

Transform Calculus and its Applications in Differential Equations
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Lecture - 39
Representation of a function as Fourier Integral

In the last lecture, we have covered how to find out the Fourier transform of some functions as well as what is the Fourier transform of Dirac delta function and some useful values of some given integrals. Now let us take some other types of problems.

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Example
Express the function

$$f(x) = \begin{cases} 1 & , |x| \leq 1 \\ 0 & , |x| > 1 \end{cases}$$

as a Fourier integral and hence evaluate $\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$

We want to express $f(x)$ as a Fourier integral, that means, in terms of Fourier integral representation, where $f(x)$ is defined as,

$$f(x) = \begin{cases} 1 & , |x| \leq 1 \\ 0 & , |x| > 1 \end{cases}$$

And then, whatever result we will obtain, using that result we will try to find out the value of

$$\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$$

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The image shows a handwritten derivation of the Fourier integral representation of a rectangular pulse function $f(t)$. The function is defined as $f(t) = 1$ for $|t| < 1$ and $f(t) = 0$ for $|t| > 1$. The derivation starts with the general formula for the Fourier integral:

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right] d\alpha$$

Since $f(t) = 1$ for $|t| < 1$ and 0 otherwise, the inner integral becomes:

$$= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-1}^1 1 \cdot \cos \alpha(t-x) dt \right] d\alpha$$

The inner integral is evaluated as:

$$= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(t-x)}{\alpha} \right]_{t=-1}^1 d\alpha$$

Finally, the outer integral is evaluated:

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(1-x) - \sin \alpha(-1-x)}{\alpha} \right] d\alpha$$

Given function $f(x)$ can be rewritten as,

$$f(x) = \begin{cases} 1 & , -1 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

We had discussed about the Fourier integral representation (FIR) in the introductory lectures of the Fourier transform. FIR of a function $f(x)$ is given as,

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right] d\alpha$$

Since, the function takes the value 1 in the interval $[-1,1]$ and 0 otherwise, therefore,

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-1}^1 1 \cdot \cos \alpha(t-x) dt \right] d\alpha$$

If we evaluate the bracketed integral, we will obtain,

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(t-x)}{\alpha} \right]_{t=-1}^1 d\alpha \\ &= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(1-x) - \sin \alpha(-1-x)}{\alpha} \right] d\alpha \end{aligned}$$

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Handwritten derivation on a whiteboard:

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(1-x) + \sin \alpha(1+x)}{\alpha} \right] d\alpha$$

$$= \frac{1}{\pi} \cdot 2 \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda \quad \text{--- (1)}$$

① represents $f(x)$ as Fourier Integral

$$\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

So, the last integral can be written as

$$f(x) = \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(1-x) + \sin \alpha(1+x)}{\alpha} \right] d\alpha$$

If we use the trigonometric identity, $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$, then,

$$f(x) = \frac{2}{\pi} \int_{\alpha=0}^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha$$

If we replace α by λ in the above relation, then we have,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda \quad (1)$$


This is the Fourier integral representation for the given function $f(x)$. From (1), we have

$$\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x)$$


$$= \begin{cases} \frac{\pi}{2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

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Solution: By Fourier integral formula

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right] d\alpha \\ &= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\int_{t=-1}^1 \cos \alpha(t-x) dt \right] d\alpha \\ &= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(t-x)}{\alpha} \right]_{t=-1}^1 d\alpha \end{aligned}$$


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$$\begin{aligned} \Rightarrow f(x) &= \frac{1}{\pi} \int_{\alpha=0}^{\infty} \left[\frac{\sin \alpha(1-x) + \sin \alpha(1+x)}{\alpha} \right] d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda \quad (1) \end{aligned}$$


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∴ (1) represents $f(x)$ as Fourier integral.

$$\text{From (1), } \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

So, this particular example explains how a function can be represented in terms of the Fourier integral and using that, how to find out the value of certain integrals.

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Example
Find the Fourier cosine transform of e^{-at^2}

Solution:

$$\mathcal{F}_c[e^{-at^2}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} \cos \alpha t dt = I \quad (\text{say}) \quad (1)$$
$$\Rightarrow \frac{dI}{d\alpha} = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} t e^{-at^2} \sin \alpha t dt \quad [\text{Differentiating w.r.t. } \alpha]$$
$$= \frac{1}{2a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin \alpha t d(e^{-at^2})$$

Now, let us take another example where we will try to find the Fourier cosine transform of e^{-at^2} .

