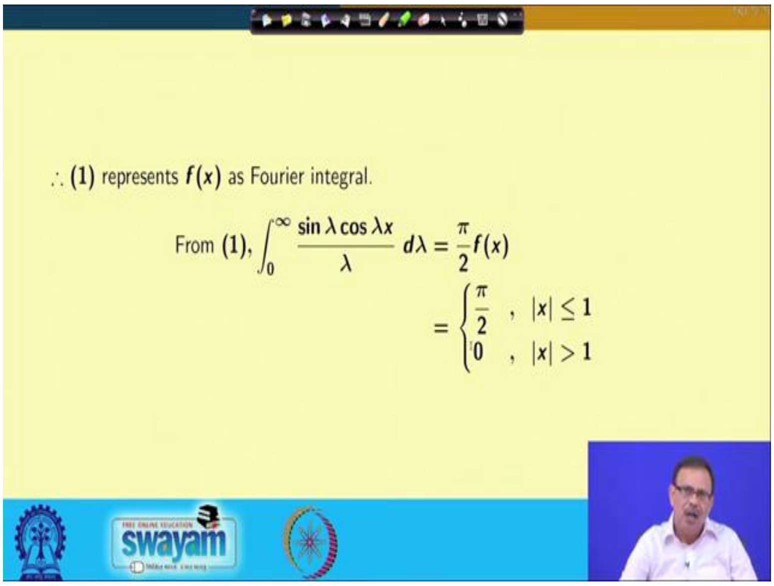

So, this particular example explains how a function can be represented in terms of the Fourier integral and using that, how to find out the value of certain integrals.

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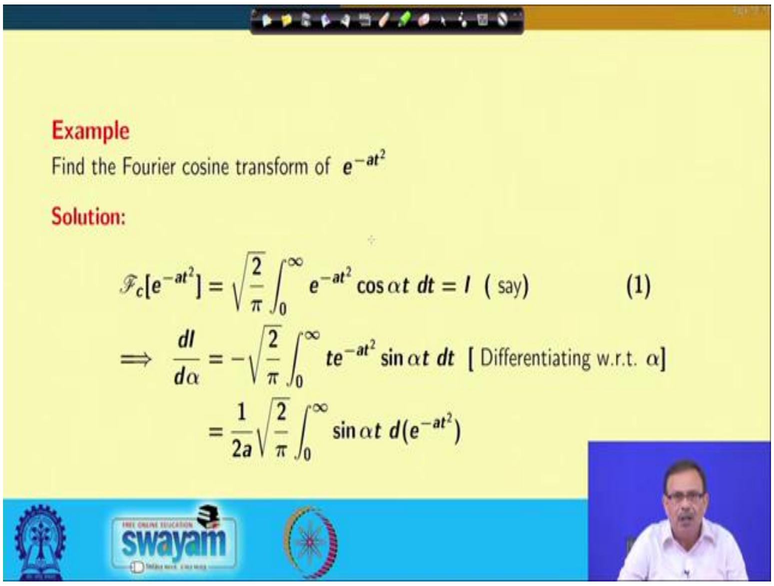

Now, let us take another example where we will try to find the Fourier cosine transform of e^{-at^2}.

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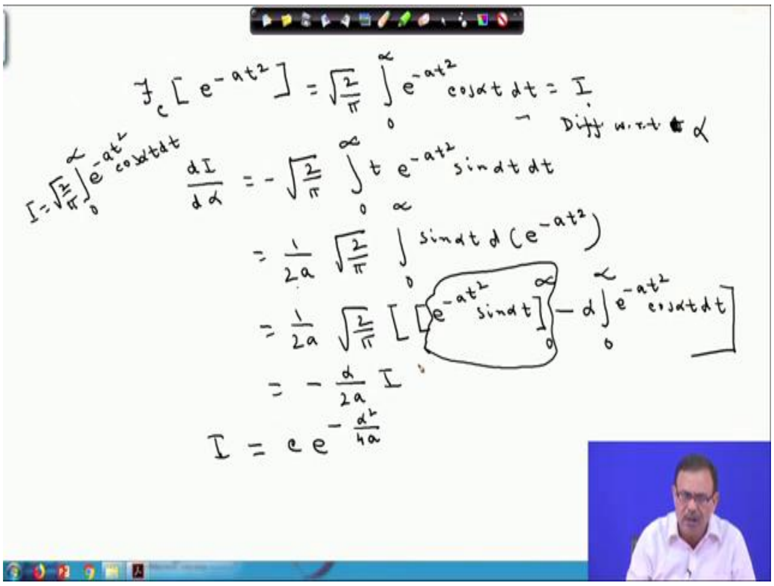

