Transform Calculus and its Application in Differential Equations
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Lecture – 35
Fourier Transform of Convolution of two functions

In this lecture, let us start with the Convolution of Fourier transform. We had defined the convolution in case of Laplace transform also and using the convolution properties, we were able to find out the Laplace transform of various functions. First, we will define here, what the convolution of two functions is and then we will see some properties of convolution of Fourier transform.

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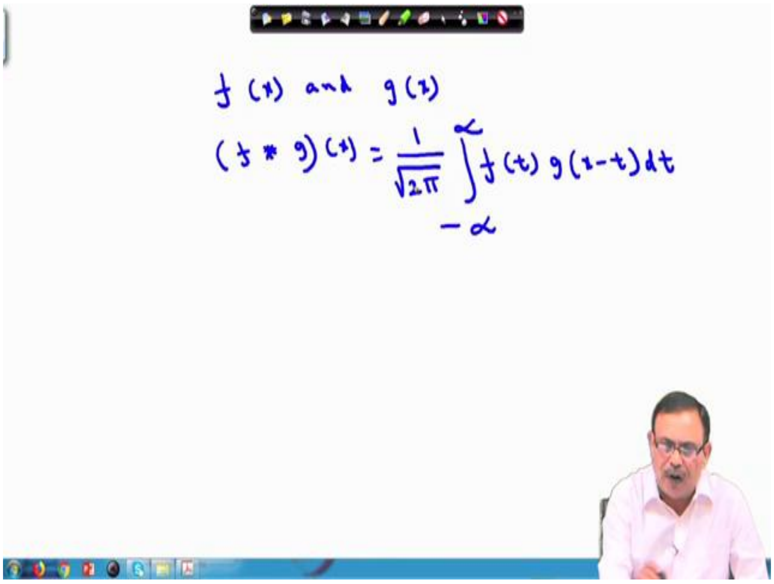

If two functions $f(x)$ and $g(x)$ are given to us, then the convolution is defined as,

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x - t) dt$$

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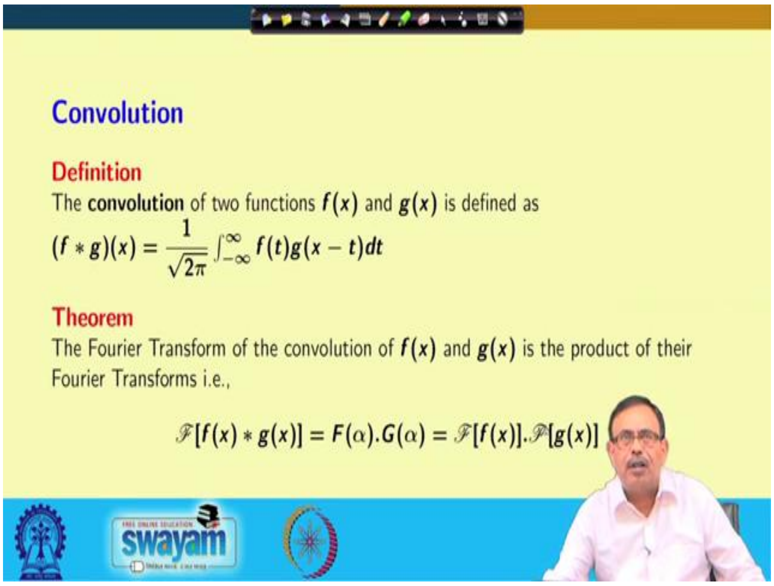

Now, let us come to a theorem which states that the Fourier transform of convolution of two functions is the product of Fourier transform of those two functions i.e., $\mathcal{F}[f(x) * g(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)] = F(\alpha) \cdot G(\alpha)$ So, this is similar to the convolution theorem in case of Laplace transform. In Laplace transform also, same thing happened that Laplace transform of convolution of two functions $f(x)$ and $g(x)$ is equal to the product of Laplace transform of $f(x)$ and Laplace transform of $g(x)$.

