Transform Calculus and its Applications in Differential Equations
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Lecture - 32
Fourier Transform of Derivative and Integral of a Function

In the previous two lectures, we have discussed the properties of Fourier transform, Fourier sine transform and Fourier cosine transform. And also we have discussed the applications of these properties that is if we know the Fourier transform of a function, then using the properties of Fourier transform, how we can find out the Fourier transform of some other complicated functions. So, let us discuss some more properties and their applications.

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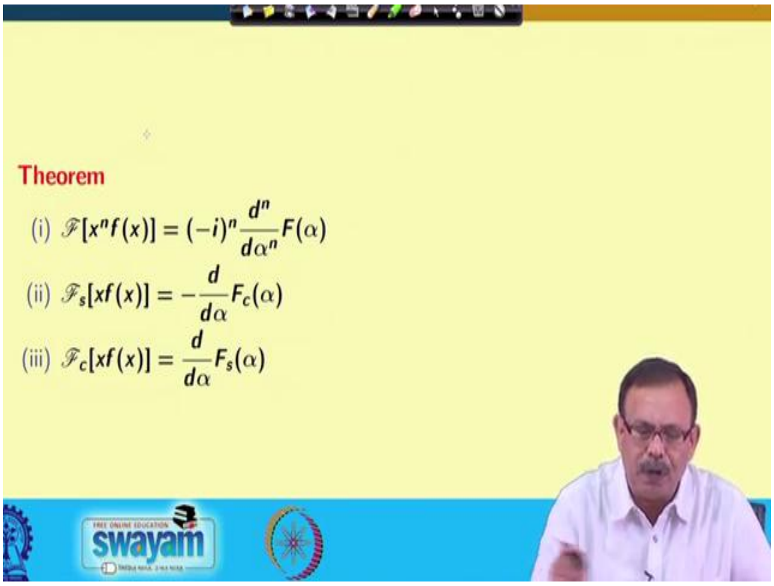

Firstly, we discuss about the effects on Fourier transform when a function is multiplied by x^n. Next we check the effects on Fourier sine transform and Fourier cosine transform also, if the function is multiplied by x only.

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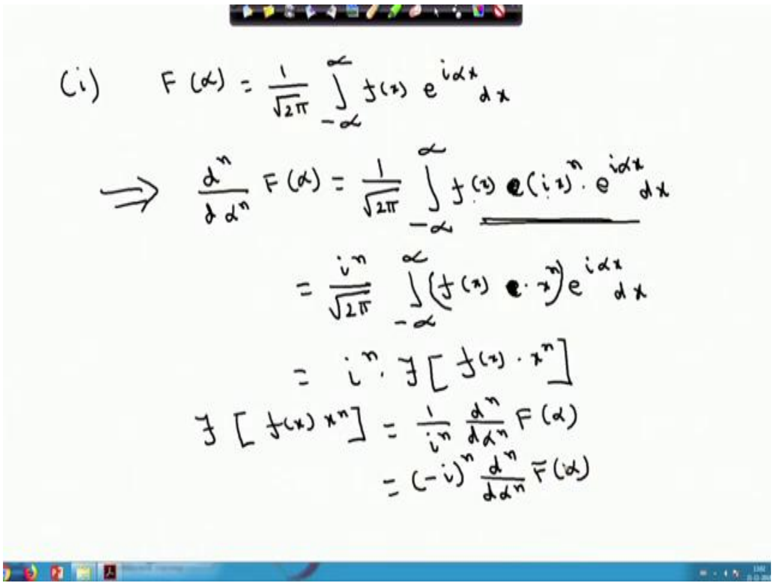

Let us see the first one. From the definition, we know,

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

If we differentiate both sides of the above equation n times with respect to α (i.e., using differentiation under integration), then we get,

$$\begin{aligned} \frac{d^n}{d\alpha^n} F(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot (ix)^n \, e^{i\alpha x} dx \\ &= \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x) \, e^{i\alpha x} dx \\ &= i^n \mathcal{F}[x^n f(x)] \end{aligned}$$

$$\therefore \mathcal{F}[x^n f(x)] = \frac{1}{i^n} \frac{d^n}{d\alpha^n} F(\alpha) = (-i)^n \, \frac{d^n}{d\alpha^n} F(\alpha)$$

This completes the proof of the first property.

