Transform Calculus and its Application in Differential Equations
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Lecture - 33
Applications of Properties of Fourier Transform – 1

In the last lectures, we have been going through the properties of Fourier transform and their applications. So, let us go through some more properties.

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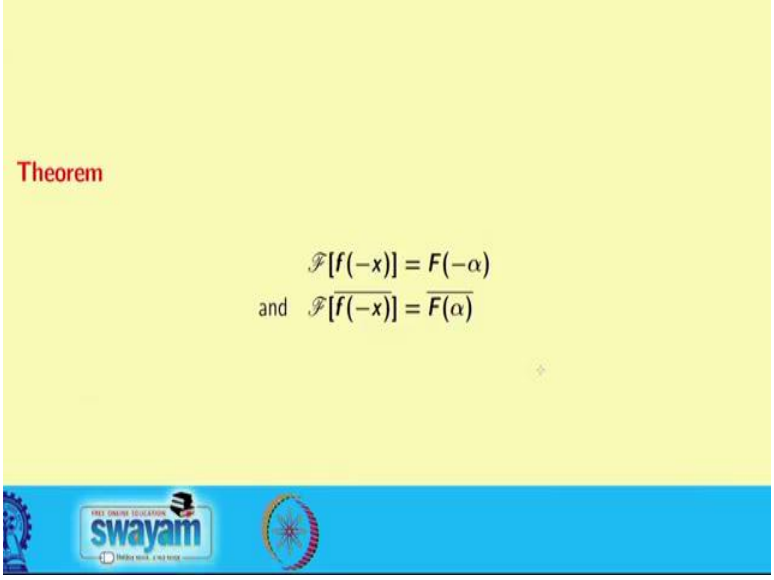

The next one is Fourier transform of $f(-x)$ equals $F(-\alpha)$ and Fourier transform of $\overline{f(-x)}$ is equal to $\overline{F(\alpha)}$.

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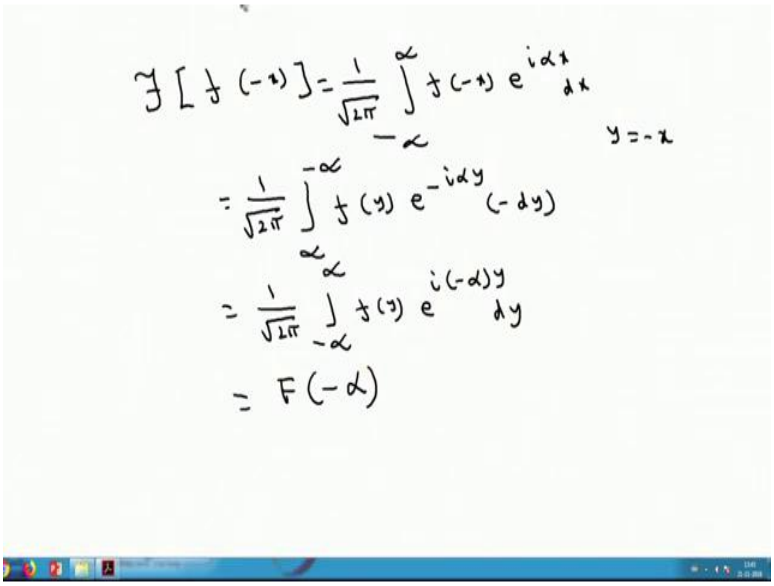

Let us see the proof of the first one first, then we will go to the second one. From definition we have,

$$\mathcal{F}[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x) e^{i\alpha x}\, dx$$

Now, we cannot keep $f(-x)$. So, we are assuming, say $y = -x$ in the above equation so that $dx = -dy$. Therefore as x approaches ∞, then y will approach to $-\infty$ and as x approaches $-\infty$, then y will approach to ∞. So, the limit will be from ∞ to $-\infty$

$$\therefore \mathcal{F}[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(y) e^{-i\alpha y}\, (-dy)$$

Now if we make the limit from $-\infty$ to ∞, then we can write down as

$$\therefore \mathcal{F}[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{i(-\alpha)y}\, dy$$
$$= F(-\alpha)$$

This completes the proof.

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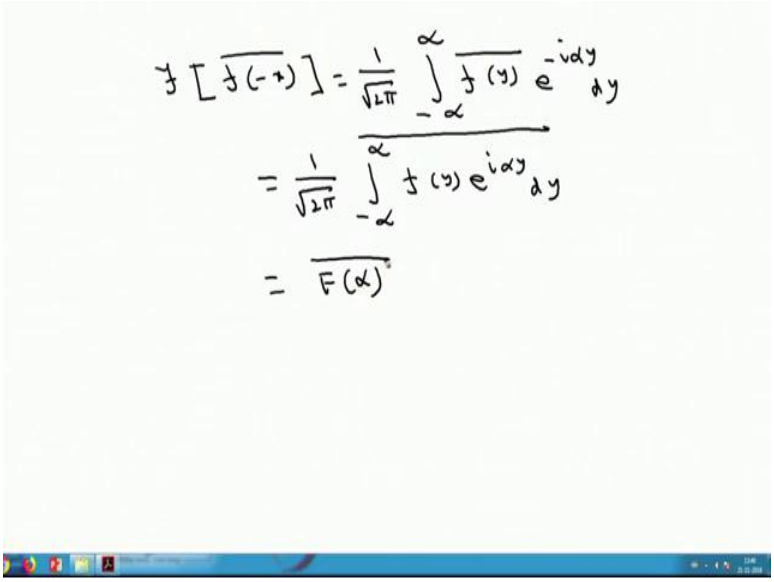