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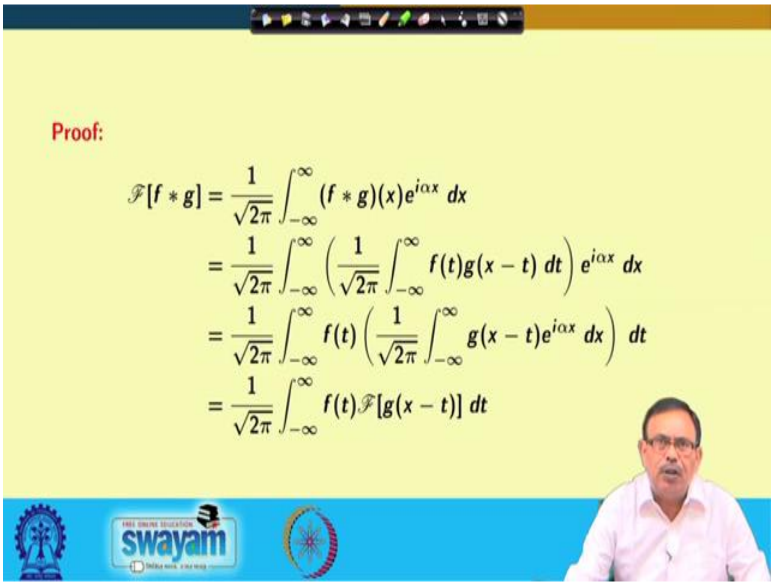

So, we will go through the proof of this one now.

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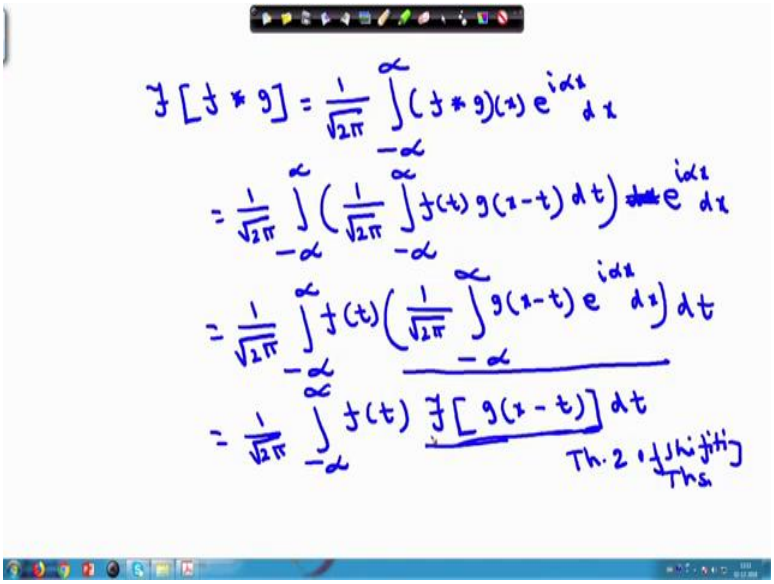

From the definition of Fourier transform, we have,

$$\mathcal{F}[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x)\, e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt \right) e^{i\alpha x} dx$$

In this double integration, f is a function of t only, but g is a function of x and t. So, if we change the order of integration, we get,

$$\mathcal{F}[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t) e^{i\alpha x} dx \right) dt$$

Now, the integration $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t) e^{i\alpha x} dx$ is nothing but the Fourier transform of $g(x-t)$. Therefore, from the above equation, we get,

$$\mathcal{F}[f * g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, \mathcal{F}[g(x-t)] dt$$

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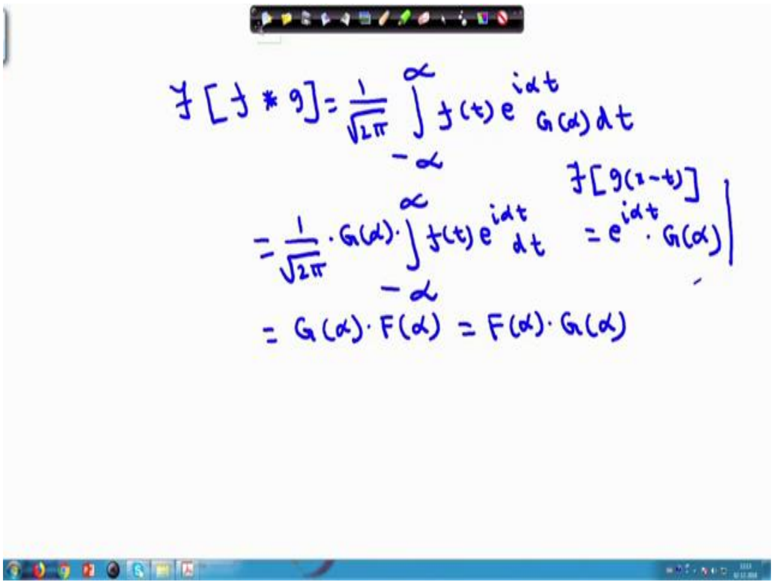

From the shifting theorem for Fourier transform, we have $\mathcal{F}[g(x-t)] = e^{i\alpha t}\mathcal{F}[g(x)]$.

