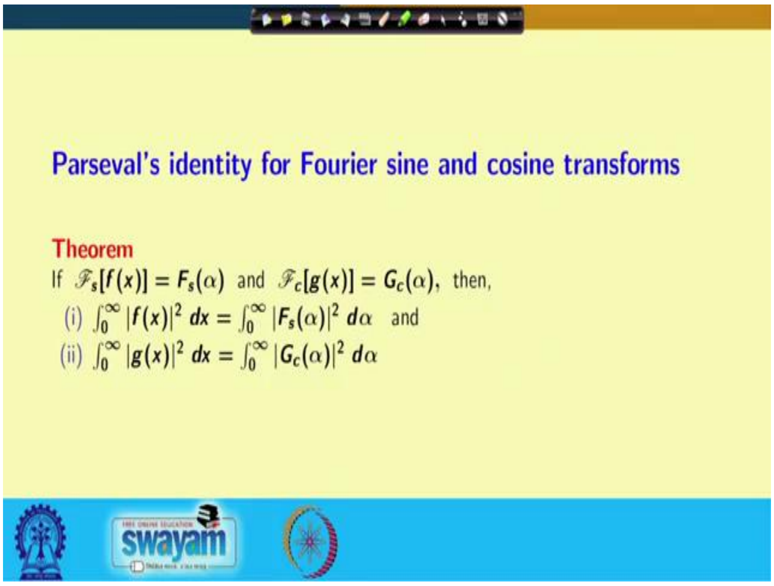
Transform Calculus and its Application in Differential Equations

Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 36
Parseval's Identity and its Application

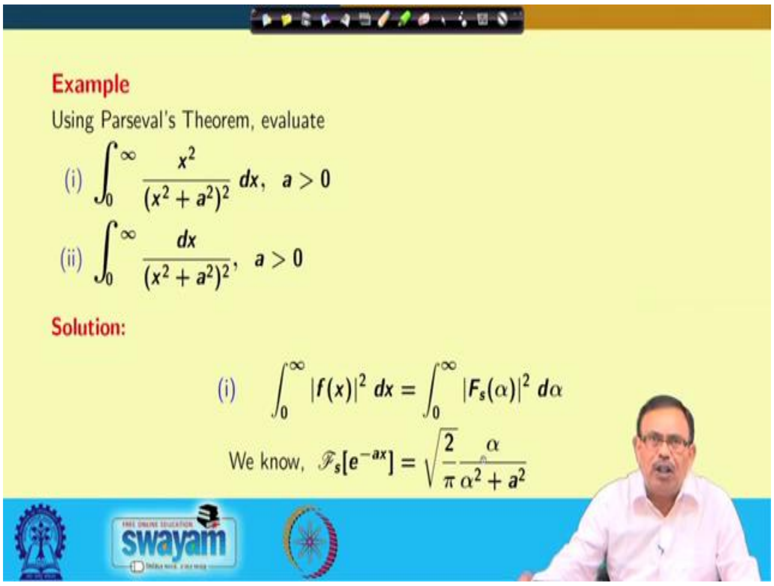
In the last lecture, we have studied the definition of convolution and the Fourier transform of convolution of two functions as well as we have discussed the Parseval's identity for Fourier transform. Now, let us see the Parseval's identity for the Fourier cosine and sine transforms also in a similar manner.

(Refer Slide Time: 00:45)



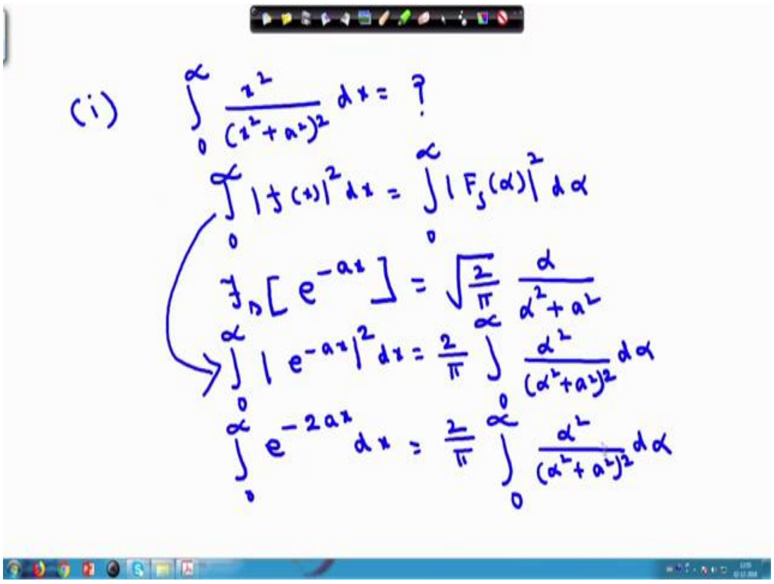
Parseval's identity for Fourier cosine and sine transforms are given in the above slide. The proof is not given, because the proofs are similar to those we have done for the case of Parseval's identity for Fourier transform.