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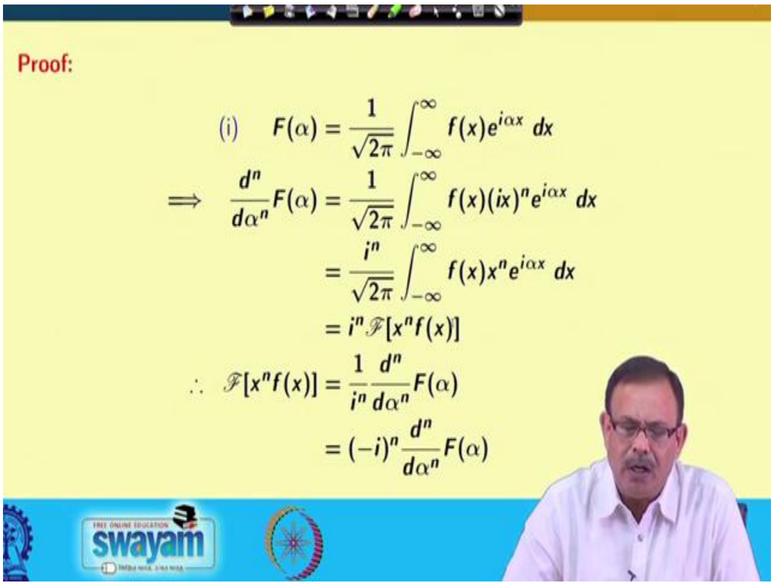

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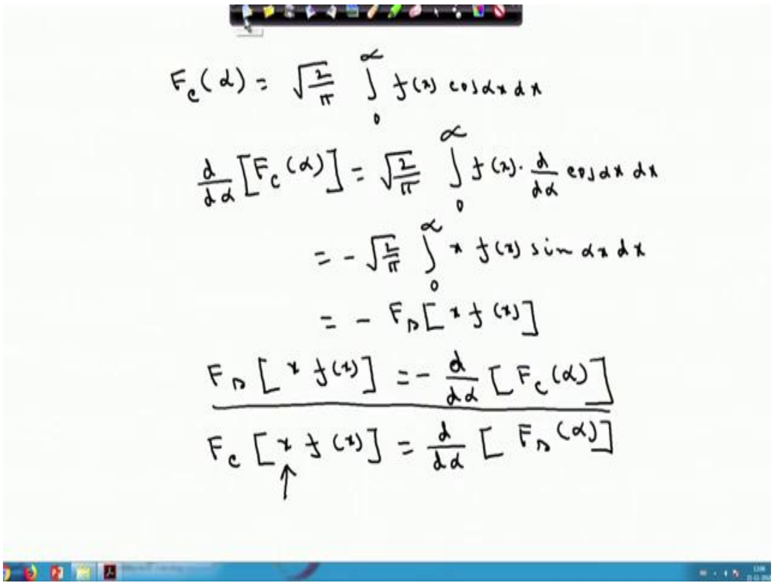

Let us now see the effects on Fourier sine transform. To check this, we will start with the definition of Fourier cosine transform of $f(x)$.

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\alpha x\, dx$$

If we differentiate both sides of the above equation with respect to α (i.e., differentiation under integration), then we get,

$$\frac{d}{d\alpha}F_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot(-x)\sin\alpha x\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\sin\alpha x\,dx$$

And this expression on the right side is nothing but the Fourier sine transform of $xf(x)$. Therefore, we can write down

$$\therefore \mathcal{F}_s[xf(x)] = -\frac{d}{d\alpha}F_c(\alpha)$$

So, this completes the proof. In the same fashion, we can prove the third property also.

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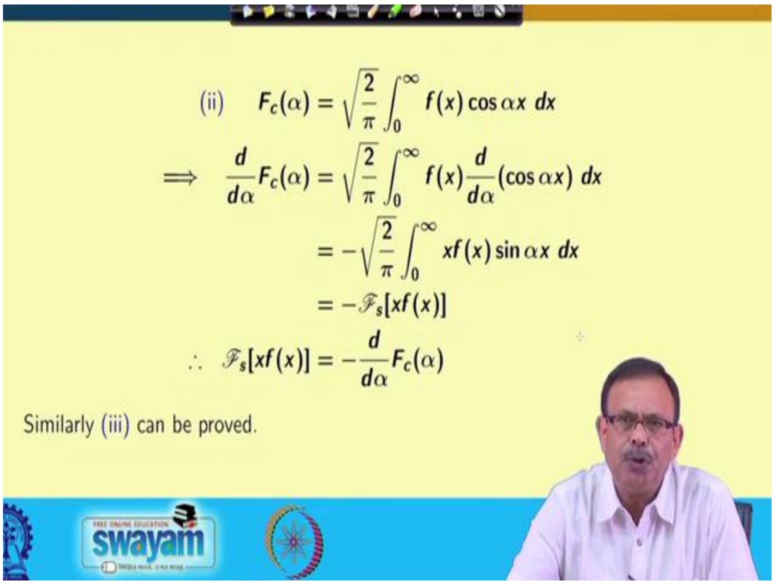

