**Transform Calculus and its Applications in Differential Equations**
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**Lecture – 37**
**Evaluation of Definite Integrals using Properties of Fourier Transform**

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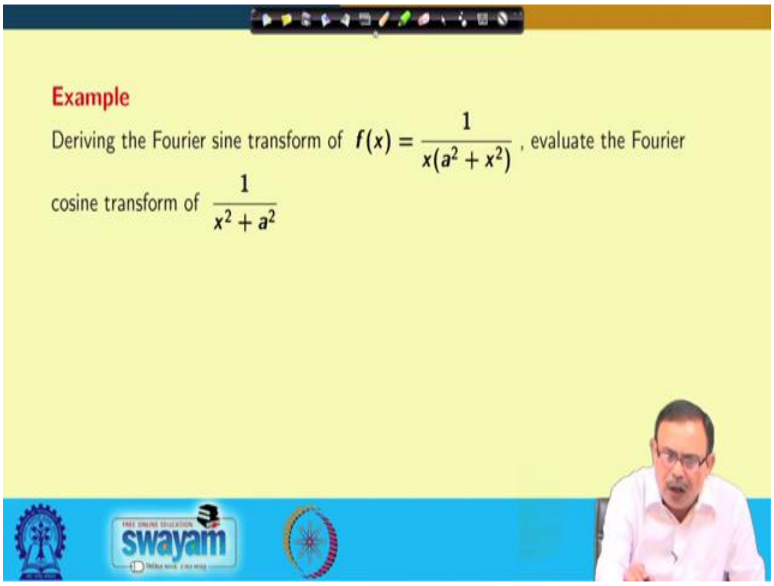


In the last lecture, we have observed that by using Parseval's identity, we can easily evaluate the value of an integral. So, effectively we can use the Laplace transform or Fourier transform or their properties for evaluation of various complicated integrals. Let us see some other examples here.

We want to derive the Fourier sine transform of $f(x)$, where $f(x)$ is defined as,

$$f(x) = \frac{1}{x(a^2 + x^2)}$$

Also we want to find out the Fourier cosine transform of $\frac{1}{x^2+a^2}$

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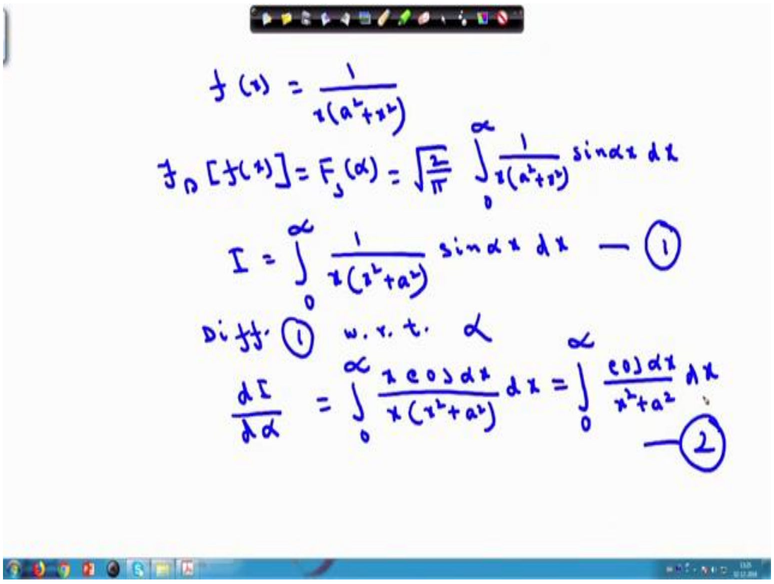

From the definition of Fourier sine transform, we can write down,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x(a^2 + x^2)} \sin \alpha x \; dx$$

What we have done earlier to evaluate the integrals is that we have used various substitution techniques to make it simple, so that we can evaluate the integral.