(Refer Slide Time: 01:16)



Now, let us see the applications of Parseval's theorem. Here we want to evaluate the value of the integrals $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx$ and $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$, where $a > 0$ holds for both the cases.

(Refer Slide Time: 02:54)



So, let us solve the first problem. From Parseval's identity for Fourier sine transform, we know,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(\alpha)|^2 d\alpha$$

If we recall, the Fourier sine transform of e^{-ax} which we have already calculated previously, is given as,

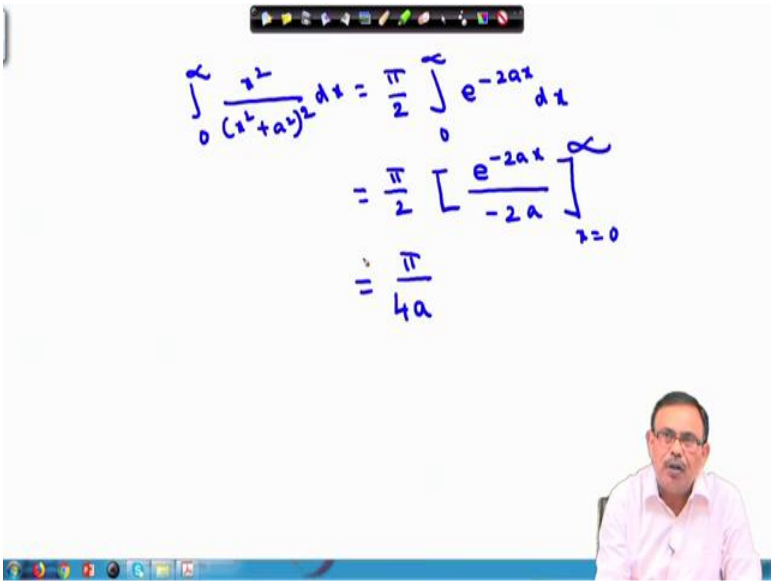
$$\mathcal{F}_s[e^{-ax}] = F_s(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{\alpha}{\alpha^2 + a^2}$$

So, using Parseval's identity for Fourier sine transform with $f(x) = e^{-ax}$, we get,

$$\int_0^\infty |e^{-ax}|^2 dx = \frac{2}{\pi}\int_0^\infty \left|\frac{\alpha}{\alpha^2 + a^2}\right|^2 d\alpha$$

$$\Rightarrow \int_0^\infty e^{-2ax} dx = \frac{2}{\pi}\int_0^\infty \frac{\alpha^2}{(\alpha^2 + a^2)^2}\, d\alpha$$

(Refer Slide Time: 06:44)



So, by replacing the parameter α on the right hand side by x , we will get,

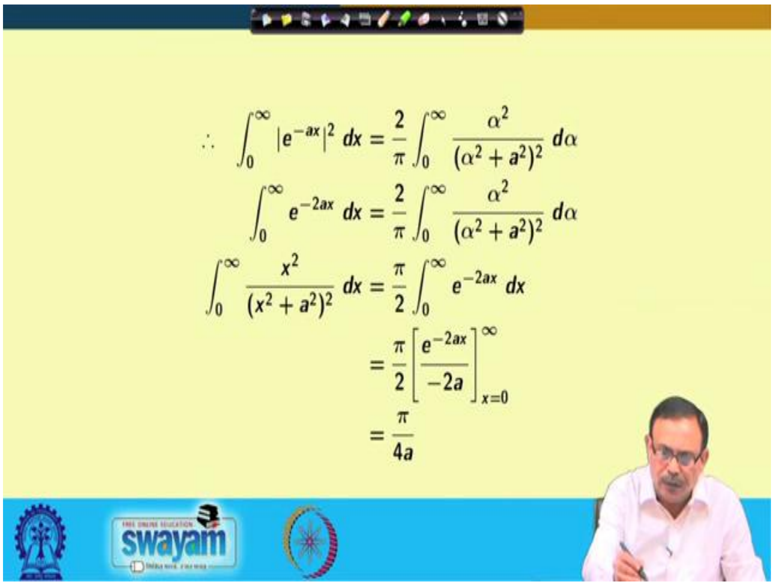
$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^2}\, dx = \frac{\pi}{2}\int_0^\infty e^{-2ax} dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_{x=0}^{\infty}$$

