Transform Calculus and its Applications in Differential Equations
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Lecture – 38
Fourier Transform of Dirac Delta Function

In the last few lectures, what we have done is the Fourier transform, Fourier sine transform, Fourier cosine transform of functions, their various properties and also we have done the Fourier transform of derivative of a function or integration of a function. Afterwards, we have studied the convolution theorem, we have discussed the Parseval's identity and we have also seen how to evaluate an integral using Fourier transform.

In this particular lecture, we will try to solve some more problems to study how to find out the Fourier transform of various functions or how to evaluate the integrals using the Fourier transform.

So, let us consider one example first.

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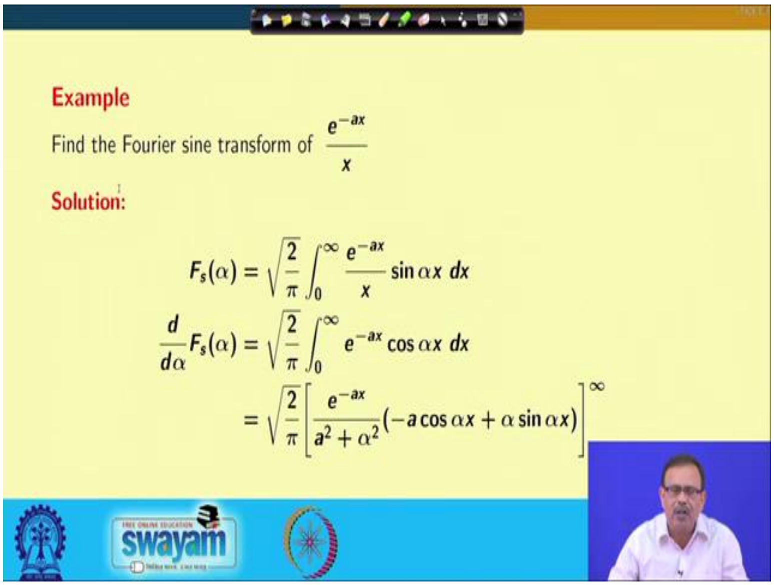

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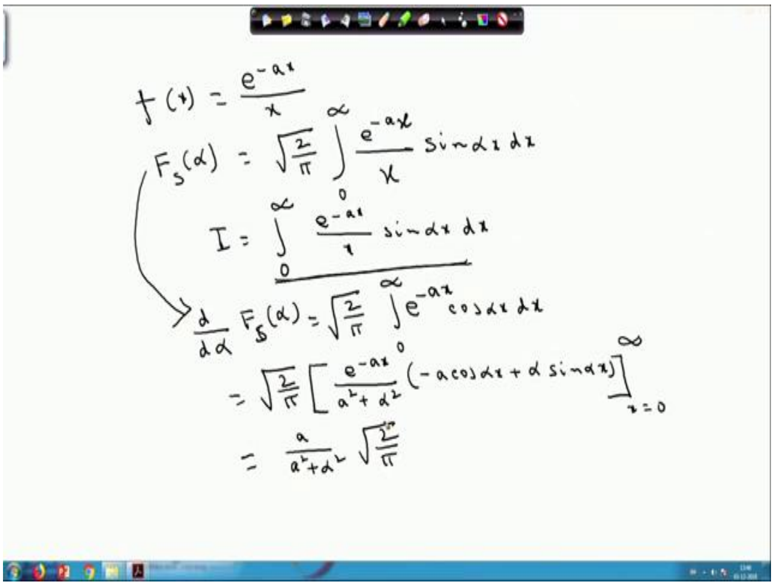

Here we want to find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$. From the definition, we have,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin \alpha x \ dx \tag{1}$$

To evaluate this integral, we will use similar approach as used for the last problem (in last lecture). First we will differentiate the above integral with respect to α. Whenever we will differentiate it with respect to α, x will be treated as constant. Therefore, we have,

$$\frac{d}{d\alpha} F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos \alpha x \ dx$$