Let us give some other approach now for evaluation of this particular integral. Suppose

$$I = \int_0^\infty \frac{1}{x(a^2 + x^2)} \sin \alpha x \; dx \tag{1}$$

Now, we will use the differentiation under the sign of integration with respect to $\alpha$. If we differentiate both sides of (1) with respect to $\alpha$, then $x$ will be treated as constant. So, differentiating both sides of (1) with respect to $\alpha$, we will obtain,

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\cos \alpha x}{a^2 + x^2} \, dx \tag{2}$$

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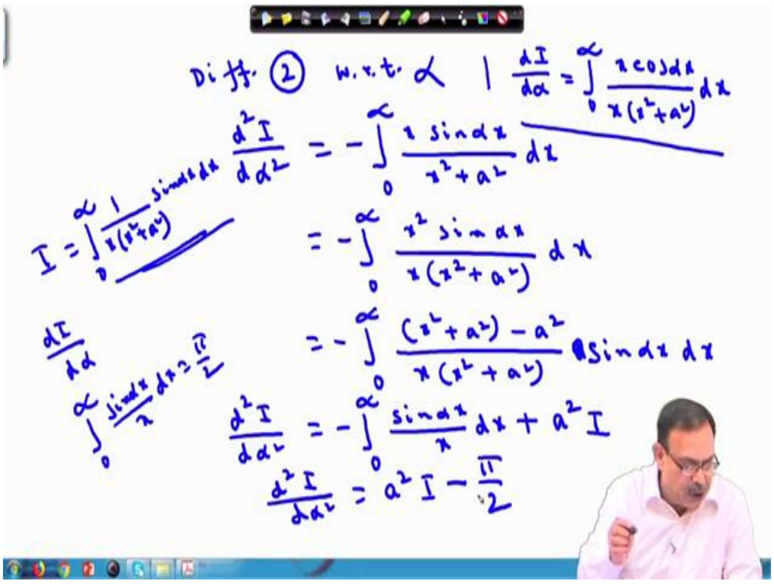

If, we differentiate both sides of (2) again with respect to $\alpha$ under the sign of integration, then we will obtain,

$$\frac{d^2 I}{d\alpha^2} = -\int_0^\infty \frac{x \sin \alpha x}{a^2 + x^2}\, dx$$
$$= -\int_0^\infty \frac{x^2 \sin \alpha x}{x(a^2 + x^2)}\, dx$$
$$= -\int_0^\infty \frac{(x^2 + a^2) - a^2}{x(a^2 + x^2)} \sin \alpha x\, dx$$
$$= -\int_0^\infty \frac{\sin \alpha x}{x} dx + a^2 \int_0^\infty \frac{\sin \alpha x}{x(a^2 + x^2)} dx$$
$$= -\int_0^\infty \frac{\sin \alpha x}{x} dx + a^2 I$$

If we recall, we had calculated the value of the integral $\int_0^\infty \frac{\sin \alpha x}{x} dx = \frac{\pi}{2}$.

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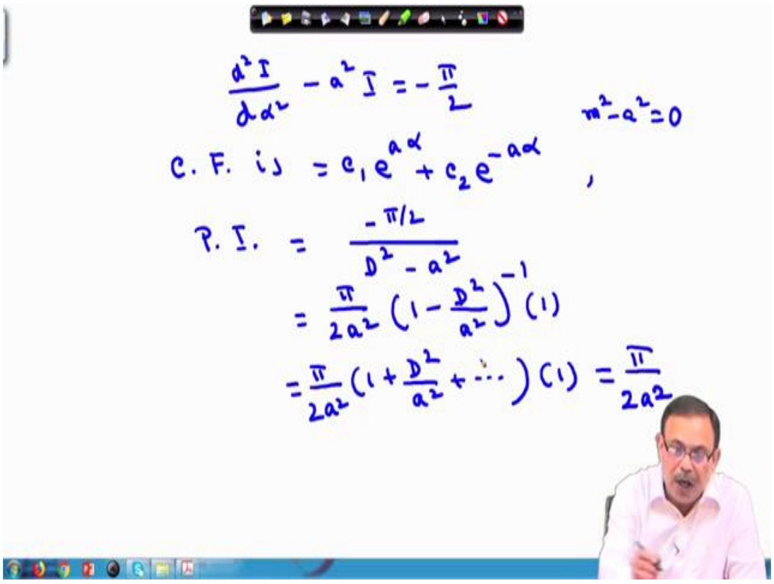