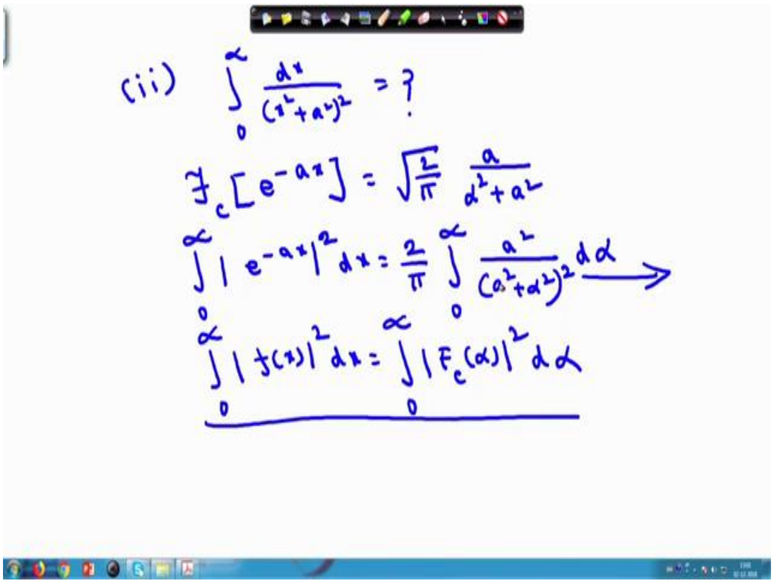
$$= \frac{\pi}{4a}$$

Thus, we have obtained the value of the integral using the Parseval's identity for Fourier sine transform.

(Refer Slide Time: 09:25)



(Refer Slide Time: 10:42)



Now, let us see the second integral.

From Parseval's identity for Fourier cosine transform, we know,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(\alpha)|^2 d\alpha$$

If we recall, the Fourier cosine transform of e^{-ax} which also we calculated previously, is given as,

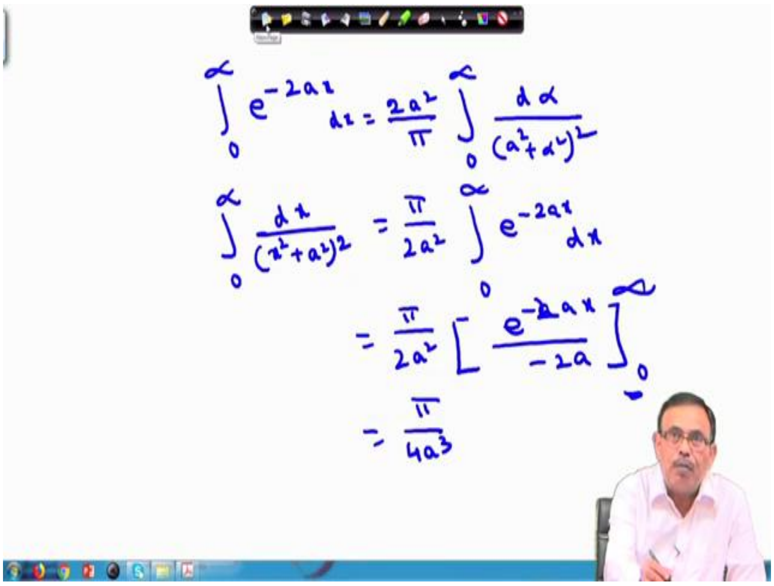
$$\mathcal{F}_c[e^{-ax}] = F_c(\alpha) = \sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2 + a^2}$$

So, using Parseval's identity for Fourier cosine transform with $f(x) = e^{-ax}$, we get,

$$\int_0^\infty |e^{-ax}|^2 dx = \frac{2}{\pi}\int_0^\infty \left|\frac{a}{\alpha^2 + a^2}\right|^2 d\alpha$$

$$\Rightarrow \int_0^\infty e^{-2ax} dx = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(\alpha^2 + a^2)^2}\, d\alpha$$