(Refer Slide Time: 11:52)

$$\mathcal{F}_c [e^{-at^2}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} \cos at dt = I$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} \cos at dt$$

$$\frac{dI}{d\alpha} = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} t e^{-at^2} \sin at dt$$

$$= -\frac{1}{2a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin at d(e^{-at^2})$$

$$= -\frac{1}{2a} \sqrt{\frac{2}{\pi}} \left[e^{-at^2} \sin at \Big|_0^{\infty} - \int_0^{\infty} e^{-at^2} \cos at dt \right]$$

$$= -\frac{1}{2a} \sqrt{\frac{2}{\pi}} \left[0 - \int_0^{\infty} e^{-at^2} \cos at dt \right]$$

$$= -\frac{1}{2a} \sqrt{\frac{2}{\pi}} I$$

$$I = e^{-\frac{a}{4a}}$$

From the definition of Fourier cosine transform, we have,

$$\mathcal{F}_c[e^{-at^2}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} \cos at dt = I \text{ (say)} \quad (2)$$

Now if we differentiate both sides of (2) with respect to α , i.e., if we use differentiation under the sign of integration, t will be treated as constant and we will obtain,

$$\frac{dI}{d\alpha} = -\sqrt{\frac{2}{\pi}} \int_0^{\infty} t e^{-at^2} \sin at dt$$

Now, using $d(e^{-at^2}) = -2at e^{-at^2} dt$, in the above equation, we get,

$$\frac{dI}{d\alpha} = \frac{1}{2a} \sqrt{\frac{2}{\pi}} \int_{t=0}^{\infty} \sin at d(e^{-at^2})$$

Using integration by parts, we will obtain,

$$\frac{dI}{d\alpha} = \frac{1}{2a} \sqrt{\frac{2}{\pi}} \left([e^{-at^2} \sin at]_{t=0}^{\infty} - \alpha \int_{t=0}^{\infty} e^{-at^2} \cos at dt \right)$$

If we put the limits in the first part, then it will be zero and from the second part, we will have,

$$\frac{dI}{d\alpha} = -\frac{\alpha}{2a}I$$

We can directly solve the above first order ODE as

$$I = ce^{-\frac{\alpha^2}{4a}} \quad (3)$$

where c is the constant of integration whose value we have to find out.

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$$I = ce^{-\frac{\alpha^2}{4a}} \quad (2)$$

$$\alpha = 0, \quad [I]_{\alpha=0} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} dt \quad \sqrt{at} = z$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-z^2} dz$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2a}}$$

$$(2) \Rightarrow \frac{1}{\sqrt{2a}} = c \quad I =$$

$$\mathcal{L}[e^{-at^2}] = I = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}}$$

Whenever $\alpha = 0$, from (2), we get

$$I(0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} dt$$

If we substitute $\sqrt{at} = z$ in the above integral so that $dt = \frac{1}{\sqrt{a}} dz$, then we have,

$$I(0) = \sqrt{\frac{2}{\pi a}} \int_0^{\infty} e^{-z^2} dz$$

We know that,

$$\int_0^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$$

$$\therefore I(0) = \frac{1}{\sqrt{2a}}$$

If put $\alpha = 0$ in (3) and use $I(0) = \frac{1}{\sqrt{2a}}$, then we have,

$$c = \frac{1}{\sqrt{2a}}$$

$$\therefore \mathcal{F}_c[e^{-at^2}] = I = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}}$$

So, without evaluating the integral directly, by differentiating, we are converting the given integral in the form of some ODE which can be solved very easily and we ultimately obtain the value of the integral.

(Refer Slide Time: 21:30)

$$\begin{aligned} \Rightarrow \frac{dI}{d\alpha} &= \frac{1}{2a} \sqrt{\frac{2}{\pi}} \left[\left[e^{-at^2} \sin \alpha t \right]_0^{\infty} - \alpha \int_0^{\infty} e^{-at^2} \cos \alpha t dt \right] \\ &= -\frac{\alpha}{2a} I \\ \Rightarrow I &= ce^{-\frac{\alpha^2}{4a}} \quad \text{[Integrating the above relation]} \quad (2) \end{aligned}$$

When $\alpha = 0$, (1) gives

$$I_{\alpha=0} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at^2} dt$$

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The slide displays the following mathematical derivation:

$$\begin{aligned}\therefore I_{\alpha=0} &= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-z^2} dz \quad [\text{put } \sqrt{at} = z] \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{a}} \frac{\sqrt{\pi}}{2} \\ &= \frac{1}{\sqrt{2a}}\end{aligned}$$

From (2), $\frac{1}{\sqrt{2a}} = c$

$$\therefore I = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}}$$

The slide also features the Swayam logo and a small video inset of the presenter in the bottom right corner.