So, we are obtaining one ordinary differential equation of second order as,

$$\frac{d^2I}{d\alpha^2} - a^2 I = -\frac{\pi}{2}$$

The solution of the above ODE will contain two components, complementary function (CF) and particular integral (PI). For this ODE, auxiliary equation is given as,

$$m^2 - a^2 = 0$$

Roots of the above auxiliary equation are $\pm a$. Therefore, CF of the above ODE is,

$$CF = c_1 e^{a\alpha} + c_2 e^{-a\alpha}$$

where $c_1$ and $c_2$ are the integration constants. And the particular integral is given by

$$PI = \frac{1}{D^2 - a^2}\left(-\frac{\pi}{2}\right)$$
$$= \frac{\pi}{2a^2}\left(1 - \frac{D^2}{a^2}\right)(1)$$
$$= \frac{\pi}{2a^2}$$

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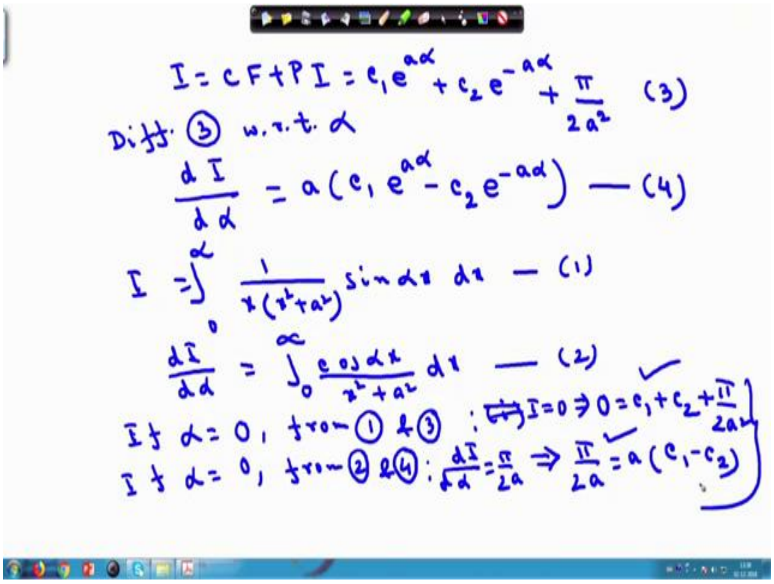


So, we have obtained CF and PI. Therefore, we can write,

$$I = CF + PI = c_1 e^{a\alpha} + c_2 e^{-a\alpha} + \frac{\pi}{2a^2} \tag{3}$$

Next, we have to find out the values of the constants $c_1$ and $c_2$. We have two constants for which we need two conditions. For this, we will find $\frac{dI}{d\alpha}$ from (3) as,

$$\frac{dI}{d\alpha} = a(c_1 e^{a\alpha} - c_2 e^{-a\alpha}) \tag{4}$$

If we put $\alpha = 0$ in (1), then we have,

$$I(0) = \int_0^\infty \frac{1}{x(a^2 + x^2)} \cdot 0 \, dx = 0$$

If we put $\alpha = 0$ in (3), then we get,

$$I(0) = c_1 + c_2 + \frac{\pi}{2a^2}$$

Therefore, at $\alpha = 0,$ (1) and (3) imply,

$$c_1 + c_2 = -\frac{\pi}{2a^2} \tag{5}$$

If we put $\alpha = 0$ in (2), then we have,

$$\frac{dI}{d\alpha}(0) = \int_0^\infty \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\left[\tan^{-1}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2a}$$

