Transform Calculus and its Applications in Differential Equations
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Lecture – 34
Applications of Properties of Fourier Transform – II

In the last few lectures, we have studied the Fourier transform, Fourier cosine transform and Fourier sine transform. We have seen their properties and that if we know the Fourier transform of a function, then we can evaluate the Fourier transform of the derivative of the function as well.

We have also seen how to find out the Fourier transform of the integral of a function and using these properties, how to find out the Fourier transform of various complicated functions has also been discussed. In this particular lecture, let us go through some more examples, so that we can understand how to find out the Fourier transform or Fourier sine transform or Fourier cosine transform of a function in a much better way. Or if we know the Fourier transform of a function, then using the inverse Fourier transform, how to find out the function, that also we are going to find out.

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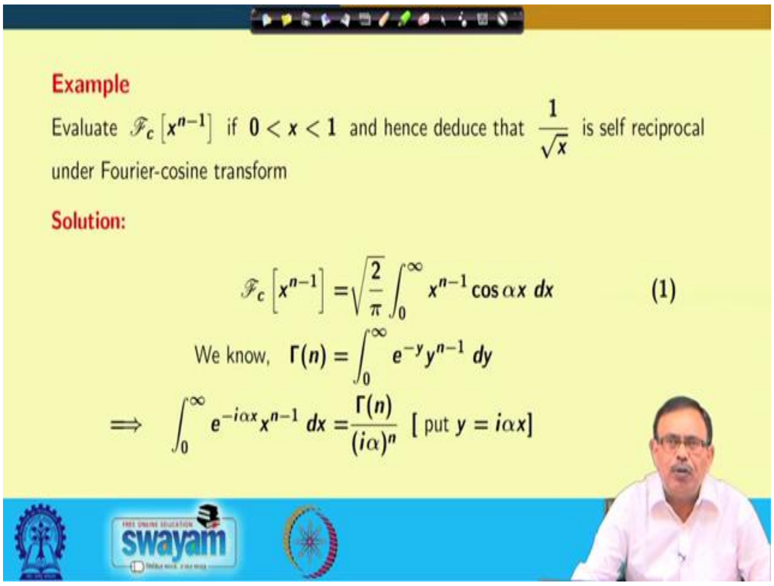

So, let us see the first example. Suppose we want to evaluate the Fourier cosine transform of x^{n-1}. And once we have obtained the Fourier cosine transform of x^{n-1}, then from there we have to show that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

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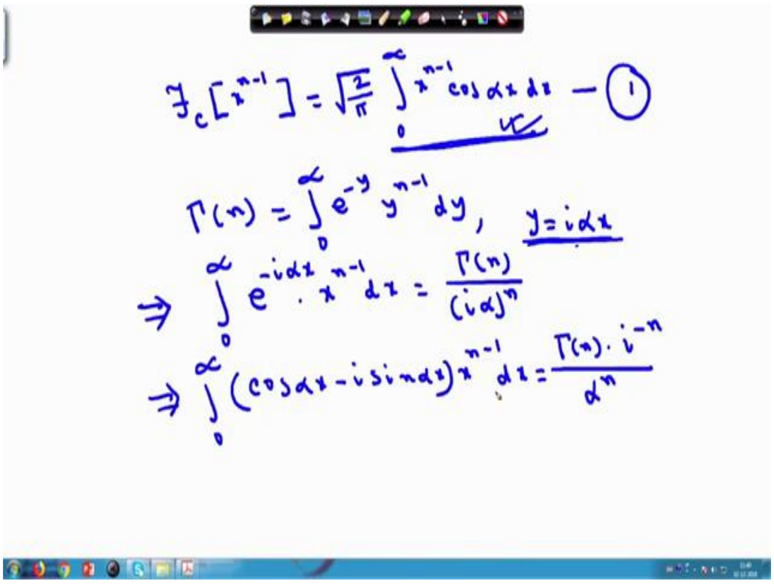

From the definition of Fourier cosine transform, we have,

$$\mathcal{F}_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos \alpha x \, dx \tag{1}$$

Now, we have to evaluate this particular integral. So, we try to find out some exponential function which can be extended in the form of $\cos x$ and $\sin x$ and whose value is known to us. So, let us start with the definition of Gamma function.

$$\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$$

If we substitute $y = i\alpha x$, then we will get

$$\Gamma(n) = \int_0^\infty e^{-i\alpha x} (i\alpha x)^{n-1} (i\alpha) dx$$

$$\Rightarrow \frac{\Gamma(n)}{(i\alpha)^n} = \int_0^\infty e^{-i\alpha x} x^{n-1} dx$$

So, we can write down

$$\int_0^\infty [\cos \alpha x - i \sin \alpha x] x^{n-1} dx = \frac{\Gamma(n)\, i^{-n}}{\alpha^n}$$

So it is clear now, why we started with the gamma function. If we take the real part of left hand side integral, then this is nothing but the required integral, which we have to find out over here.