Similarly, for the second one,

$$\therefore \mathcal{F}\left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)} e^{i\alpha x}\, dx$$

Now if we follow the same steps to get $f(y)$ from $f(-x)$, we will obtain,

$$\therefore \mathcal{F}\left[\overline{f(-x)}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(y)} e^{-i\alpha y}\, dy$$

$$= \left[\overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{\iota\alpha y}\, dy}\right]$$

$$= \overline{F(\alpha)}$$

So, this completes the proof of the theorem.

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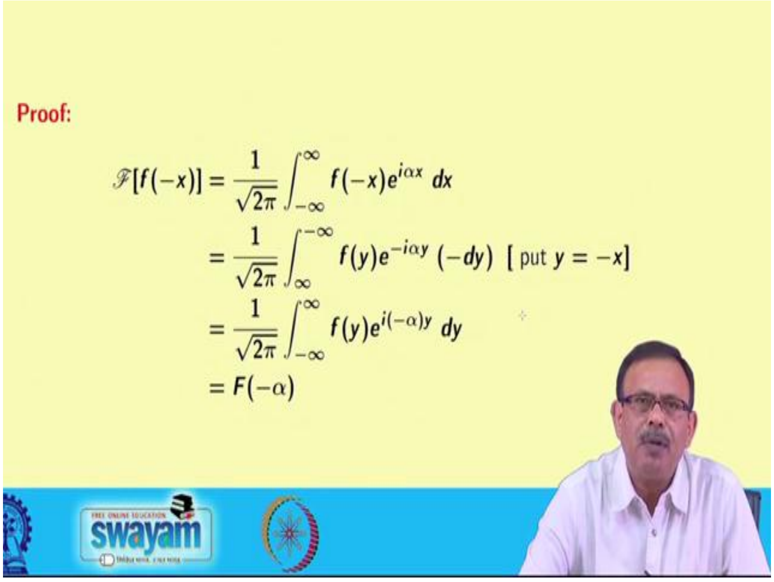

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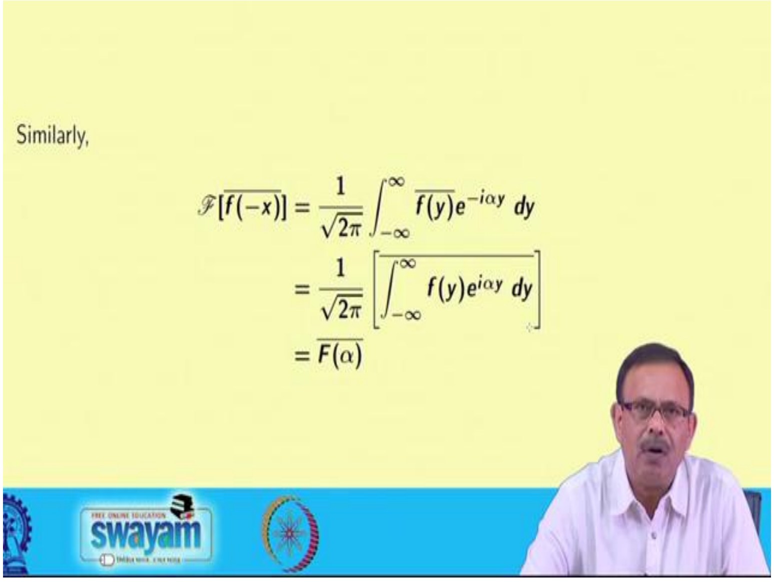

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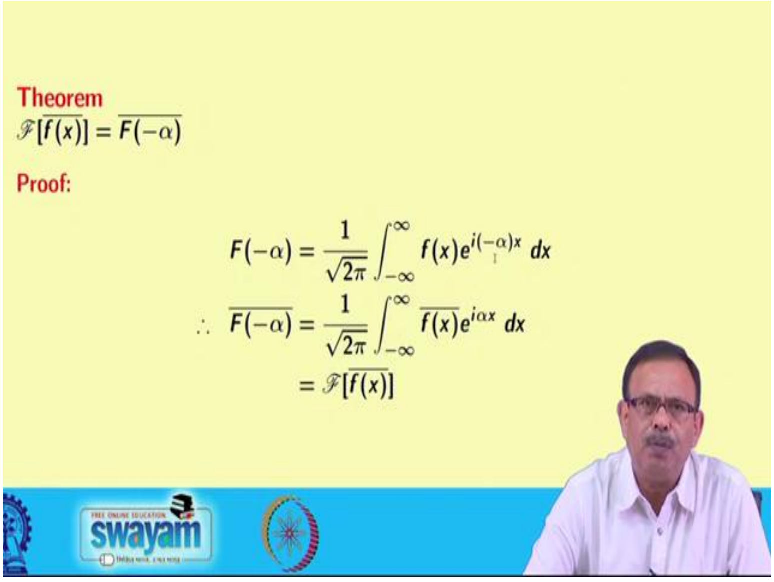