So, using this we get,

$$\mathcal{F}[f * g] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\mathcal{F}[g(x)]dt$$

Now, $\mathcal{F}[g(x)] = G(\alpha)$ is independent of variable t. So, we can take $\mathcal{F}[g(x)]$ outside the integration and rewrite the above equation as,

$$\mathcal{F}[f * g] = G(\alpha) \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}dt$$
$$= G(\alpha) \cdot F(\alpha)$$

This completes the proof.

From here, we can draw one corollary, that is

$$\mathcal{F}^{-1}[F(\alpha) \cdot G(\alpha)] = (f * g)(x) = \mathcal{F}^{-1}[F(\alpha)] * \mathcal{F}^{-1}[G(\alpha)]$$

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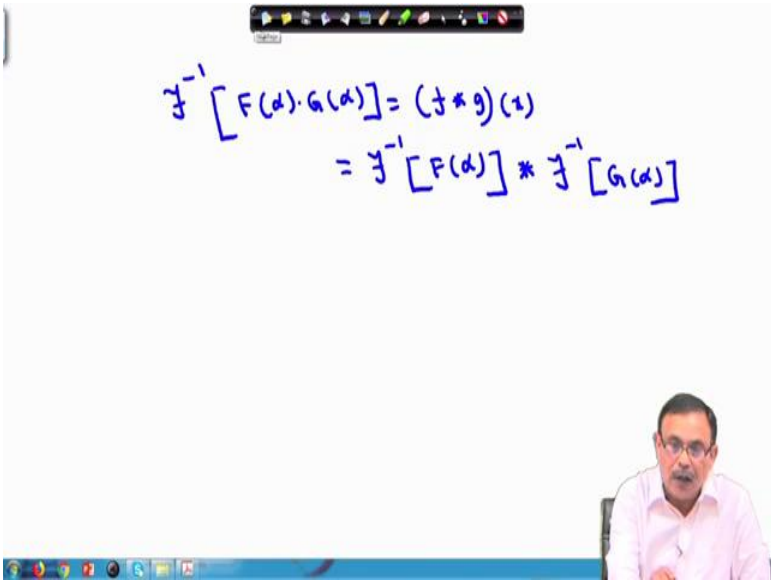

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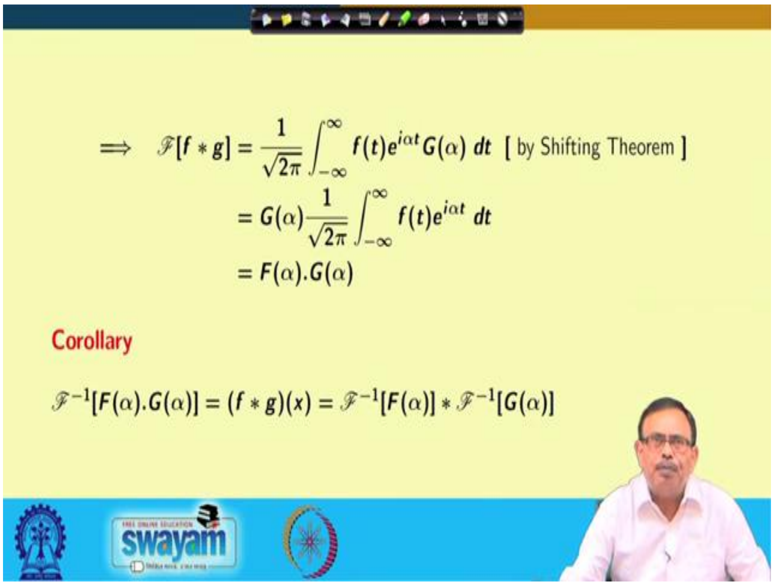