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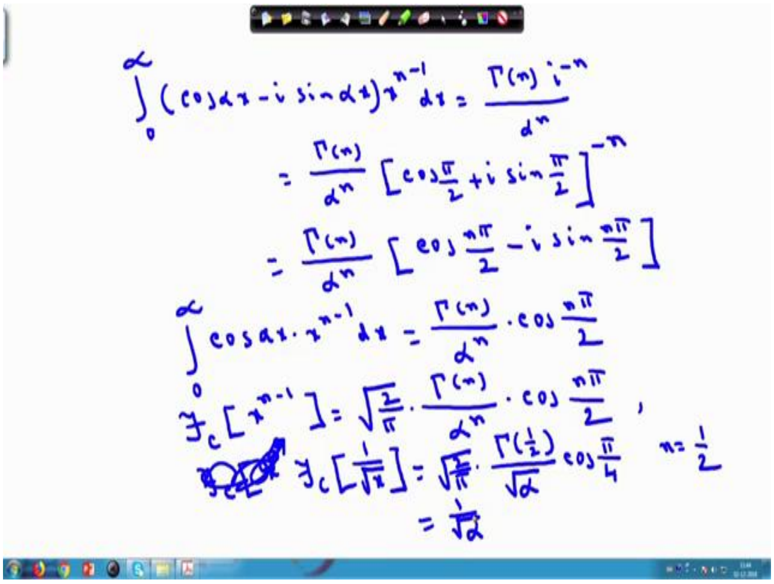

Now, we have to simplify this function.

$$\int_0^\infty [\cos \alpha x - i \sin \alpha x] x^{n-1} dx = \frac{\Gamma(n)}{\alpha^n} \left[\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right]^{-n}$$

$$= \frac{\Gamma(n)}{\alpha^n} \left[\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right]$$

Now comparing the real parts from both sides, we have,

$$\int_0^\infty x^{n-1} \cos \alpha x \, dx = \frac{\Gamma(n)}{\alpha^n} \cos\frac{n\pi}{2}$$

$$\therefore \mathcal{F}_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(n)}{\alpha^n} \cos\frac{n\pi}{2}$$

For the second part of the problem, if we substitute $n = \frac{1}{2}$ in $\mathcal{F}_c[x^{n-1}]$, then, it will become

$$\therefore \mathcal{F}_c\left[x^{\frac{1}{2}-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{\alpha^{\frac{1}{2}}}\cos\frac{\pi}{4}$$

$$\Rightarrow \mathcal{F}_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{\sqrt{\pi}}{\sqrt{\alpha}}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{\alpha}}$$

Therefore $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine transform.

So, please note that whenever some integrals are given, using some suitable known results, we can find out the solution or we can find out the value of some other integral also.

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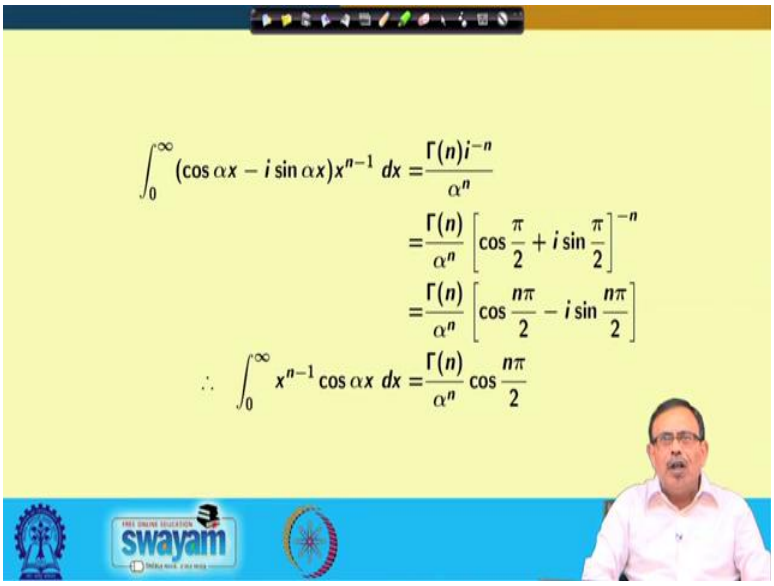