And, using normal integration approach we can integrate right hand side

$$\frac{d}{d\alpha} F_s(\alpha) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + \alpha^2} (-a \cos \alpha x + \alpha \sin \alpha x) \right]_{x=0}^{\infty}$$

$$\Rightarrow \frac{d}{d\alpha} F_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \alpha^2}$$

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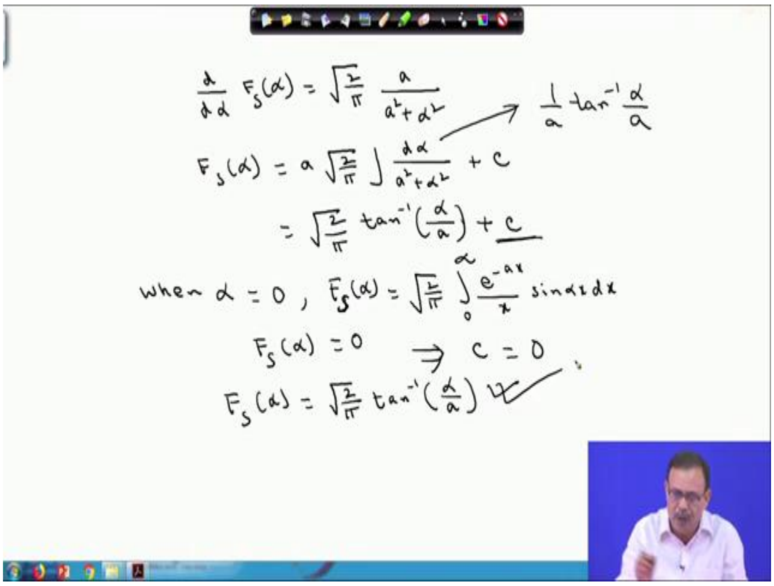

Basically, we have obtained a first order ODE. We can solve this directly as,

$$F_s(\alpha) = a\sqrt{\frac{2}{\pi}}\int \frac{d\alpha}{a^2+\alpha^2} + c$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{\alpha}{a} + c$$

From the definition of Fourier sine transform, we have,

$$F_s(0) = 0$$

$$\therefore c = 0$$

This implies

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{\alpha}{a}$$

This gives us the desired result.

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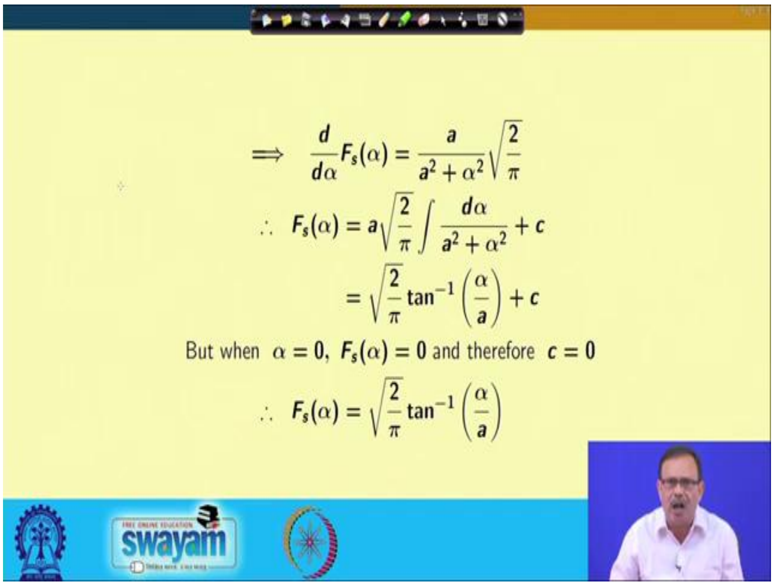

So, let us take the next problem.