From the definition of Fourier cosine transform, we have,

$$\mathcal{F}_c\left[e^{-at^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at^2} \cos \alpha t \; dt = I \text{ (say)} \tag{2}$$

Now if we differentiate both sides of (2) with respect to α, i.e., if we use differentiation under the sign of integration, t will be treated as constant and we will obtain,

$$\frac{dI}{d\alpha} = -\sqrt{\frac{2}{\pi}} \int_0^\infty t\, e^{-at^2} \sin \alpha t \; dt$$

Now, using $d\left(e^{-at^2}\right) = -2at\, e^{-at^2} dt$, in the above equation, we get,

$$\frac{dI}{d\alpha} = \frac{1}{2a}\sqrt{\frac{2}{\pi}} \int_{t=0}^\infty \sin \alpha t \; d\left(e^{-at^2}\right)$$

Using integration by parts, we will obtain,

$$\frac{dI}{d\alpha} = \frac{1}{2a}\sqrt{\frac{2}{\pi}} \left(\left[e^{-at^2} \sin \alpha t\right]_{t=0}^\infty - \alpha \int_{t=0}^\infty e^{-at^2} \cos \alpha t \; dt \right)$$

If we put the limits in the first part, then it will be zero and from the second part, we will have,

$$\frac{dI}{d\alpha} = -\frac{\alpha}{2a}I$$

We can directly solve the above first order ODE as

$$I = ce^{-\frac{\alpha^2}{4a}} \qquad (3)$$

where c is the constant of integration whose value we have to find out.

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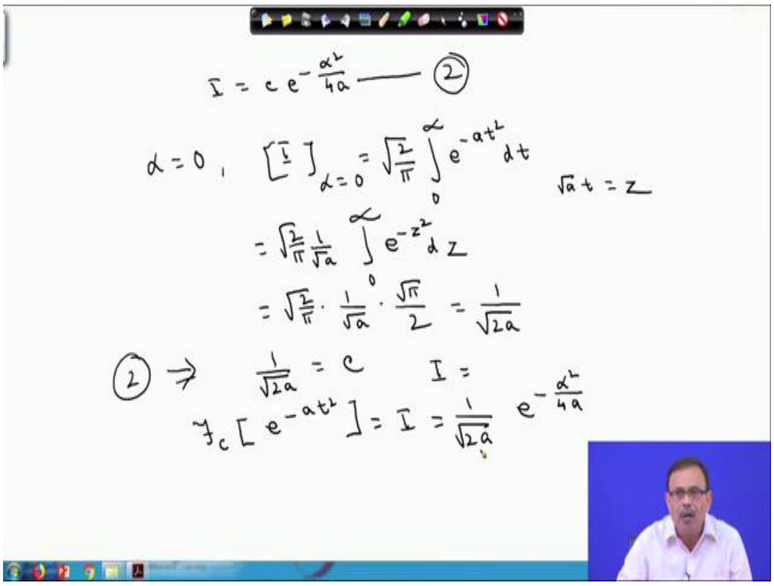

Whenever $\alpha = 0$, from (2), we get

$$I(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-at^2}dt$$

If we substitute $\sqrt{a}t = z$ in the above integral so that $dt = \frac{1}{\sqrt{a}}dz$, then we have,

$$I(0) = \sqrt{\frac{2}{\pi a}}\int_0^\infty e^{-z^2}dz$$

We know that,

$$\int_0^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$$

$$\therefore I(0) = \frac{1}{\sqrt{2a}}$$

If put $\alpha = 0$ in (3) and use $I(0) = \frac{1}{\sqrt{2a}}$, then we have,

$$c = \frac{1}{\sqrt{2a}}$$

$$\therefore \mathcal{F}_c\left[e^{-at^2}\right] = I = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}}$$

So, without evaluating the integral directly, by differentiating, we are converting the given integral in the form of some ODE which can be solved very easily and we ultimately obtain the value of the integral.