Now, the next theorem is Fourier transform of $\overline{f(x)}$ equals $\overline{F(-\alpha)}$. If we see the earlier theorem that is $\mathcal{F}[f(-x)] = F(-\alpha)$ and $\mathcal{F}\left[\overline{f(-x)}\right] = \overline{F(\alpha)}$ whereas, this one is $\mathcal{F}\left[\overline{f(x)}\right] = \overline{F(-\alpha)}$.

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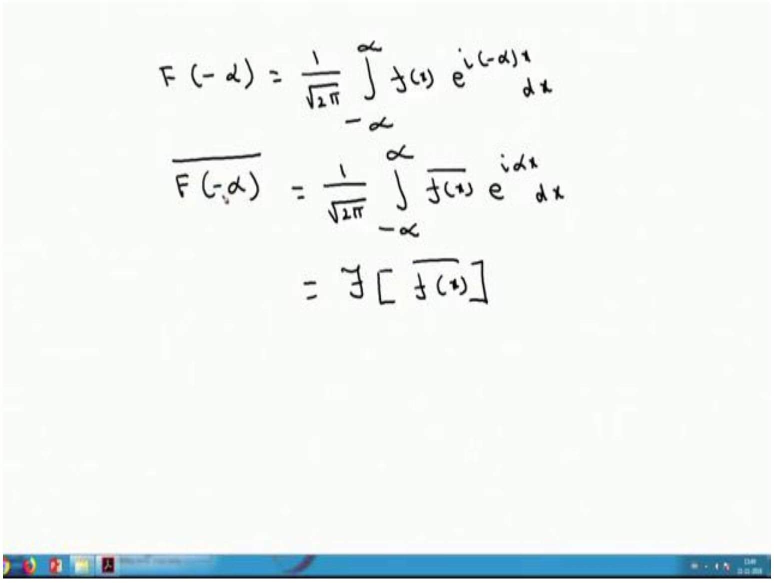

To prove this one again, we are starting with the definition, that is,

$$F(-\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(-\alpha)x} dx$$

If we take conjugate on both sides, then we have,

$$\overline{F(-\alpha)} = \overline{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{\iota(-\alpha)x}dx\right]} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)}e^{\iota\alpha x}dx = \mathcal{F}\left[\overline{f(x)}\right]$$

So this completes the proof.

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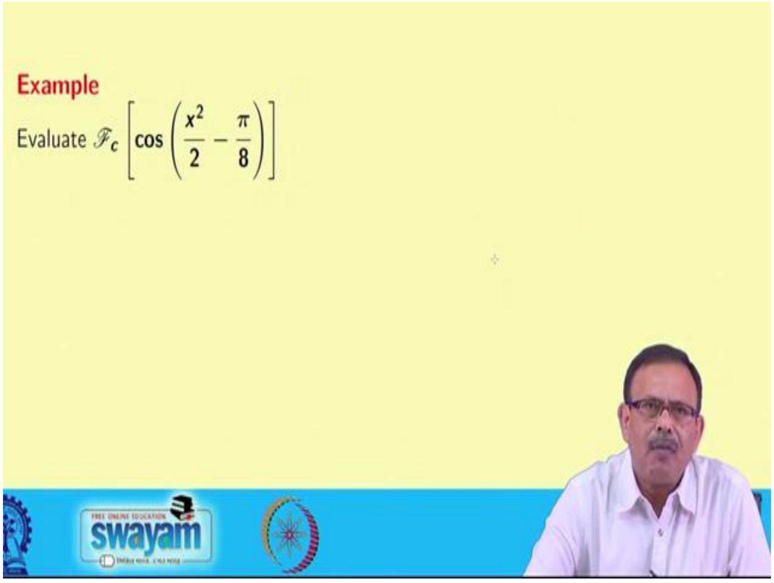

Let us see some examples now. Suppose, we want to evaluate the Fourier cosine transform of $\cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right)$. Instead of solving this problem directly, let us do it in some other way, where we will use various properties and we will try to find out the Fourier cosine transform of this function.