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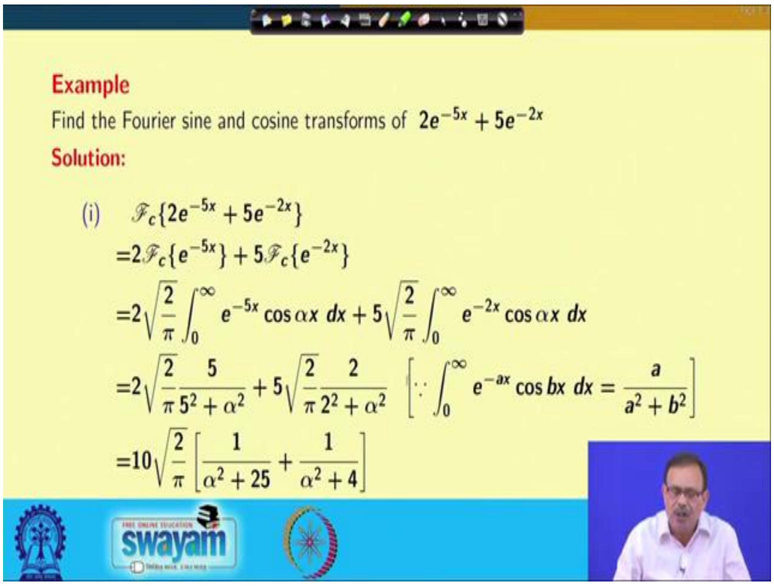

Here, we want to find the Fourier sine and cosine transforms of $2e^{-5x} + 5e^{-2x}$.

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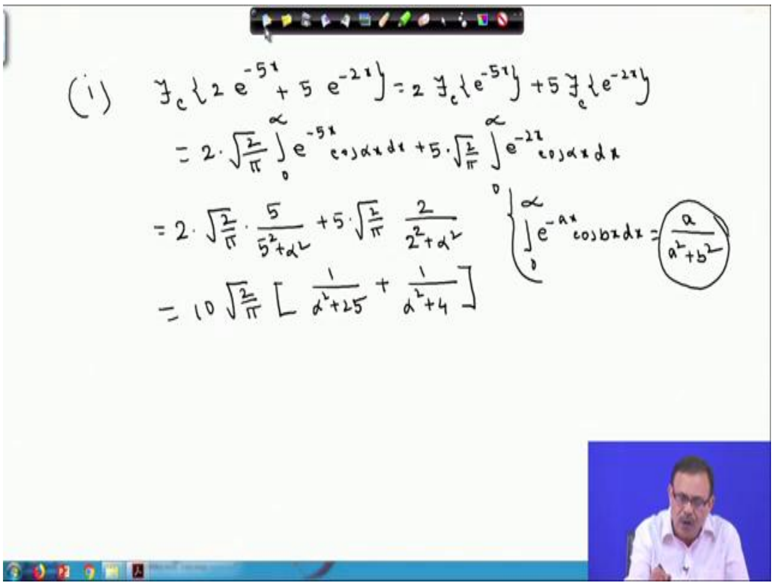

First we will find out Fourier cosine transform of the function. Since Fourier cosine transform is a linear transform, therefore,

$$\mathcal{F}_c[2e^{-5x} + 5e^{-2x}] = 2\mathcal{F}_c[e^{-5x}] + 5\mathcal{F}_c[e^{-2x}]$$

We know that,

$$\mathcal{F}_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2 + a^2}$$

$$\therefore \mathcal{F}_c[2e^{-5x} + 5e^{-2x}] = 2\mathcal{F}_c[e^{-5x}] + 5\mathcal{F}_c[e^{-2x}]$$

$$= 2\sqrt{\frac{2}{\pi}}\frac{5}{\alpha^2 + 5^2} + 5\sqrt{\frac{2}{\pi}}\frac{2}{\alpha^2 + 2^2}$$

$$= 10\sqrt{\frac{2}{\pi}}\left[\frac{1}{\alpha^2 + 25} + \frac{1}{\alpha^2 + 4}\right]$$