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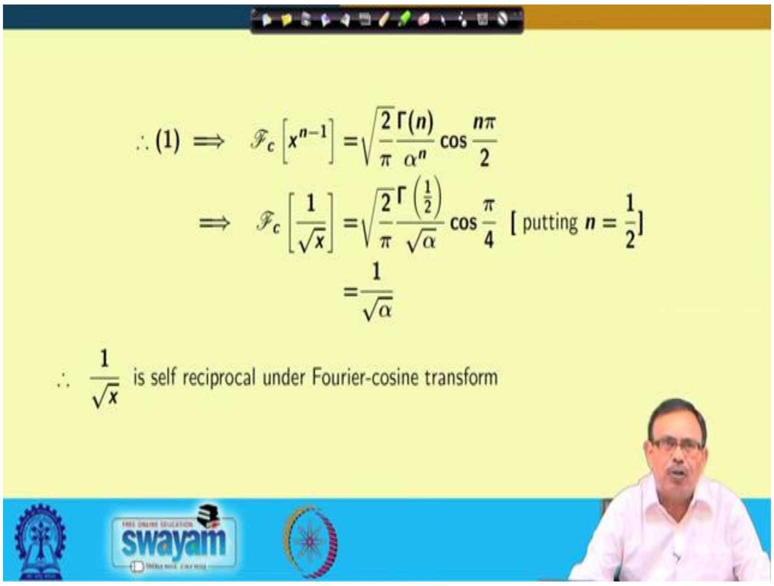

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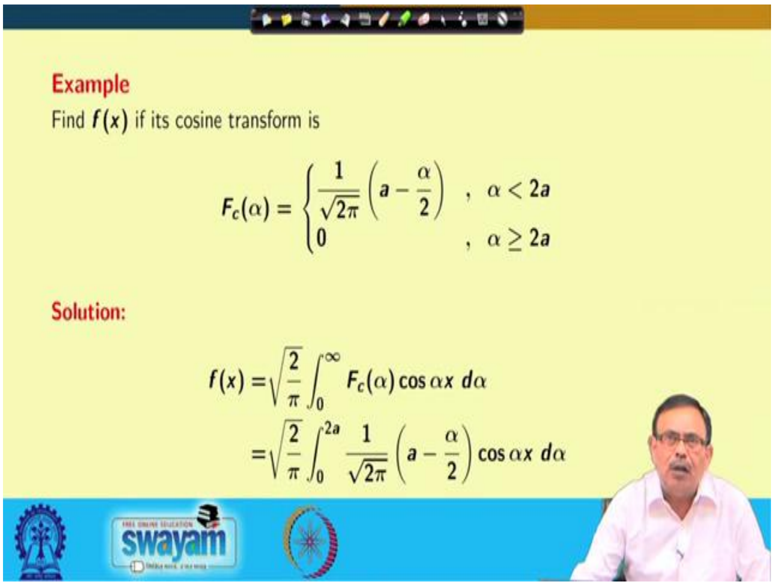

Now, let us see the next example. We want to find out the function $f(x)$ when the Fourier cosine transform of the function is given as

$$F_c(\alpha) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{\alpha}{2}\right) & , \text{ if } \alpha < 2a \\ 0 & , \text{ if } \alpha \geq 2a \end{cases}$$

So, using inverse Fourier cosine transform formula, we have to find out the value of $f(x)$.

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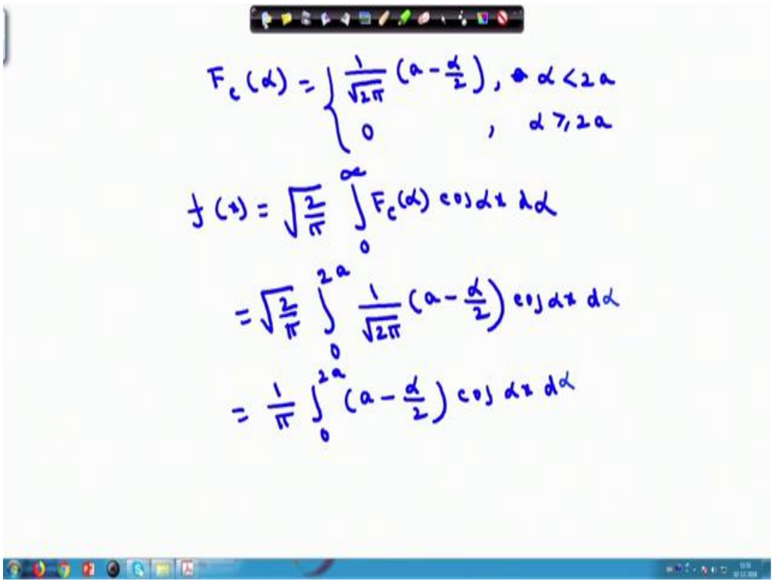