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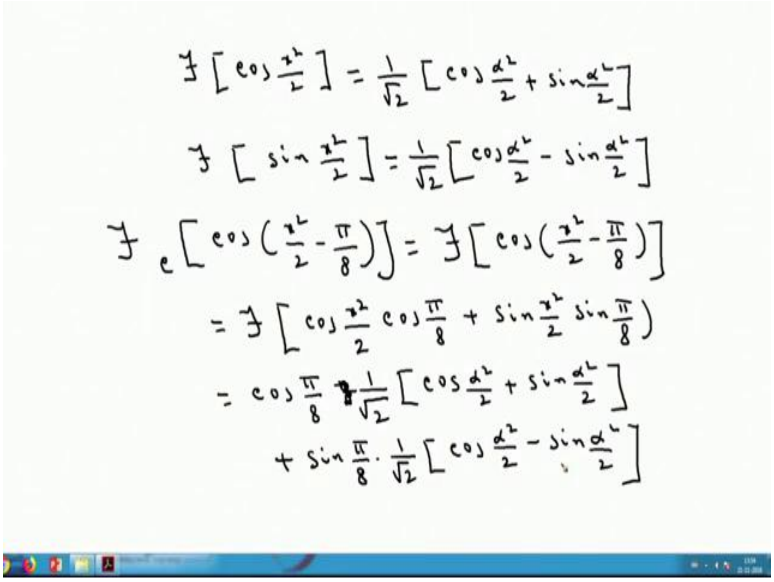

As already derived, we know that

$$\mathcal{F}\left[\cos\frac{x^2}{2}\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} + \sin\frac{\alpha^2}{2}\right]$$

$$\mathcal{F}\left[\sin\frac{x^2}{2}\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} - \sin\frac{\alpha^2}{2}\right]$$

Again if we recall, we have proved that for an even function, the Fourier transform and the Fourier cosine transform are the same. Since $\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)$ is an even function, so we can say that,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right] = \mathcal{F}\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right]$$

Now using the formula for $\cos(A+B)$ we get,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right] = \cos\frac{\pi}{8}\cdot\mathcal{F}\left[\cos\frac{x^2}{2}\right] + \sin\frac{\pi}{8}\cdot\mathcal{F}\left[\sin\frac{x^2}{2}\right]$$

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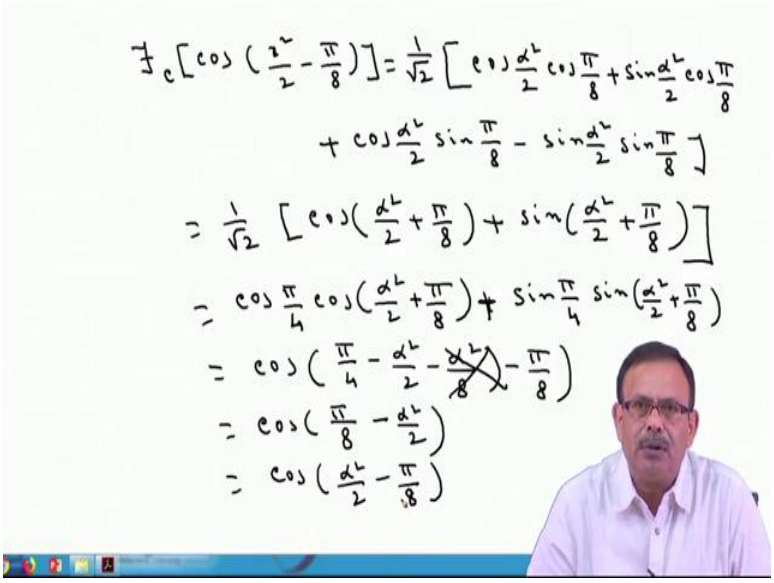

If we substitute the values of $\mathcal{F}\left[\cos\frac{x^2}{2}\right]$ and $\mathcal{F}\left[\sin\frac{x^2}{2}\right]$, then we get,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right]=\cos\frac{\pi}{8}\cdot\frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2}+\sin\frac{\alpha^2}{2}\right]+\sin\frac{\pi}{8}\cdot\frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2}-\sin\frac{\alpha^2}{2}\right]$$

$$=\frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2}\cos\frac{\pi}{8}+\sin\frac{\alpha^2}{2}\cos\frac{\pi}{8}+\cos\frac{\alpha^2}{2}\sin\frac{\pi}{8}-\sin\frac{\alpha^2}{2}\sin\frac{\pi}{8}\right]$$

$$=\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)+\sin\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)\right]$$

$$=\cos\frac{\pi}{4}\cos\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)+\sin\frac{\pi}{4}\sin\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)$$

$$=\cos\left(\frac{\alpha^2}{2}+\frac{\pi}{8}-\frac{\pi}{4}\right)$$

$$=\cos\left(\frac{\alpha^2}{2}-\frac{\pi}{8}\right)$$

Therefore, the function is self-reciprocal with respect to Fourier cosine transform, because, Fourier cosine transform of the function is the function itself.