(Refer Slide Time: 13:00)



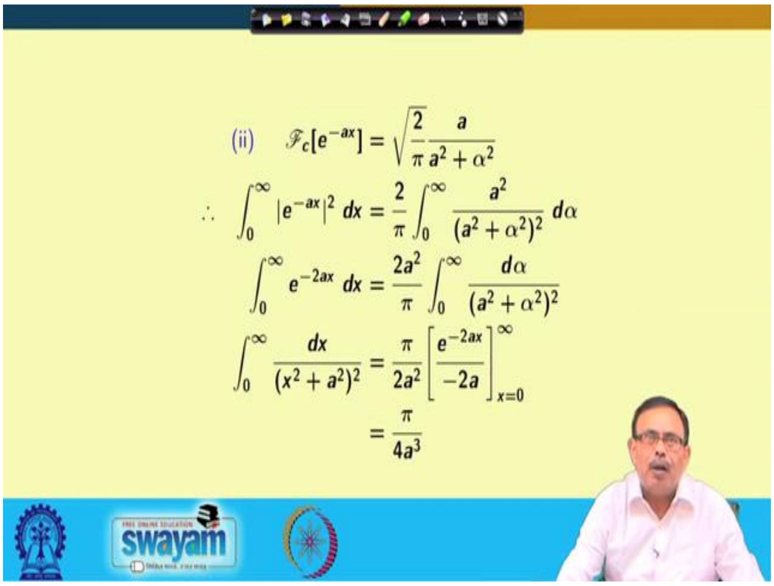
So, here also, by replacing the parameter α on the right hand side by x , we will get,

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{2a^2}\int_0^\infty e^{-2ax} dx$$

$$= \frac{\pi}{2a^2}\left[\frac{e^{-2ax}}{-2a}\right]_{x=0}^{\infty}$$

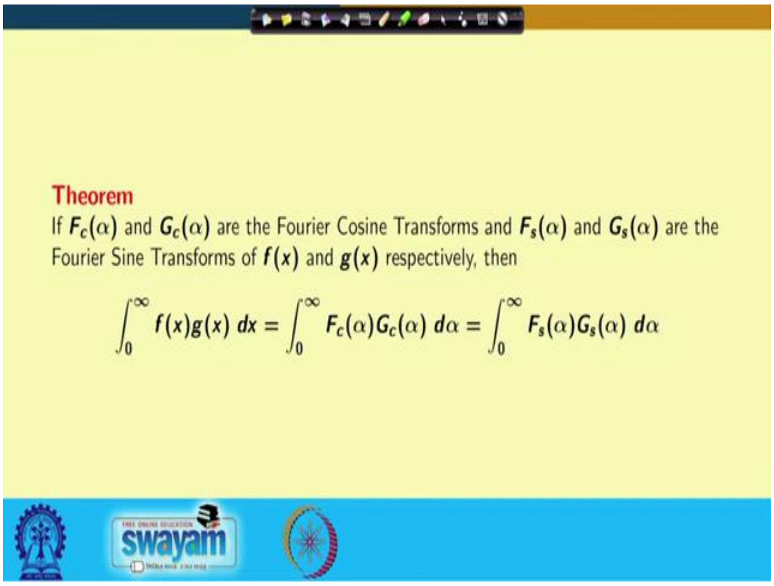
$$= \frac{\pi}{4a^3}$$

(Refer Slide Time: 14:40)



So, following similar procedure, for evaluation of various integral values, we can use the properties of Fourier transform, we can use the convolution, we can use the Parseval's identity, as required.

(Refer Slide Time: 16:52)