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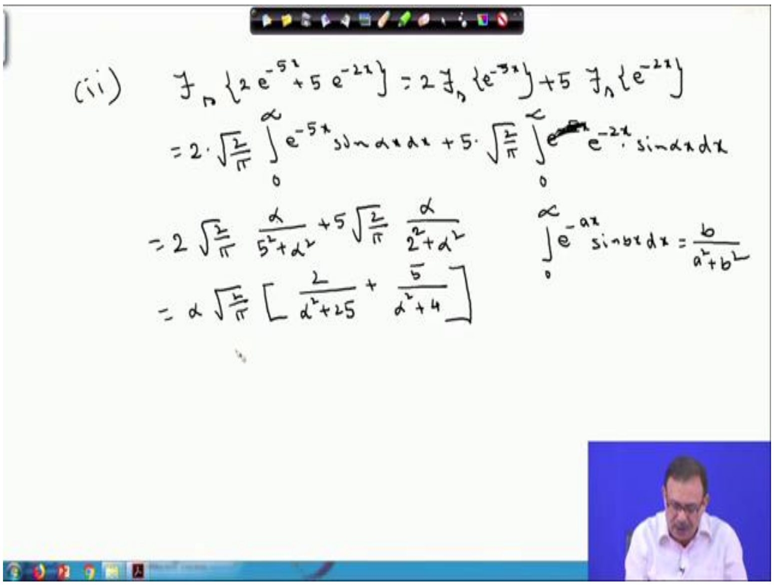

Now, we will find out Fourier sine transform of the function. Since Fourier sine transform is also a linear transform, then,

$$\mathcal{F}_s[2e^{-5x} + 5e^{-2x}] = 2\mathcal{F}_s[e^{-5x}] + 5\mathcal{F}_s[e^{-2x}]$$

We know that,

$$\mathcal{F}_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + a^2}$$

$$\therefore \mathcal{F}_s[2e^{-5x} + 5e^{-2x}] = 2\mathcal{F}_s[e^{-5x}] + 5\mathcal{F}_s[e^{-2x}]$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + 5^2} + 5\sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + 2^2}$$

$$= \alpha\sqrt{\frac{2}{\pi}} \left[\frac{2}{\alpha^2 + 25} + \frac{5}{\alpha^2 + 4}\right]$$

This gives the desired result.

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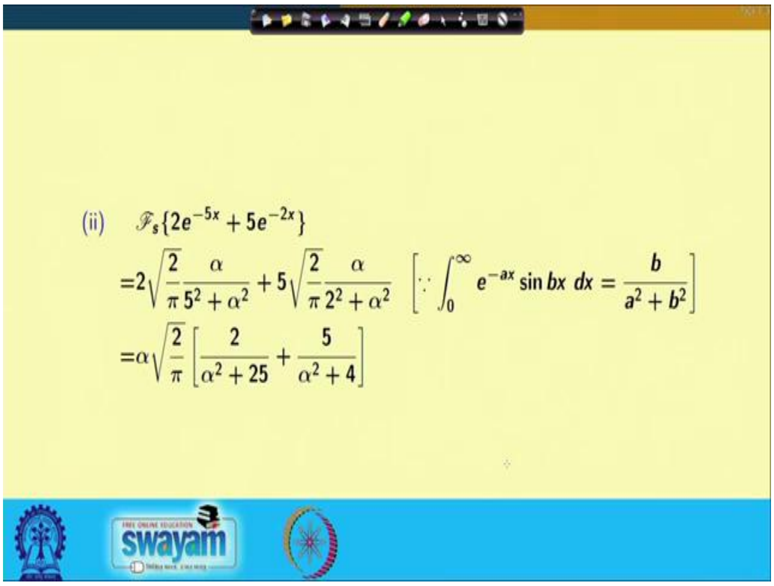

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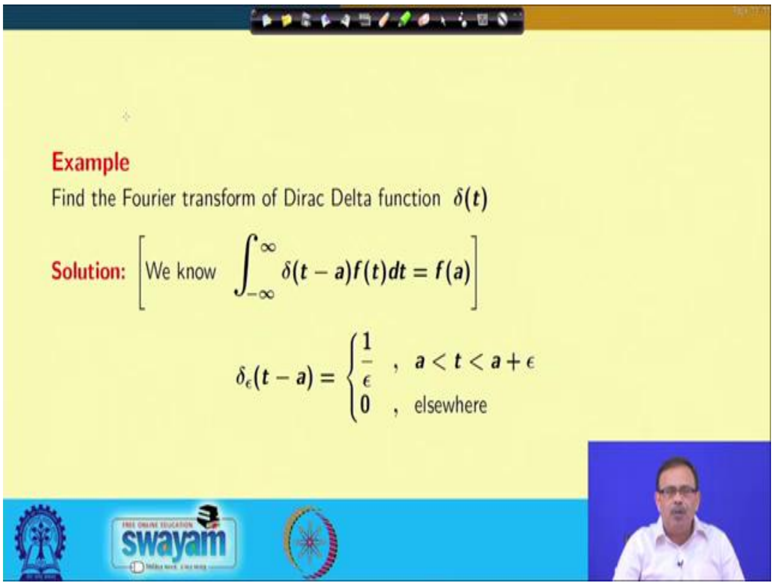