If we put $\alpha = 0$ in (4), then we get,

$$\frac{dI}{d\alpha}(0) = a(c_1 - c_2)$$

Therefore, at $\alpha = 0$, (2) and (4) imply,

$$c_1 - c_2 = \frac{\pi}{2a^2} \qquad (6)$$

Now, solving (5) and (6) for $c_1$ and $c_2$, we get,

$$c_1 = 0, \qquad c_2 = -\frac{\pi}{2a^2}$$

$$\therefore I = \frac{\pi}{2a^2}(1 - e^{-a\alpha})$$

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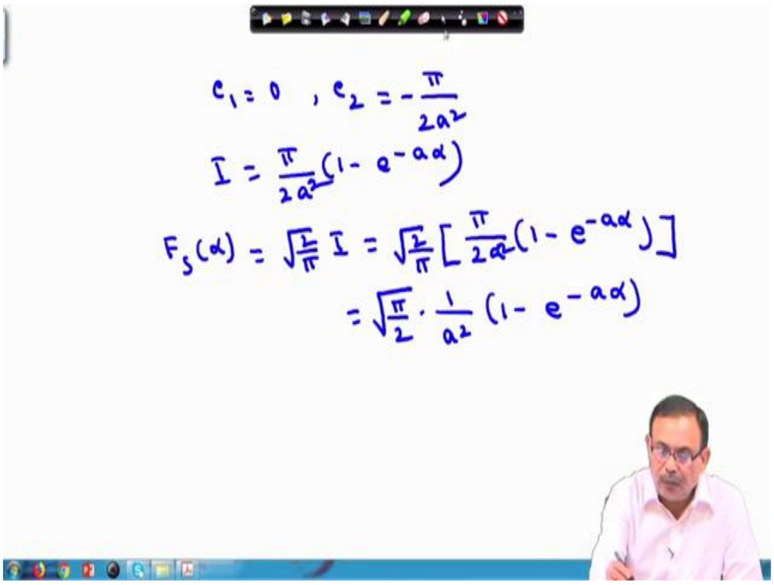

Therefore, the Fourier sine transform of the given function is

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2a^2}(1 - e^{-a\alpha}) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{a^2}(1 - e^{-a\alpha})$$

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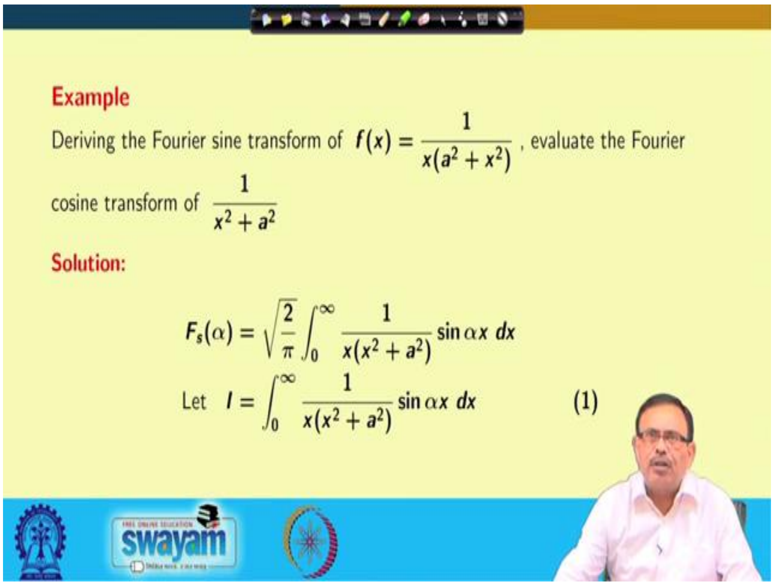


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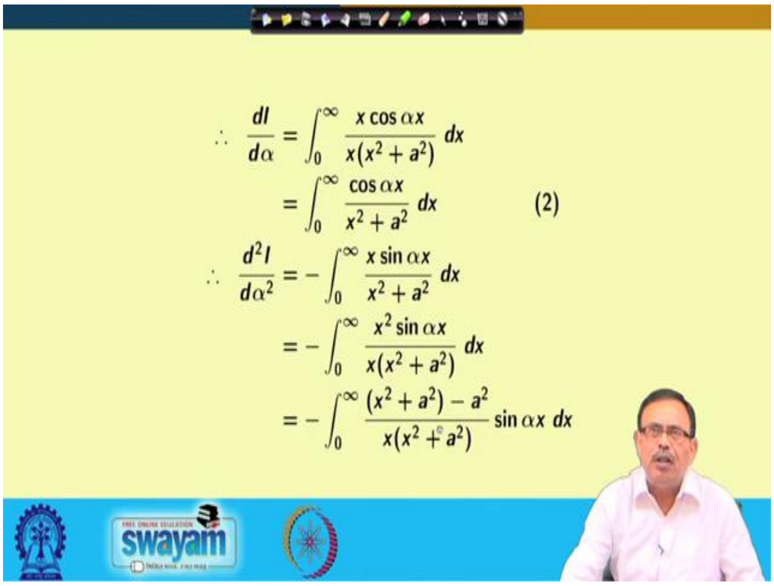