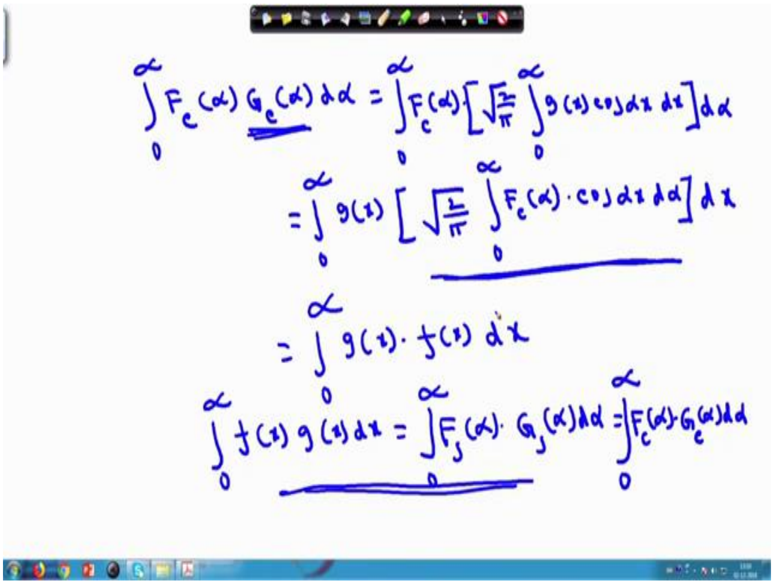


Now, let us see another theorem. This is an interesting theorem which states that, if $F_c(\alpha)$ and $G_c(\alpha)$ are the Fourier cosine transforms and $F_s(\alpha)$ and $G_s(\alpha)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then,

$$\int_0^\infty f(x)\, g(x)dx = \int_0^\infty F_c(\alpha)G_c(\alpha)d\alpha = \int_0^\infty F_s(\alpha)G_s(\alpha)d\alpha$$

Let us see the proof of this theorem. We will prove it one by one.

(Refer Slide Time: 18:06)



The proof is very simple. We will start from the right hand side and use the definition of Fourier cosine transform, i.e.

$$\int_0^\infty F_c(\alpha)G_c(\alpha)d\alpha = \int_0^\infty F_c(\alpha)\left[\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos\alpha x\, dx\right]d\alpha$$

Now changing the order of integration on the right side, we get,

$$\int_0^\infty F_c(\alpha)G_c(\alpha)d\alpha = \int_0^\infty g(x)\left[\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\alpha)\cos\alpha x\, d\alpha\right]dx$$

Now, the bracketed term in the right hand side is nothing but the inverse Fourier cosine transform of $f(x)$. Therefore we have,

$$\int_0^\infty F_c(\alpha)G_c(\alpha)d\alpha = \int_0^\infty f(x)\, g(x)dx$$

Using the same procedure, we can obtain,

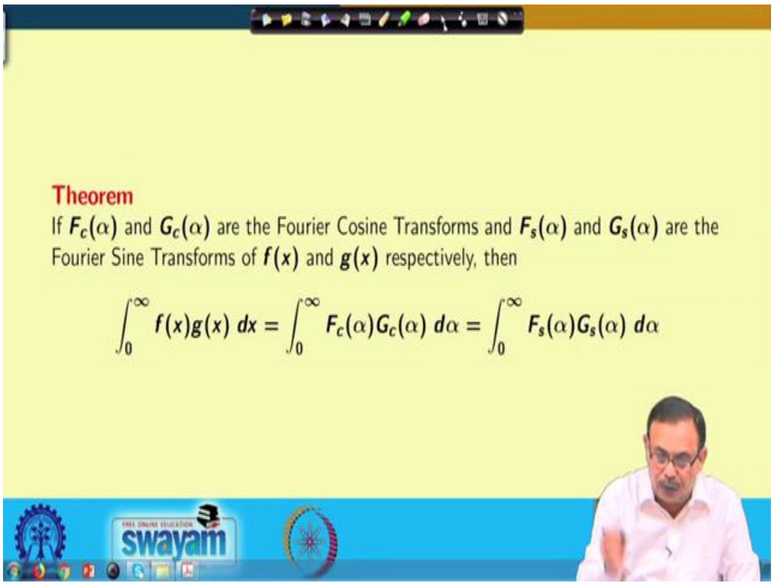
$$\int_0^\infty F_s(\alpha)G_s(\alpha)d\alpha = \int_0^\infty f(x)\, g(x)dx$$