So, here, we know the Fourier cosine transform of $f(x)$ and we have to find out the function $f(x)$ itself. So, from the definition of inverse Fourier cosine transform, we have,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha$$

Since $F_c(\alpha) = 0$ in the interval $2a$ to ∞, so we can write down,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{2a} \frac{1}{\sqrt{2\pi}} \left(a - \frac{\alpha}{2}\right) \cos \alpha x \, d\alpha$$

$$= \frac{1}{\pi} \int_0^{2a} \left(a - \frac{\alpha}{2}\right) \cos \alpha x \, d\alpha$$

So, if we evaluate the integral using integration by parts, then we have,

$$f(x) = \frac{1}{\pi}\left[\left[\left(a - \frac{\alpha}{2}\right) \frac{\sin \alpha x}{x}\right]_{\alpha=0}^{2a} + \frac{1}{2x}\int_0^{2a} \sin \alpha x \, d\alpha\right]$$

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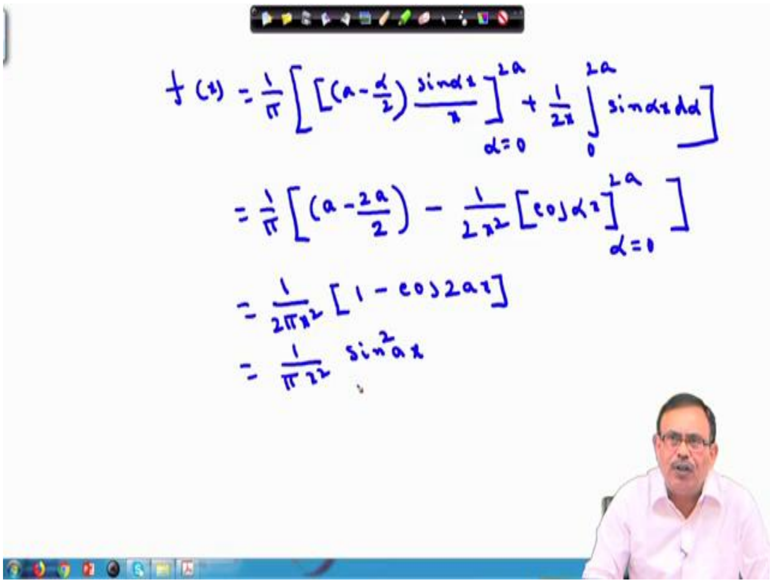

So, if we put the limits in the first part, then it will be 0 and if we integrate the second part, then we obtain,

$$f(x) = \frac{1}{\pi}\left(-\frac{1}{2x^2}[\cos\alpha x\,]_{\alpha=0}^{2a}\right)$$
$$= \frac{1}{2\pi x^2}[1 - \cos 2ax]$$
$$= \frac{\sin^2 ax}{\pi x^2}$$

So, once we know the Fourier transform or Fourier cosine transform or Fourier sine transform of a function, then using the inverse transform formula, we can find out the actual function whose Fourier transform it was.

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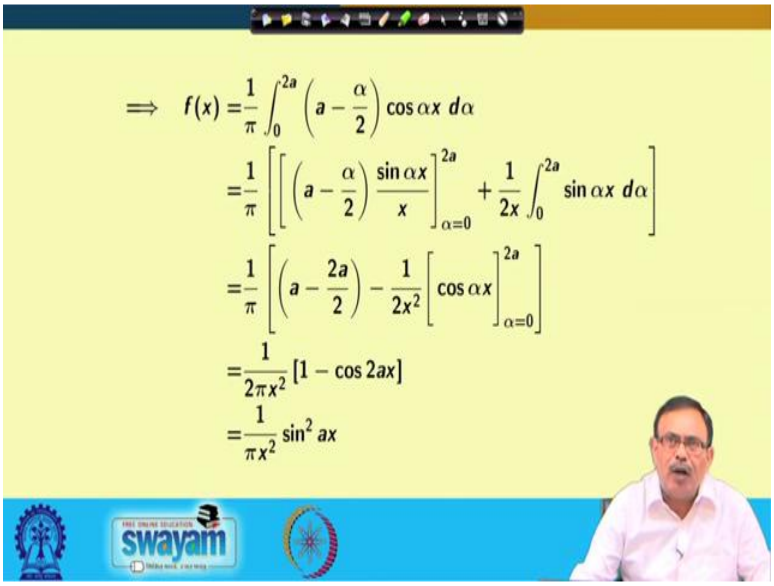