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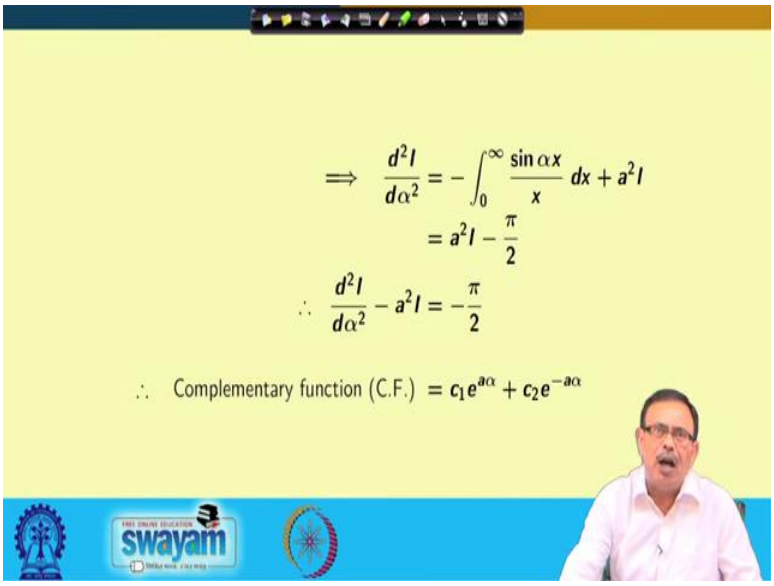


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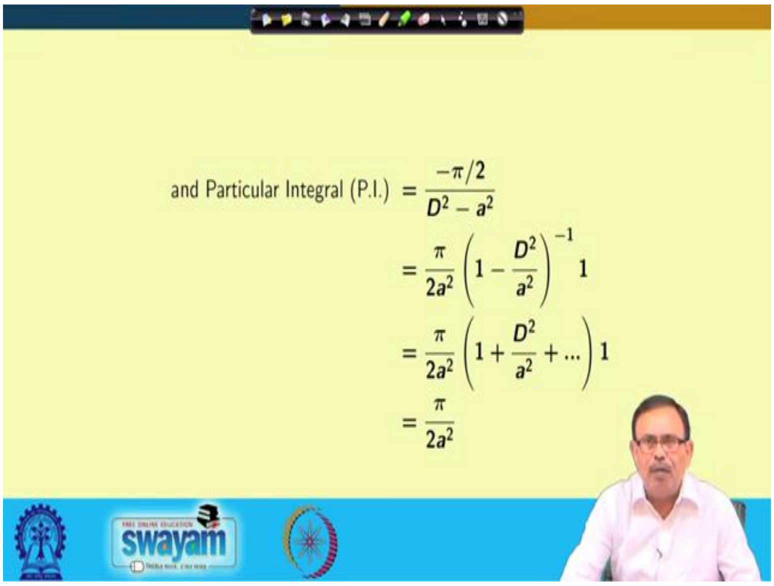

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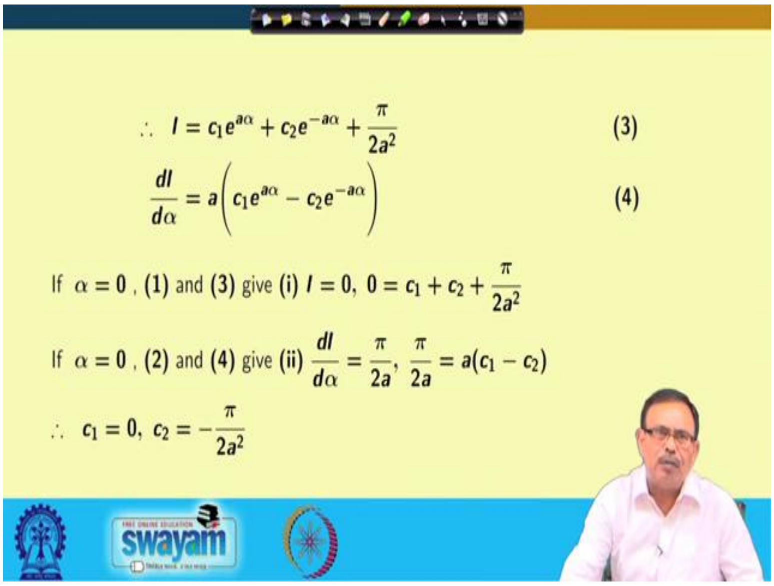