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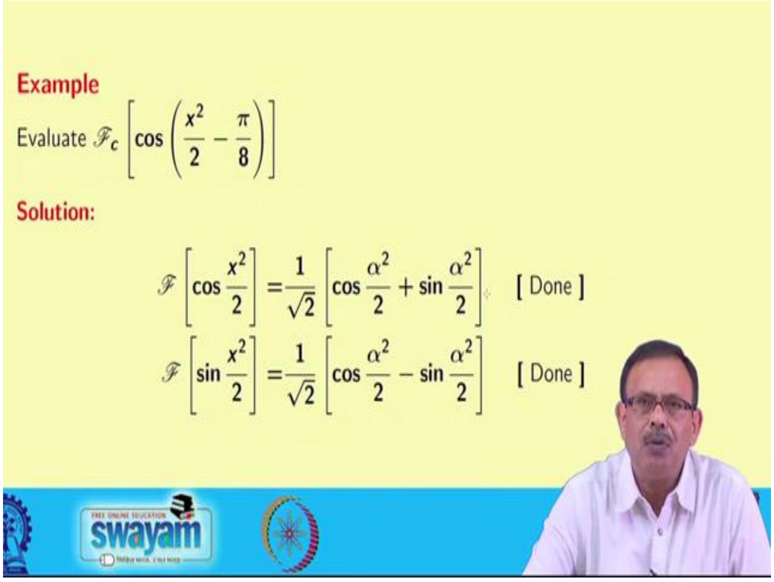

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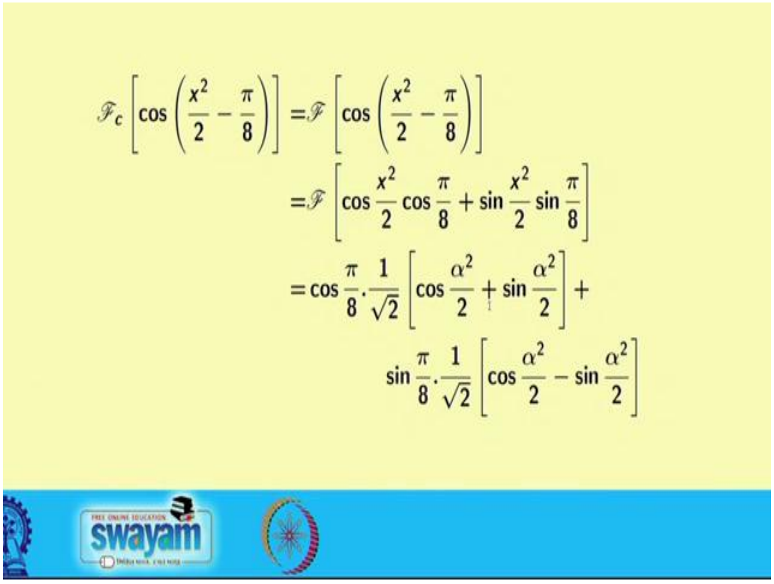

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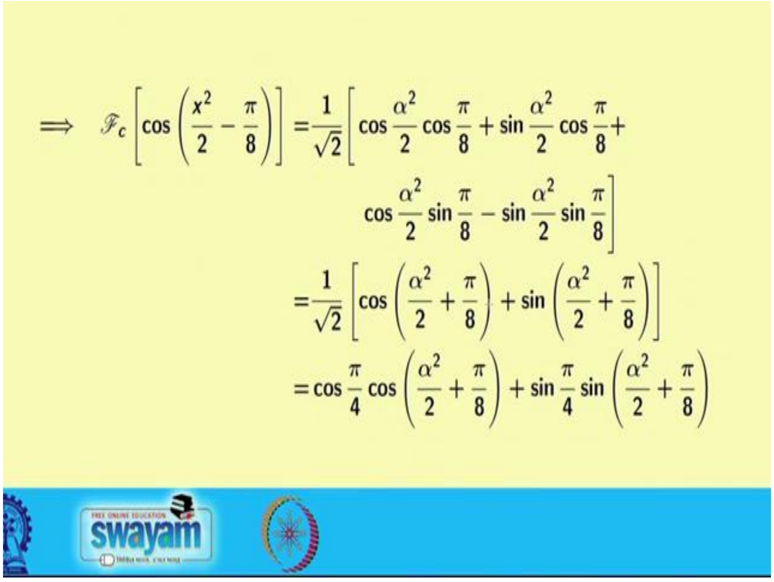

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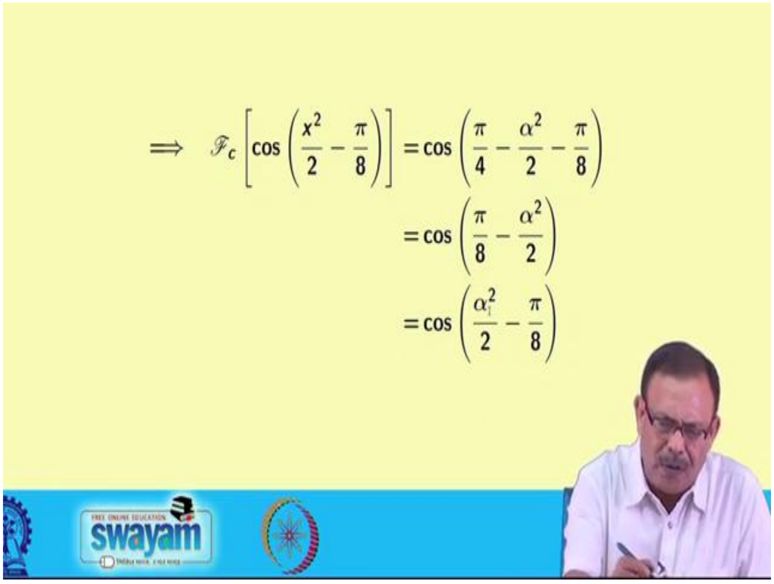