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The slide contains the following text and equations:

Example
Find the inverse Fourier transform of $e^{-t\alpha^2}$ where t is a parameter
Or, Given $F(\alpha) = e^{-t\alpha^2}$, evaluate $f(x)$

Solution: $\mathcal{F}^{-1}[e^{-t\alpha^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\alpha^2} e^{-i\alpha x} d\alpha$

Let $\alpha = -\frac{v}{\sqrt{2t}} \Rightarrow d\alpha = -\frac{dv}{\sqrt{2t}}$

$$\therefore t\alpha^2 = \frac{v^2}{2}$$

The slide also features the Swayam logo and a small video inset of the presenter in the bottom right corner.

Let us take one more example where we find the inverse Fourier transform of $e^{-t\alpha^2}$ where t is a parameter.

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The image shows a whiteboard with handwritten mathematical steps for finding the inverse Fourier transform of $e^{-t\alpha^2}$. The steps are as follows:

$$\mathcal{F}^{-1}[e^{-t\alpha^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\alpha^2} e^{-i\alpha x} d\alpha$$

$$\alpha = -\frac{v}{\sqrt{2t}} \Rightarrow d\alpha = -\frac{dv}{\sqrt{2t}}$$

$$t\alpha^2 = \frac{v^2}{2}$$

$$\mathcal{F}^{-1}[e^{-t\alpha^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} \cdot \frac{dv}{\sqrt{2t}}$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} dv$$

Below this, there is a circled note: $\frac{x}{\sqrt{2t}} = \alpha'$. At the bottom, the final result is written as:

$$\mathcal{F}^{-1}[e^{-t\alpha^2}] = e^{-\alpha'^2/2}$$

So, let us see this one. From the definition of inverse Fourier transform, we have,

$$\mathcal{F}^{-1}[e^{-t\alpha^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t\alpha^2} e^{-i\alpha x} d\alpha$$

Suppose,

$$\alpha = -\frac{v}{\sqrt{2t}}$$

$$\Rightarrow d\alpha = -\frac{dv}{\sqrt{2t}} \text{ and } t\alpha^2 = \frac{v^2}{2}$$

So, if we substitute this in the given integral, we can write down

$$\begin{aligned} \mathcal{F}^{-1}[e^{-t\alpha^2}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} \frac{dv}{\sqrt{2t}} \\ &= \frac{1}{\sqrt{2t}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} dv \end{aligned}$$

The integral on the RHS represents the Fourier transform of $e^{-\frac{v^2}{2}}$ where the kernel is $\frac{x}{\sqrt{2t}}$.

That means, we need to find the Fourier transform of $e^{-\frac{v^2}{2}}$ where $\alpha = \frac{x}{\sqrt{2t}}$.

We know that $e^{-\frac{v^2}{2}}$ is self reciprocal with respect to Fourier transform, i.e.,

$$\mathcal{F}\left[e^{-\frac{v^2}{2}}\right] = e^{-\frac{\alpha^2}{2}} \text{ where } \alpha = \frac{x}{\sqrt{2t}}$$

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The image shows a whiteboard with handwritten mathematical steps for the inverse Fourier transform of e^{-ta^2} . The steps are as follows:

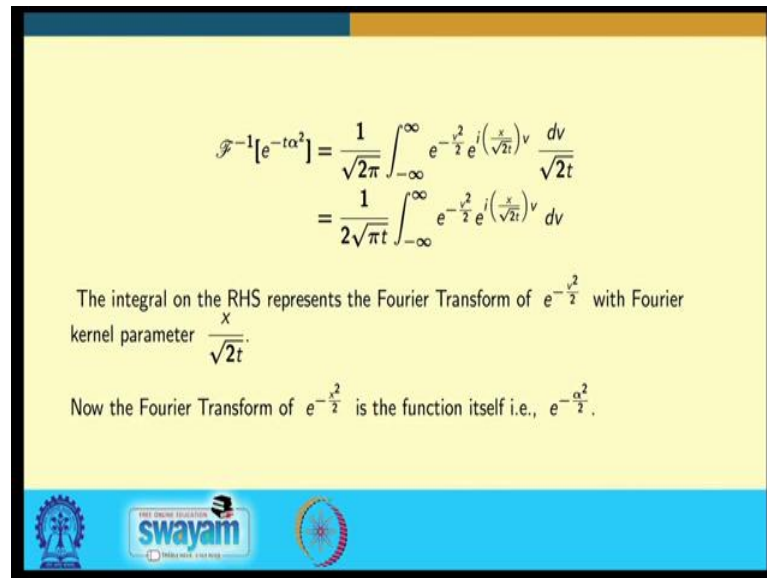
$$\begin{aligned} \mathcal{F}^{-1}[e^{-ta^2}] &= \frac{1}{\sqrt{2t}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} \cdot e^{i\left(\frac{x}{\sqrt{2t}}\right)v} dv \right] \\ &= \frac{1}{\sqrt{2t}} e^{-\frac{1}{2} \left(\frac{x}{\sqrt{2t}}\right)^2} \\ &= \frac{1}{\sqrt{2t}} e^{-\frac{1}{2} \frac{x^2}{2t}} \\ &= \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}} \end{aligned}$$

A small video inset in the bottom right corner shows a man speaking.

Therefore,




$$\begin{aligned} \mathcal{F}^{-1}[e^{-ta^2}] &= \frac{1}{\sqrt{2t}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} dv \\ &= \frac{1}{\sqrt{2t}} e^{-\frac{1}{2} \left(\frac{x}{\sqrt{2t}}\right)^2} \\ &= \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}} \end{aligned}$$

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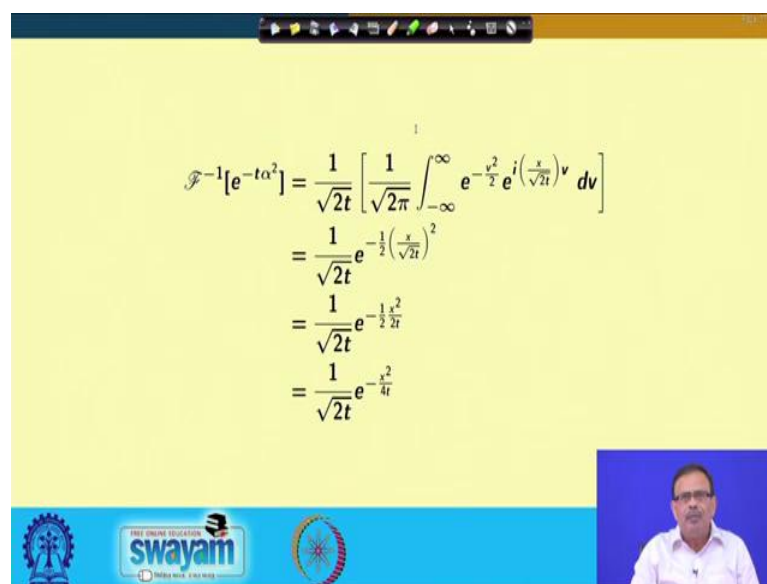





$$\mathcal{F}^{-1}[e^{-t\alpha^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} \frac{dv}{\sqrt{2t}}$$
$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} dv$$

The integral on the RHS represents the Fourier Transform of $e^{-\frac{v^2}{2}}$ with Fourier kernel parameter $\frac{x}{\sqrt{2t}}$.

Now the Fourier Transform of $e^{-\frac{v^2}{2}}$ is the function itself i.e., $e^{-\frac{\alpha^2}{2}}$.



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$$\mathcal{F}^{-1}[e^{-t\alpha^2}] = \frac{1}{\sqrt{2t}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} e^{i\left(\frac{x}{\sqrt{2t}}\right)v} dv \right]$$
$$= \frac{1}{\sqrt{2t}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2t}}\right)^2}$$
$$= \frac{1}{\sqrt{2t}} e^{-\frac{1}{2}\frac{x^2}{2t}}$$
$$= \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}$$


So, we have done various problems where various types of tricks, various techniques we have discussed by which we can compute the Fourier transform of many functions and also evaluate various integrals. Thank you.