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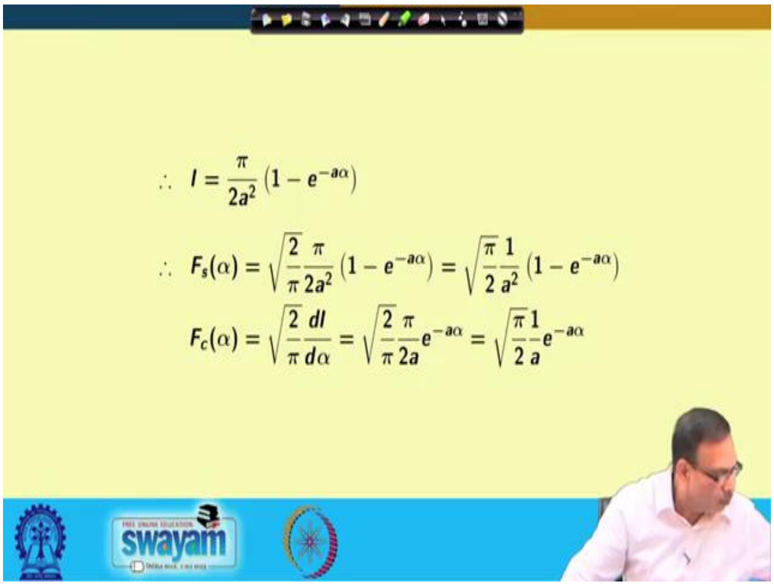

Now, we have to find out the Fourier cosine transform of $\frac{1}{x^2+a^2}$. From definition, we have,

$$\mathcal{F}_c\left[\frac{1}{x^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\cos\alpha x}{a^2+x^2}\,dx = \sqrt{\frac{2}{\pi}}\cdot\frac{dI}{d\alpha} \quad [\text{using } (2)]$$

If we put the values of $c_1$and $c_2$ in (4), then we have,

$$\frac{dI}{d\alpha} = \frac{\pi}{2a}e^{-a\alpha}$$

Therefore, Fourier cosine transform of $\frac{1}{x^2+a^2}$ is,

$$\mathcal{F}_c\left[\frac{1}{x^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2a}e^{-a\alpha} = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{a}e^{-a\alpha}$$

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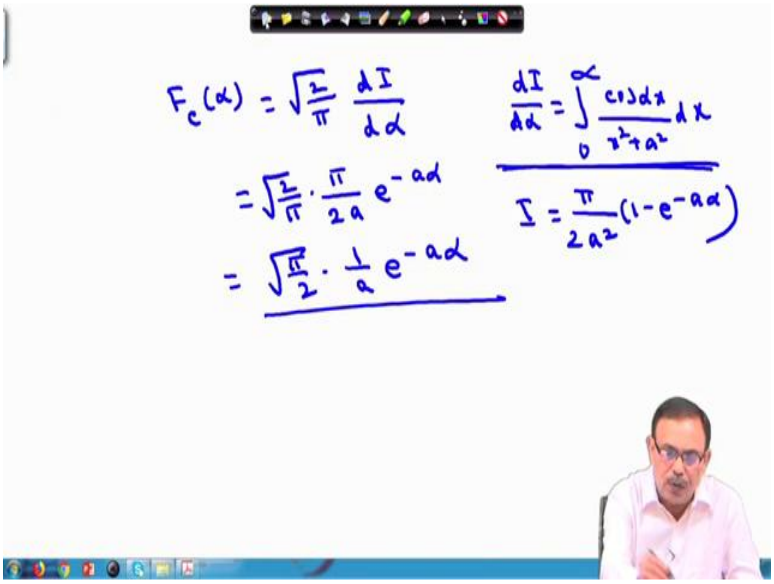

So, for evaluation of the integrals also, we can use the Fourier transform, Fourier cosine transform, Fourier sine transform and various properties of these transforms. Thank you.