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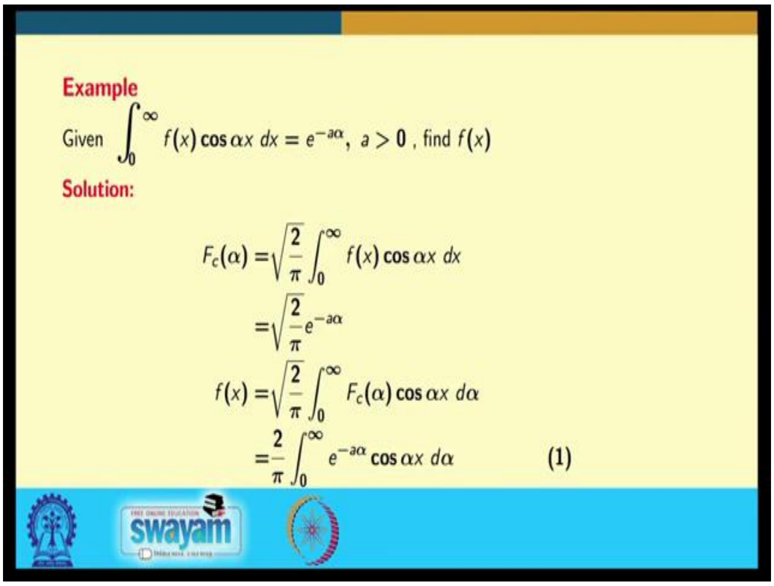

Now, let us see another example, where the value of an integral containing an unknown function is given and we have to find that function. Here we have not talked about any transform, but only we have been told that the value of the integral is known. But if we notice very minutely, what is this integral? The given integral is nothing, but the Fourier cosine transform of the function $f(x)$ with $\sqrt{\frac{2}{\pi}}$ missing.

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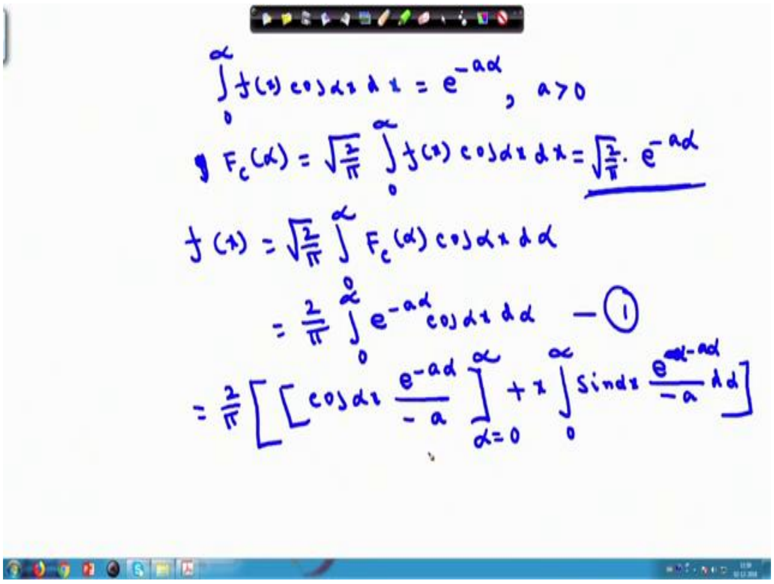

Therefore,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} e^{-a\alpha}$$

So, once we know the Fourier cosine transform of $f(x)$, using inverse Fourier cosine transform formula, we have,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha$$

So, if we substitute $F_c(\alpha)$ here, we get,

$$f(x) = \frac{2}{\pi} \int_0^\infty e^{-a\alpha} \cos \alpha x \, d\alpha \tag{1}$$