So combining the two relations, we get,

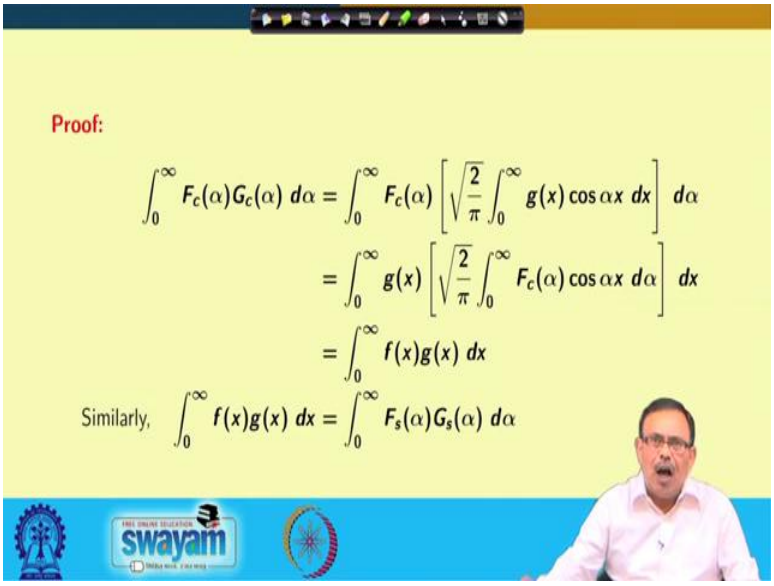
$$\int_0^\infty f(x)\, g(x)dx = \int_0^\infty F_c(\alpha)G_c(\alpha)d\alpha = \int_0^\infty F_s(\alpha)G_s(\alpha)d\alpha$$

This completes the proof.

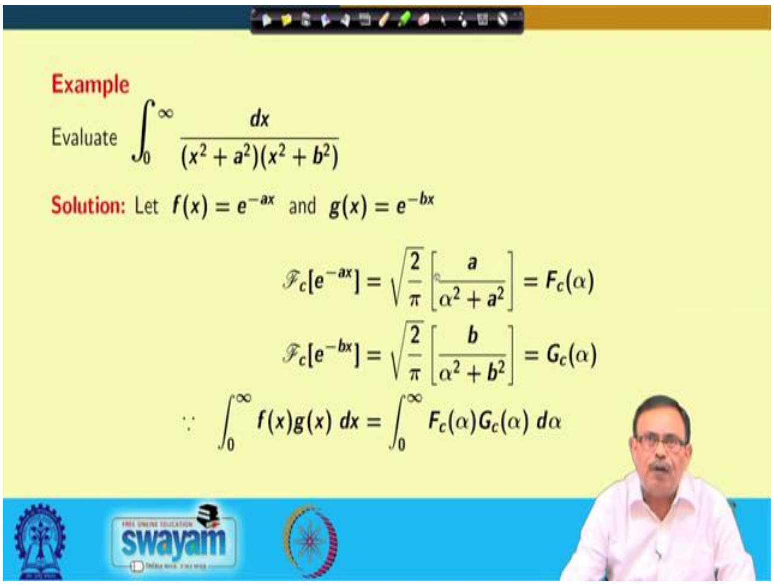
(Refer Slide Time: 22:19)



(Refer Slide Time: 22:21)



(Refer Slide Time: 23:48)



Now, let us see the application of the last theorem. We want to evaluate the value of

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

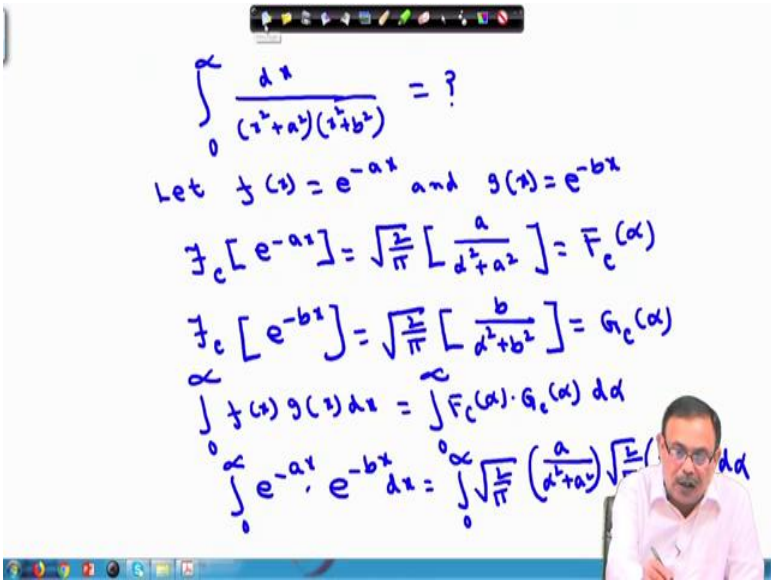
These integrals are known integrals and we have purposefully using this, so that we can understand how easily we can find out the values of different integrations using the Fourier transform or the properties of Fourier transform.