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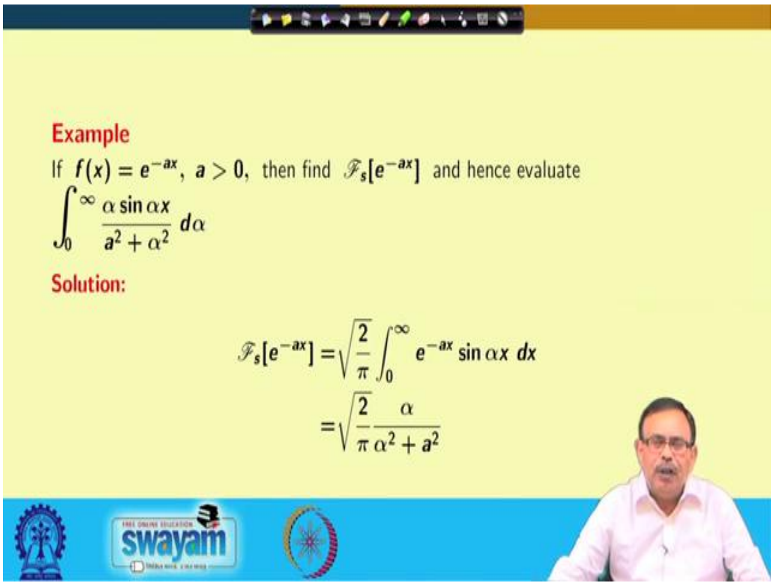

Now in the next example, we want to find the Fourier sine transform of e^{-ax}, where $a > 0$. Using the result, we will try to find out the value of the integral

$$\int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + a^2} d\alpha$$

which is very difficult to solve directly.

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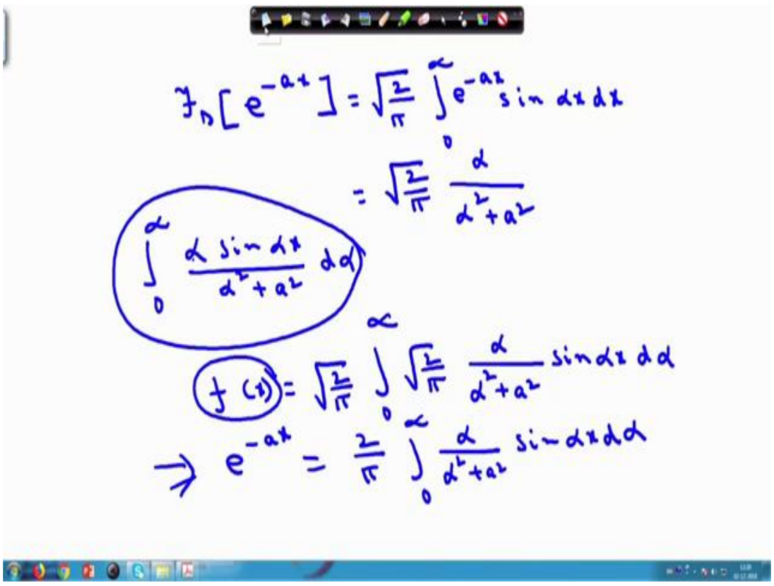

We have already obtained the Fourier sine transform of e^{-ax} in previous lectures. So here we are writing the result directly as,

$$\mathcal{F}_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + a^2}$$

If we take the inverse Fourier sine transform, then, we have,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{F}_s[e^{-ax}] \sin \alpha x \, d\alpha$$

$$\Rightarrow e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + a^2} \sin \alpha x \, d\alpha$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + a^2} d\alpha$$

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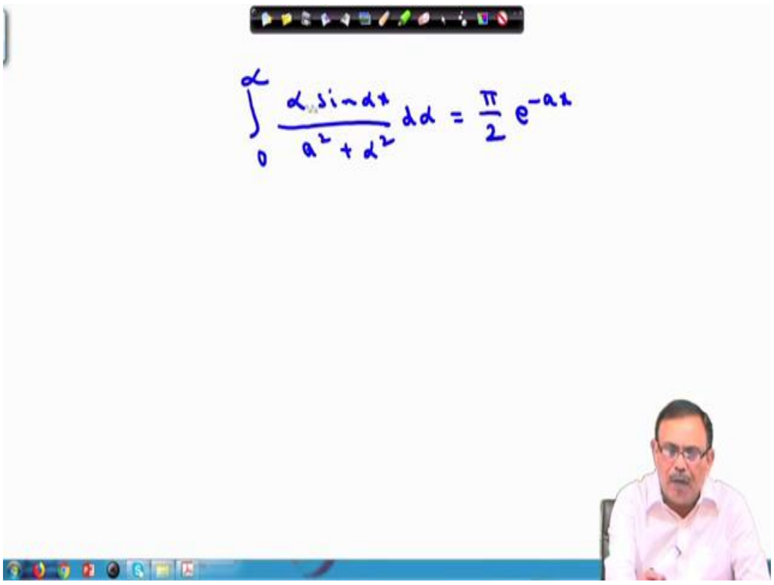