Using integration by parts, we obtain,

$$f(x) = \frac{2}{\pi} \left[\left[\cos \alpha x \, \frac{e^{-a\alpha}}{-a} \right]_{\alpha=0}^{\infty} + x \int_0^\infty \sin \alpha x \, \frac{e^{-a\alpha}}{-a} \, d\alpha \right]$$

$$= \frac{2}{\pi a} - \frac{2x}{\pi a} \int_0^\infty e^{-a\alpha} \sin \alpha x \, d\alpha$$

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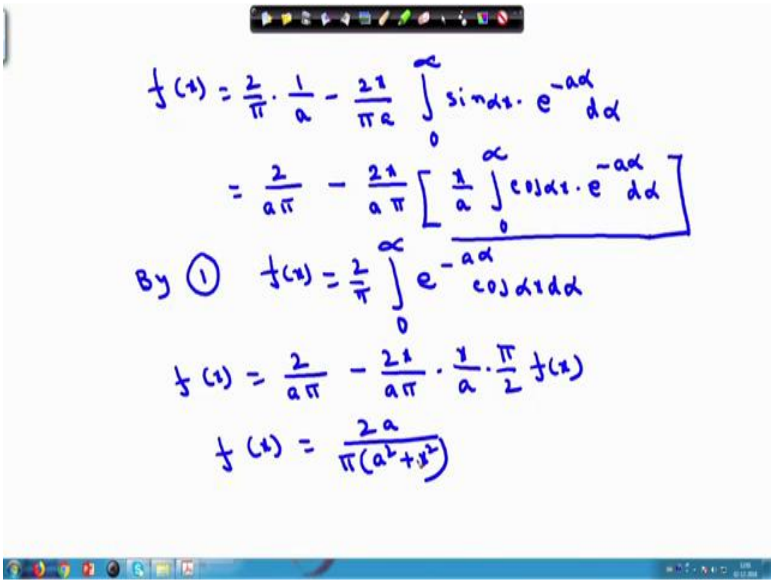

Again using integration by parts, we have,

$$f(x) = \frac{2}{\pi a} - \frac{2x}{\pi a}\left[\left[\sin \alpha x \, \frac{e^{-a\alpha}}{-a}\right]_{\alpha=0}^{\infty} + \frac{x}{a}\int_0^{\infty} e^{-a\alpha}\cos \alpha x \, d\alpha\right]$$

If we put the limits and simplify it, then we have,

$$f(x) = \frac{2}{\pi a} - \frac{2x}{\pi a} \cdot \frac{x}{a} \cdot \frac{\pi}{2} f(x) \quad [\text{from } (1)]$$

$$\therefore f(x) = \frac{2a}{\pi(a^2 + x^2)}$$

So, please note that without evaluating the given integral directly, by using inverse Fourier cosine transform, we obtained the function $f(x)$.

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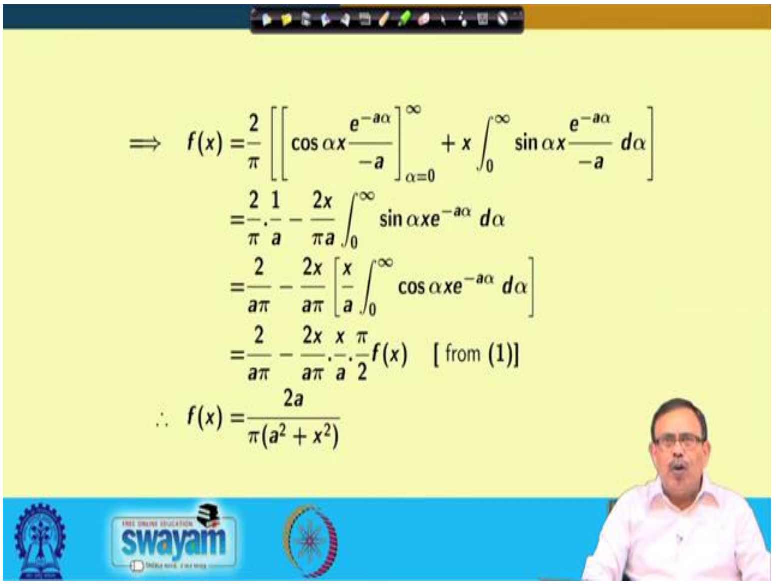

So, we have discussed here, how to find out the Fourier transform of a function or if the Fourier transform, Fourier cosine transform or Fourier sine transform of a function is given to us, how to find out the function itself using the inverse transform formula.

Thank you.