Now, let us solve this same problem using some other approach so that we will see that one problem can be solved in various ways and we will visualize the advantages of using the properties of the transform.

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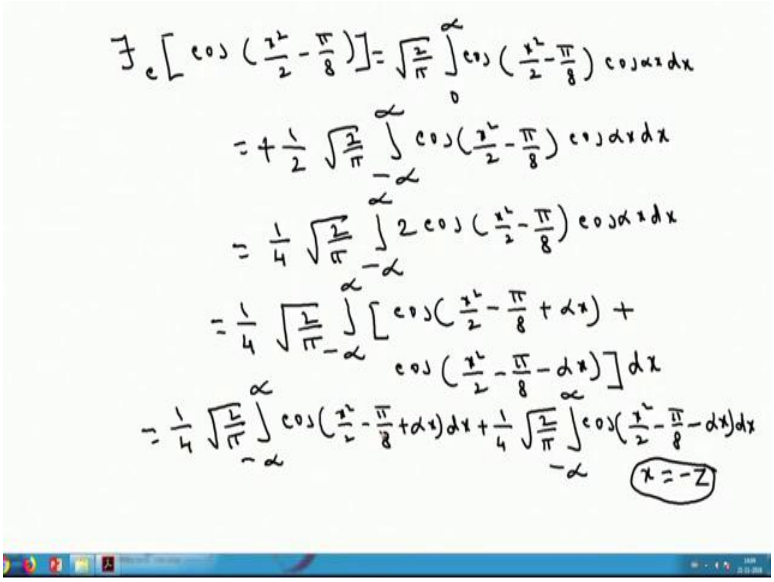

So, from the definition of Fourier cosine transform, we can write,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\cos\alpha x\, dx$$

Since $\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\cos\alpha x$ is an even function, so we can change the limit and write down as

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right]=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\cos\alpha x\, dx$$

This is possible only because the function is an even function. If we use the formula for $2\cos A\cos B$, we can rewrite the above equation as,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right]=\frac{1}{4}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty \left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}+\alpha x\right)+\cos\left(\frac{x^2}{2}-\frac{\pi}{8}-\alpha x\right)\right]dx$$

Now, we can break it into two separate integrals, that is

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right]=\frac{1}{4}\sqrt{\frac{2}{\pi}}\left[\int_{-\infty}^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}+\alpha x\right)dx+\int_{-\infty}^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}-\alpha x\right)dx\right]$$

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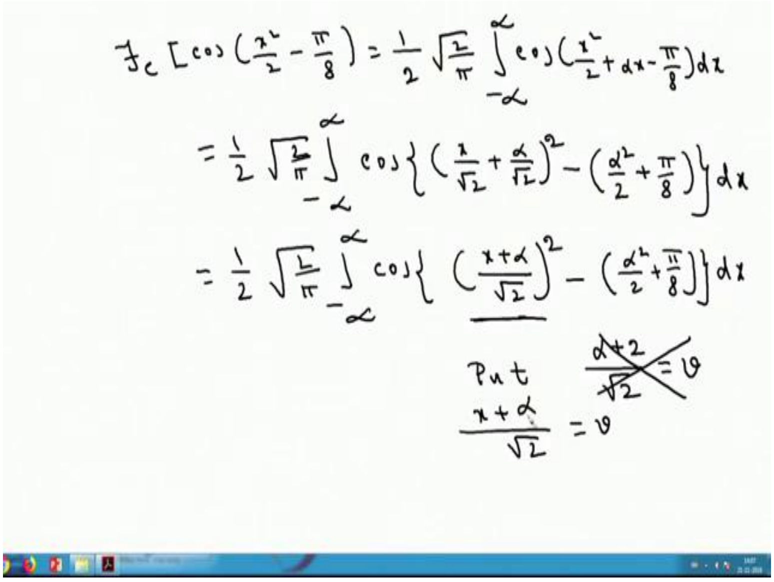

In the second integral, if we put $x = -z$ then it will be the same as the first integral. Therefore,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} \cos\left(\frac{x^2}{2} - \frac{\pi}{8} + \alpha x\right) dx$$

Now, we have to adjust it in such a way that, after adjustment it will contain a whole square and a constant term i.e.,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right)\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} \cos\left[\left(\frac{x + \alpha}{\sqrt{2}}\right)^2 - \left(\frac{\alpha^2}{2} + \frac{\pi}{8}\right)\right] dx$$