Next we will find the Fourier transform of Dirac delta function. If we recall, we have already defined Dirac delta function when we were studying the Laplace transform.

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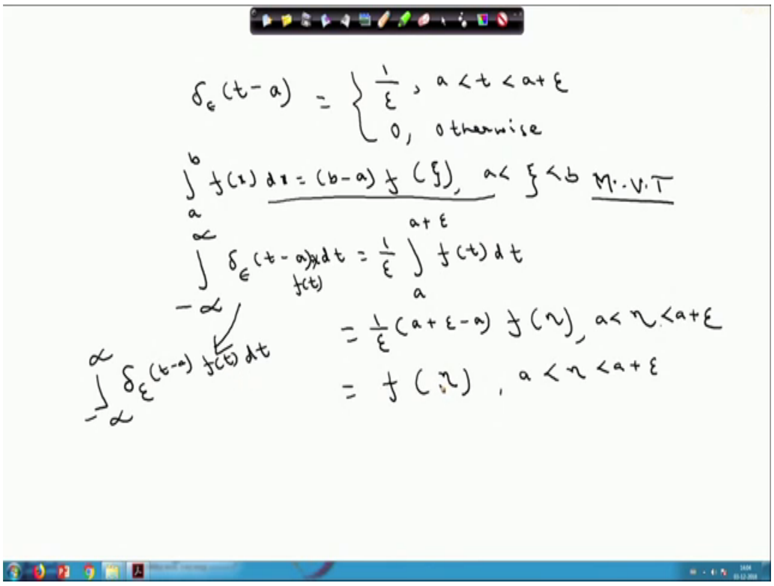

Dirac delta function is defined as,

$$\delta_\epsilon(t-a) = \begin{cases} \dfrac{1}{\epsilon} & , \quad a < t < a+\epsilon \\ 0 & , \quad \text{elsewhere} \end{cases}$$

So, as $\epsilon \to 0$, the function value will approach to infinity. Now we prove a property for Dirac delta function as:

$$\int_{-\infty}^{\infty} \delta_\epsilon(t-a) f(t)\, dt = \frac{1}{\epsilon}\int_{a}^{a+\epsilon} f(t)\, dt$$

Now, using the Mean Value Theorem for integral calculus, we obtain,

$$\int_{-\infty}^{\infty} \delta_\epsilon(t-a) f(t)\, dt = \frac{1}{\epsilon}(a+\epsilon-a) f(\eta) = f(\eta)$$

where, $a < \eta < a+\epsilon$.

As, $\epsilon \to 0$, then $\eta = a$ and we will get,

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a)$$

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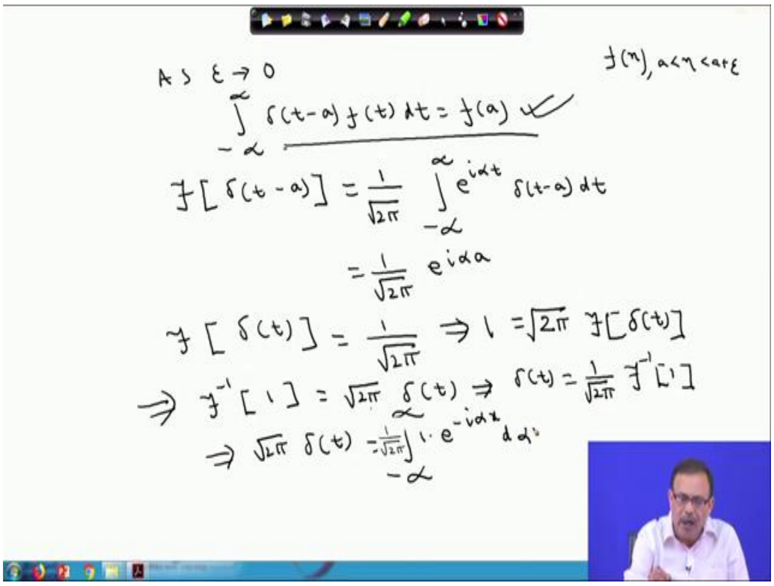