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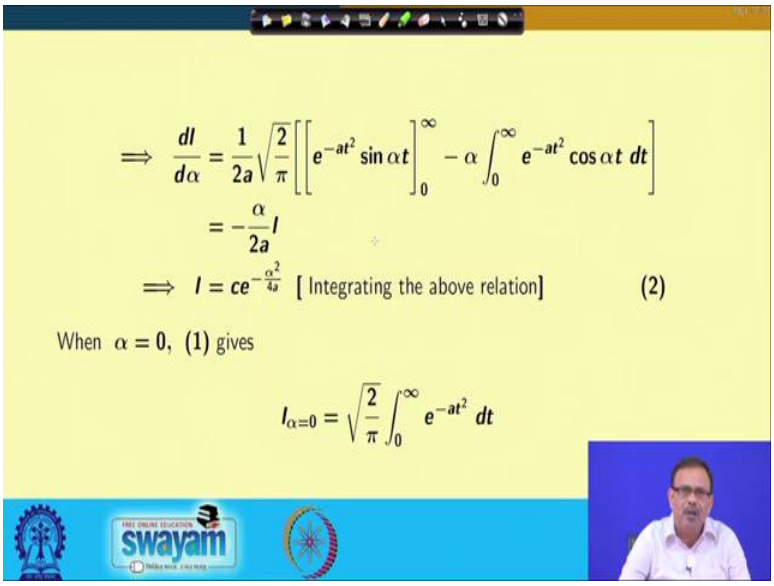

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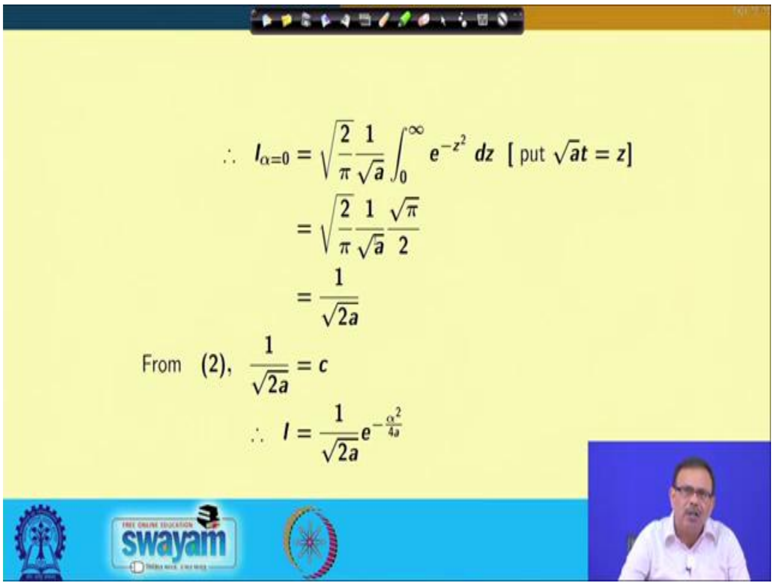

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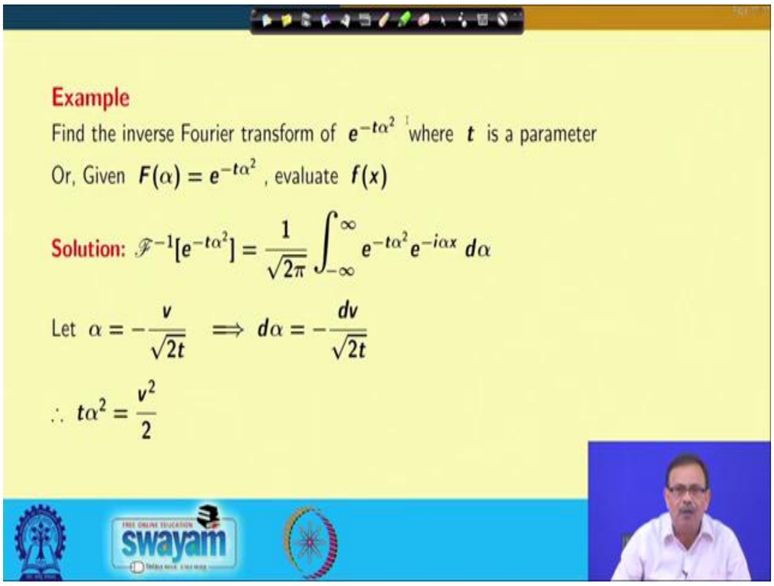

Let us take one more example where we find the inverse Fourier transform of $e^{-t\alpha^2}$ where t is a parameter.