Therefore, using inverse Fourier sine transform, we obtained the value of the integral as,

$$\int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + a^2} d\alpha = \frac{\pi}{2} e^{-ax}$$

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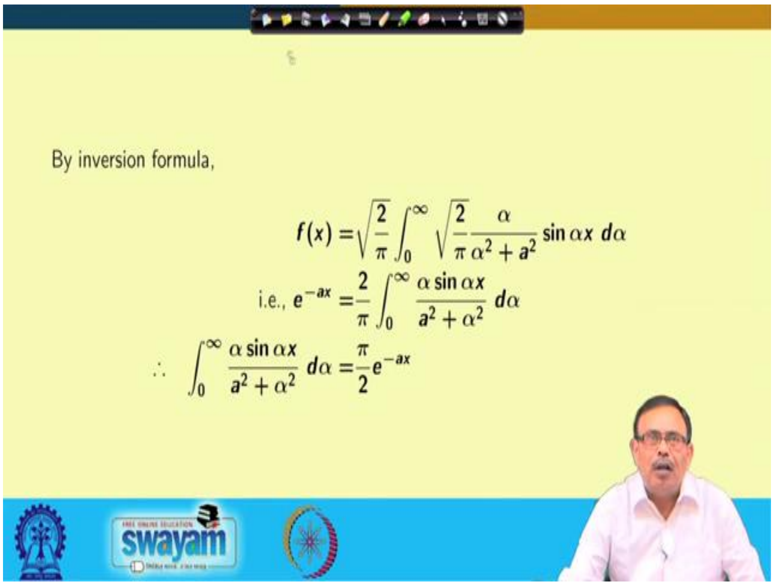

Now, let us see the next problem.

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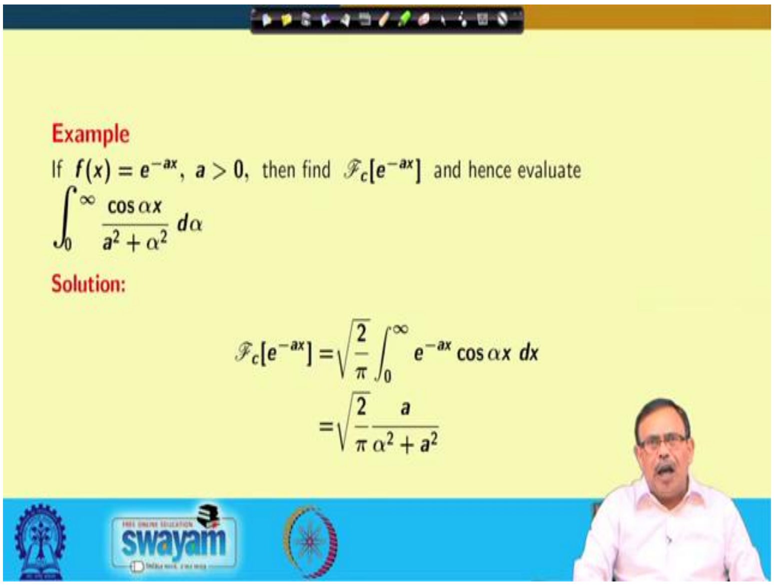

The function is same, but here we want to find out the Fourier cosine transform of e^{-ax} and from there we want to find the value of $\int_0^\infty \frac{\cos \alpha x}{\alpha^2 + a^2} d\alpha$. So, obviously, now we have understood that the technique will be similar as we have done it in the case of the earlier problem, that is first we will find out the Fourier cosine transform.

Again, we have already done what is the Fourier cosine transform of e^{-ax}. So, we will use that particular value whatever we have done earlier and using that one, we will try to evaluate the value of the given integral.