Now, we come to our original problem that is Fourier transform of $\delta(t-a)$. So, from the definition of Fourier transform, we have,

$$\mathcal{F}[\delta(t-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t-a)\, e^{i\alpha t}\, dt$$

Using the property of Dirac delta function, we have,

$$\mathcal{F}[\delta(t-a)] = \frac{e^{i\alpha a}}{\sqrt{2\pi}}$$

For $a = 0$, we will get,

$$\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}}$$

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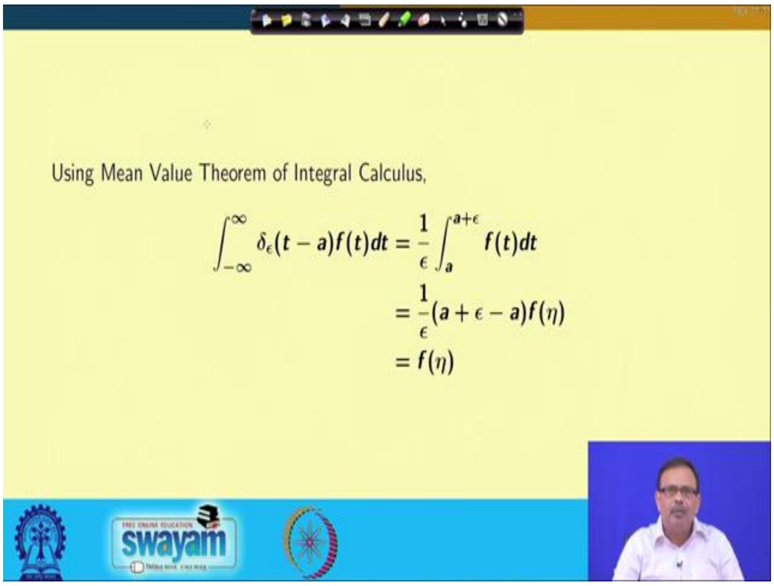

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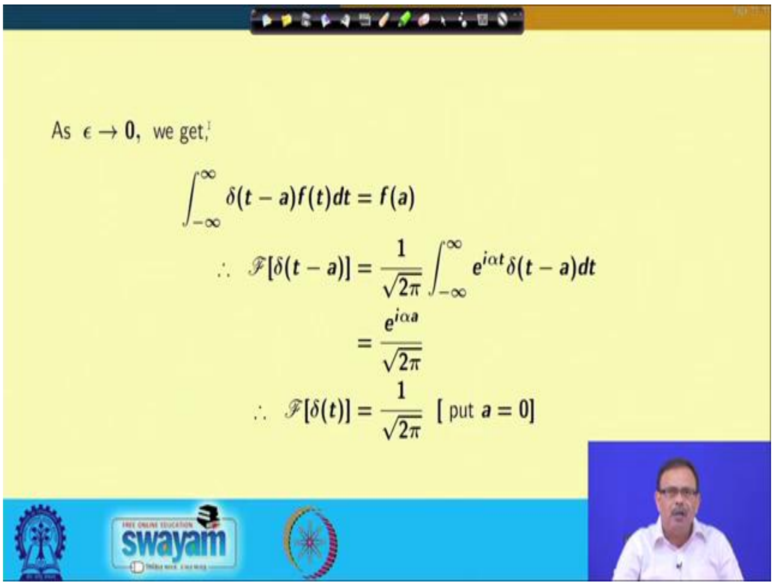

Thank you.