So, let us evaluate this integral.

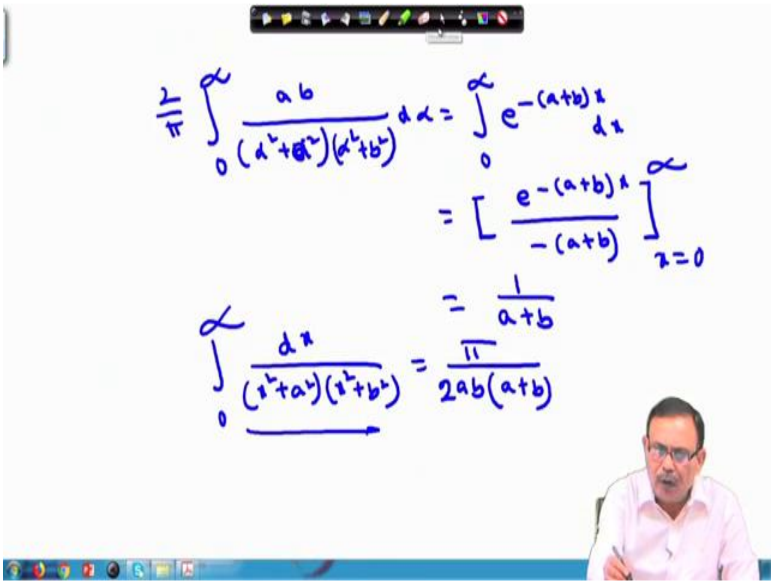
Let, $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Fourier cosine transform of these two functions are known to us and are given as,

$$\mathcal{F}_c[e^{-ax}] = F_c(\alpha) = \sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2+a^2} \quad \text{and} \quad \mathcal{F}_c[e^{-bx}] = G_c(\alpha) = \sqrt{\frac{2}{\pi}}\frac{b}{\alpha^2+b^2}$$

(Refer Slide Time: 24:38)



(Refer Slide Time: 27:38)



From the last theorem we have,

$$\int_0^\infty F_c(\alpha) G_c(\alpha) d\alpha = \int_0^\infty f(x)\, g(x) dx$$

So, putting all the values we will obtain,

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{\alpha^2 + a^2} \cdot \sqrt{\frac{2}{\pi}} \frac{b}{\alpha^2 + b^2} d\alpha = \int_0^\infty e^{-ax} e^{-bx} dx$$

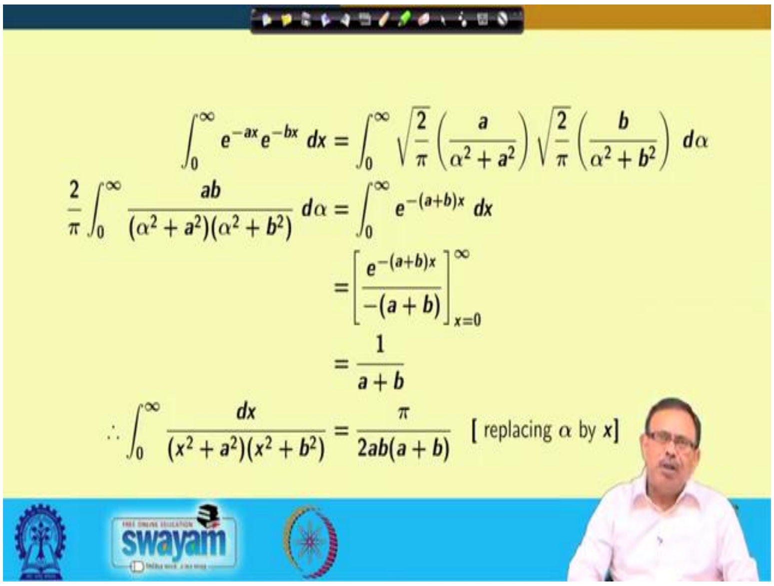
$$\Rightarrow \frac{2ab}{\pi}\int_0^{\infty} \frac{d\alpha}{(\alpha^2+a^2)(\alpha^2+b^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_{x=0}^{\infty}$$

$$\Rightarrow \int_0^{\infty} \frac{d\alpha}{(\alpha^2+a^2)(\alpha^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Now, replacing α by x on the left hand side, we get the value of the required integral as,

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

(Refer Slide Time: 29:55)



Thank you.