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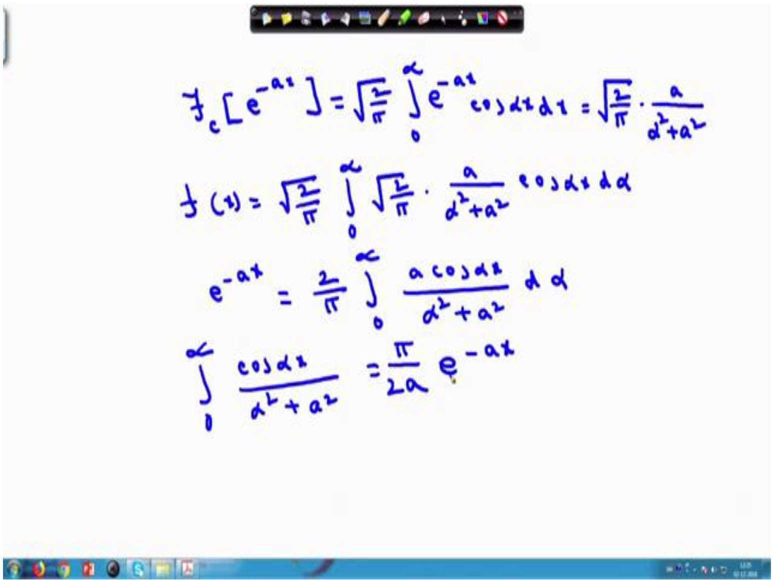

We obtained Fourier cosine transform of e^{-ax} as,

$$\mathcal{F}_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2 + a^2}$$

So, following the same procedure, if we take the inverse Fourier cosine transform, then, we have,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c[e^{-ax}]\cos\alpha x\, d\alpha$$

$$\Rightarrow e^{-ax} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2 + a^2}\cos\alpha x\, d\alpha$$

$$\Rightarrow e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\alpha x}{\alpha^2 + a^2}d\alpha$$

$$\Rightarrow \int_0^\infty \frac{\cos\alpha x}{\alpha^2 + a^2}d\alpha = \frac{\pi}{2a}e^{-ax}$$

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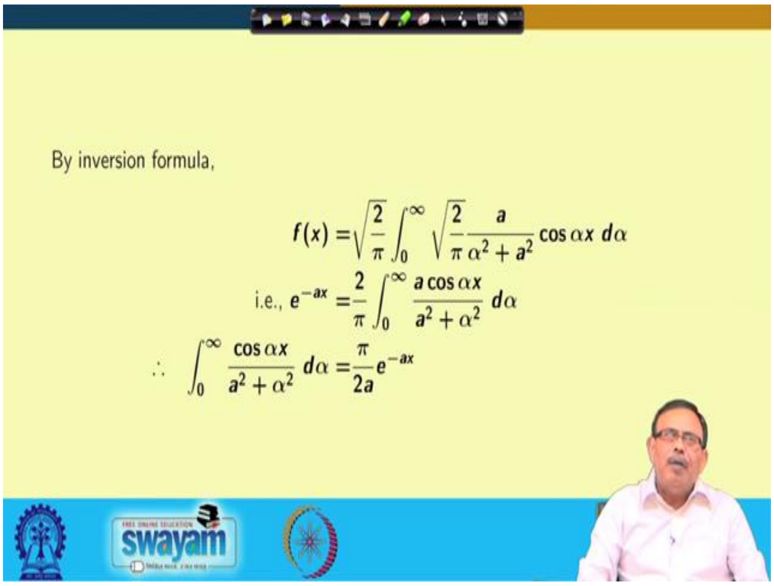

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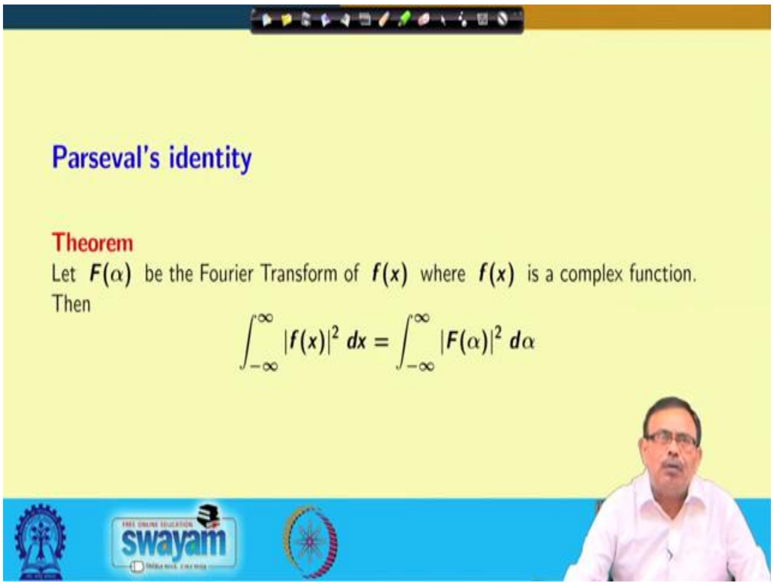

Now, let us discuss the Parseval's identity or Parseval's theorem. We did the same for the Laplace transform also. So, Parseval's identity states that,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha$$

Let us see the proof of this theorem.