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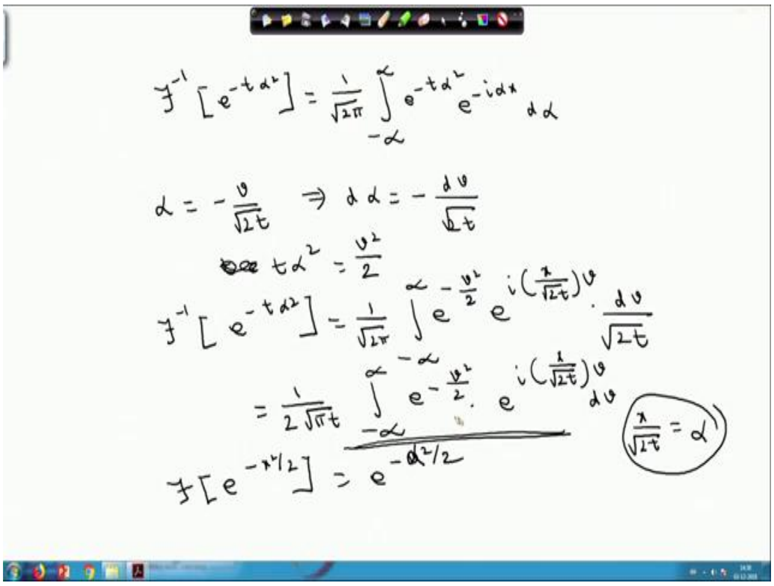

So, let us see this one. From the definition of inverse Fourier transform, we have,

$$\mathcal{F}^{-1}\left[e^{-t\alpha^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\alpha^2}\, e^{-i\alpha x}\, d\alpha$$

Suppose,

$$\alpha = -\frac{v}{\sqrt{2t}}$$

$$\Rightarrow d\alpha = -\frac{dv}{\sqrt{2t}} \quad \text{and} \quad t\alpha^2 = \frac{v^2}{2}$$

So, if we substitute this in the given integral, we can write down

$$\mathcal{F}^{-1}\left[e^{-t\alpha^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}}\, e^{i\left(\frac{x}{\sqrt{2t}}\right)v}\, \frac{dv}{\sqrt{2t}}$$

$$= \frac{1}{\sqrt{2t}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}}\, e^{i\left(\frac{x}{\sqrt{2t}}\right)v}\, dv$$

The integral on the RHS represents the Fourier transform of $e^{-\frac{v^2}{2}}$ where the kernel is $\frac{x}{\sqrt{2t}}$.

That means, we need to find the Fourier transform of $e^{-\frac{v^2}{2}}$ where $\alpha = \frac{x}{\sqrt{2t}}$.

We know that $e^{-\frac{v^2}{2}}$ is self reciprocal with respect to Fourier transform, i.e.,

$$\mathcal{F}\left[e^{-\frac{v^2}{2}}\right] = e^{-\frac{\alpha^2}{2}} \ \text{ where } \ \alpha = \frac{x}{\sqrt{2t}}$$