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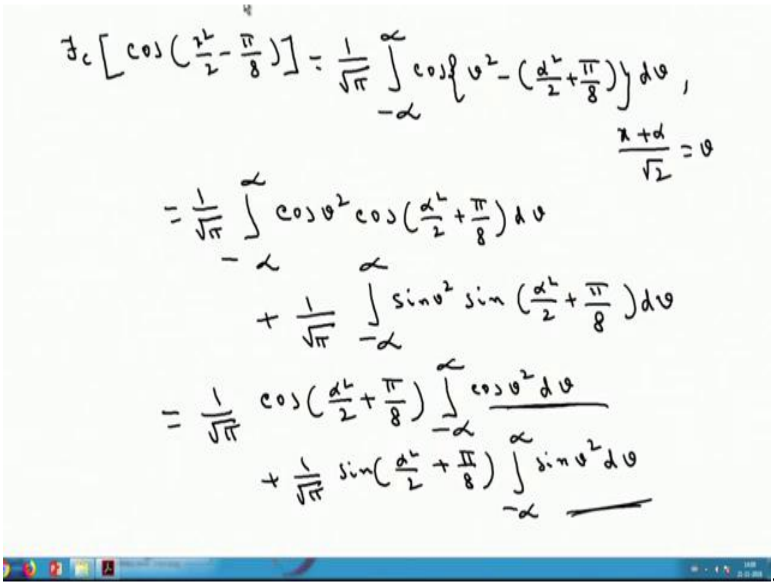

If we substitute $\frac{x+\alpha}{\sqrt{2}} = v$ so that $dx = \sqrt{2}dv$, then above equation reduces to

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right] = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\cos\left[v^2-\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)\right]dv$$

If we expand it, we get,

$$\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right] = \frac{1}{\sqrt{\pi}}\cos\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)\int_{-\infty}^{\infty}\cos v^2\,dv + \frac{1}{\sqrt{\pi}}\sin\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)\int_{-\infty}^{\infty}\sin v^2\,dv$$

Now, we have to find out the values of $\int_{-\infty}^{\infty}\cos v^2\,dv$ and $\int_{-\infty}^{\infty}\sin v^2\,dv$. So, let us see, how to find out the values of these integrals.

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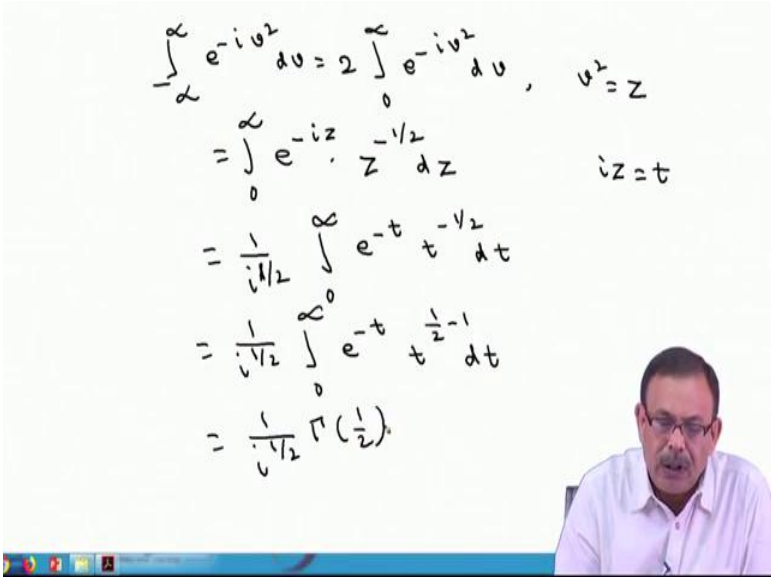

To find the above integrals, we start with

$$\int_{-\infty}^{\infty} e^{-iv^2} dv \ = 2\int_0^{\infty} e^{-iv^2} dv \quad \left(\because e^{-iv^2} \text{ is an even function of } v\right)$$

If we substitute $v^2 = z$, then we have,

$$\int_{-\infty}^{\infty} e^{-iv^2} dv \ = \int_0^{\infty} e^{-iz} z^{-1/2} \, dz$$

Again if we substitute $iz = t$, then we get,

$$\int_{-\infty}^{\infty} e^{-iv^2} dv \ = \frac{1}{i^{1/2}} \int_0^{\infty} e^{-t} t^{-1/2} \, dt$$

The integral on the right side is in the form of Gamma function. Therefore,

$$\int_{-\infty}^{\infty} e^{-iv^2} dv \ = \frac{1}{i^{1/2}} \Gamma\left(\frac{1}{2}\right)$$