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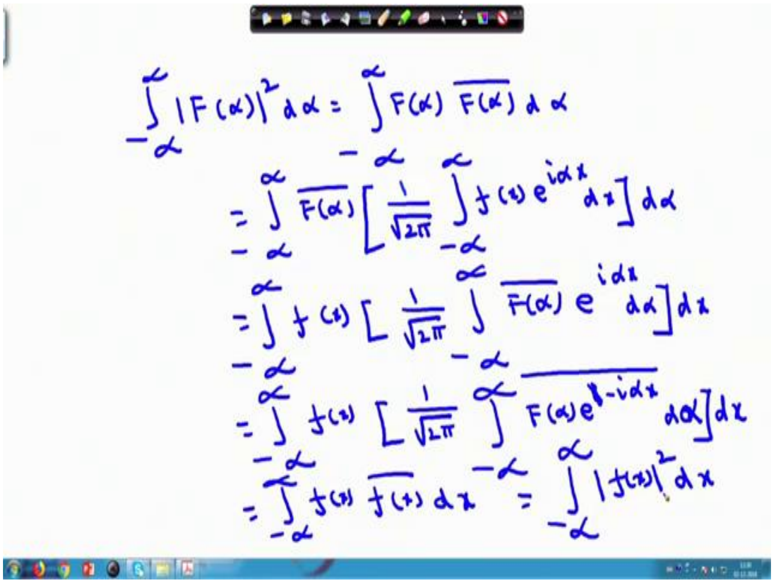

So, we are starting from the right hand side,

$$\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} F(\alpha)\, \overline{F(\alpha)} d\alpha$$

If we replace $F(\alpha)$ by the definition of Fourier transform then, we get,

$$\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} \overline{F(\alpha)} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx\right] d\alpha$$

If we change the order of integration on the right side, we get,

$$\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} f(x) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{F(\alpha)} e^{i\alpha x} d\alpha\right] dx$$

$$= \int_{-\infty}^{\infty} f(x) \left[\overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha}\right] dx$$

So, from the definition of inverse Fourier transform, we get,

$$\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} f(x)\, \overline{f(x)}\ dx$$

$$= \int_{-\infty}^{\infty} |f(x)|^2 dx$$

This completes the proof.

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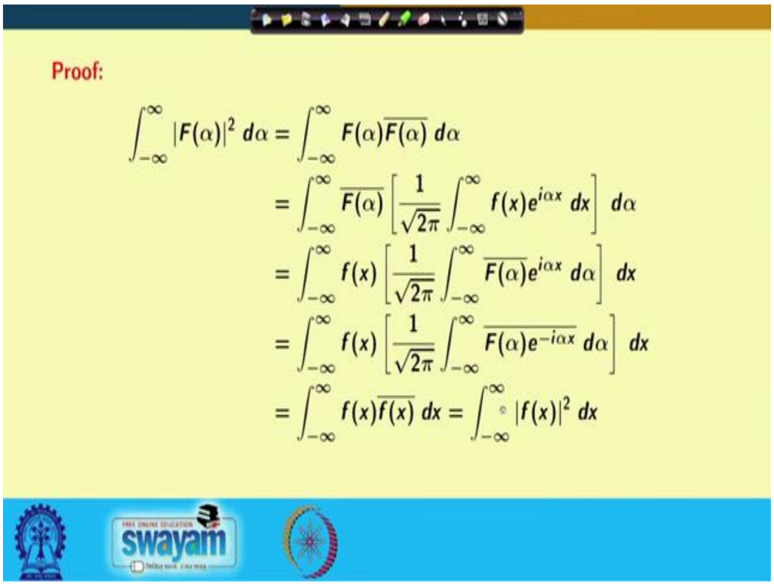

Thank you.