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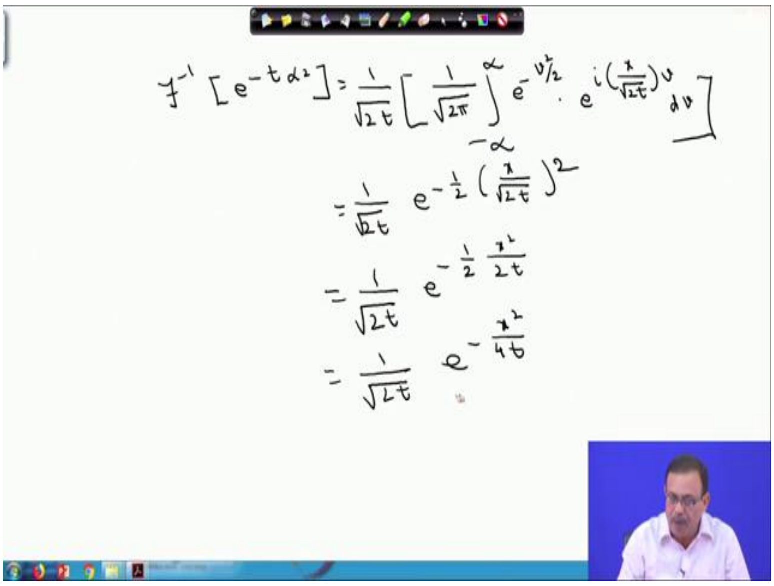

Therefore,

$$\mathcal{F}^{-1}\left[e^{-t\alpha^2}\right] = \frac{1}{\sqrt{2t}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}}\, e^{i\left(\frac{x}{\sqrt{2t}}\right)v}\, dv$$

$$= \frac{1}{\sqrt{2t}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2t}}\right)^2}$$

$$= \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}$$

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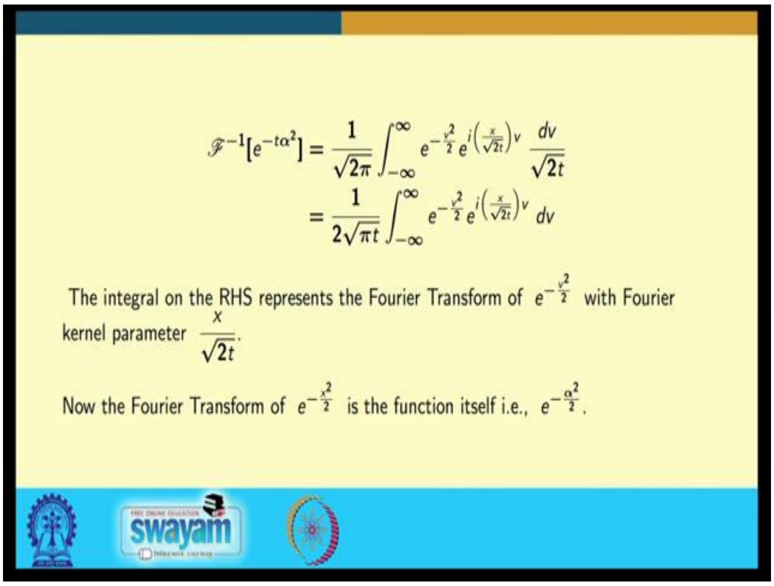

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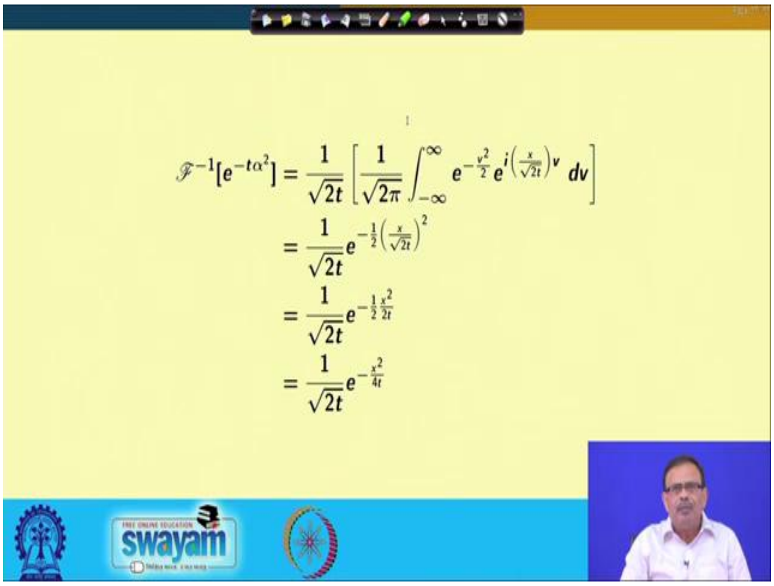

So, we have done various problems where various types of tricks, various techniques we have discussed by which we can compute the Fourier transform of many functions and also evaluate various integrals. Thank you.