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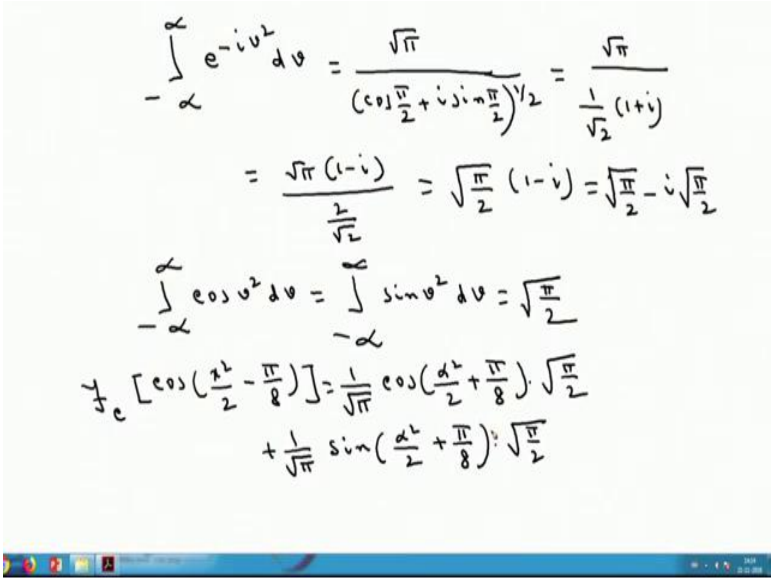

Since $i^{1/2} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i) = \frac{\sqrt{2}}{1-i}$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, then after simplification, the above equation reduces to,

$$\int_{-\infty}^{\infty} e^{-iv^2} dv = \sqrt{\frac{\pi}{2}}\,(1-i) = \sqrt{\frac{\pi}{2}} - i\sqrt{\frac{\pi}{2}}$$

So, if we write e^{-iv^2} in terms of cosine and sine, then we have,

$$\int_{-\infty}^{\infty} [\cos v^2 - i\sin v^2] dv = \sqrt{\frac{\pi}{2}} - i\sqrt{\frac{\pi}{2}}$$

Now comparing real part and imaginary part on both sides, we get,

$$\int_{-\infty}^{\infty} \cos v^2\, dv = \int_{-\infty}^{\infty} \sin v^2\, dv = \sqrt{\frac{\pi}{2}}$$

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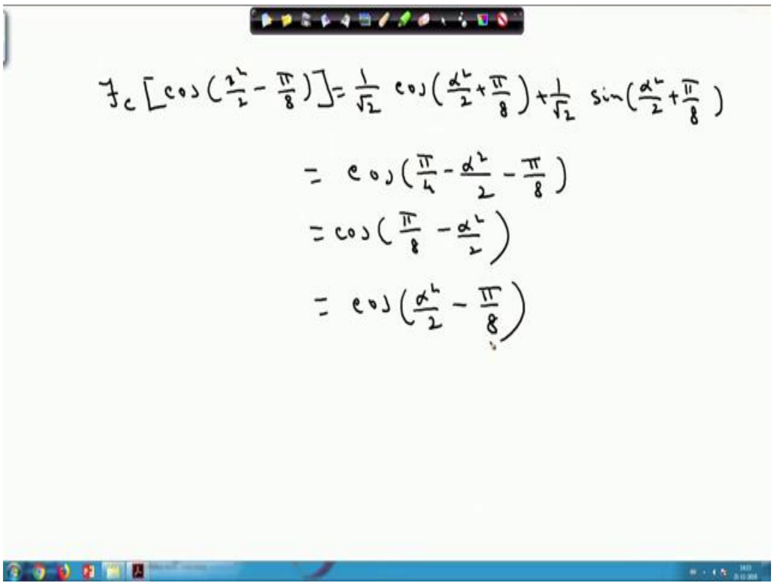

Now, we substitute the values of $\int_{-\infty}^{\infty}\cos v^2\,dv$ and $\int_{-\infty}^{\infty}\sin v^2\,dv$, so we get,

$$
\begin{aligned}
\mathcal{F}_c\left[\cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\right] &= \frac{1}{\sqrt{2}}\cos\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)+\frac{1}{\sqrt{2}}\sin\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)\\
&= \cos\frac{\pi}{4}\cos\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)+\sin\frac{\pi}{4}\sin\left(\frac{\alpha^2}{2}+\frac{\pi}{8}\right)\\
&= \cos\left(\frac{\pi}{4}-\frac{\alpha^2}{2}-\frac{\pi}{8}\right)\\
&= \cos\left(\frac{\pi}{8}-\frac{\alpha^2}{2}\right)\\
&= \cos\left(\frac{\alpha^2}{2}-\frac{\pi}{8}\right)
\end{aligned}
$$

Hence we obtain the solution which is same as the previous one. We did the problem purposefully, because we wanted to show one thing that, if we solve it using integration how much time it takes, how much effort we have to give and how many difficult integrations we have to evaluate. Whereas, if we use the properties and if we use the Fourier transform of known functions, then very easily we can find out the Fourier transform or Fourier cosine transform of complicated functions also.

Thank you.