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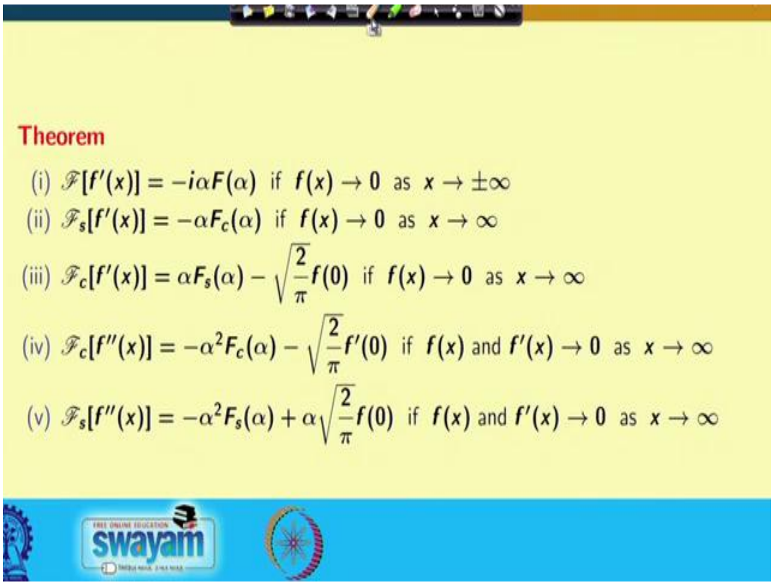

Now we will study how to find the Fourier transform of derivatives of a function. We have studied similar properties whenever we were dealing with the Laplace transform. In these cases, we add an extra condition only i.e., $f(x)$ approaches 0 as $x \to \infty$. So, whenever we are taking the second derivative, then not only $f(x)$ but also its first derivative $f'(x)$, both should approach 0 whenever $x \to \infty$. So, let us see the proofs of the theorems one after another.

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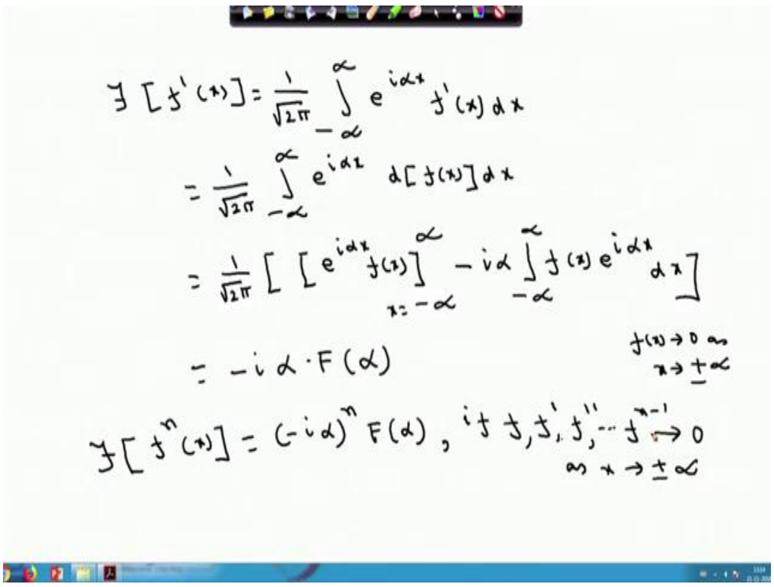

Firstly, we have to prove that

$$\mathcal{F}[f'(x)] = -i\alpha\, F(\alpha) \;\text{ if }\; f(x) \to 0 \;\text{ as } x \to \pm\infty$$

We are starting with the left hand side that is

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} \frac{d}{dx}\, [f(x)]\, dx$$

If we take $e^{i\alpha x}$ as the first function and $\frac{d}{dx}\,[f(x)]$ as the second function and we use integration by parts, then we have,

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \left(\left[e^{i\alpha x} f(x) \right]_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx \right)$$

Since $f(x) \to 0$ as $x \to \pm\infty$, so the first term on the right hand side of the above equation will vanish. Therefore, we have,

$$\mathcal{F}[f'(x)] = -i\alpha \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx \right)$$
$$= -i\alpha\, F(\alpha)$$

If we proceed in the same manner, then we will obtain,

$$\mathcal{F}[f^{n}(x)] = (-i\alpha)^{n}\, F(\alpha) \;\text{ provied }\; f(x), f'(x), \cdots, f^{n-1}(x) \to 0 \;\text{ as } x \to \pm\infty$$

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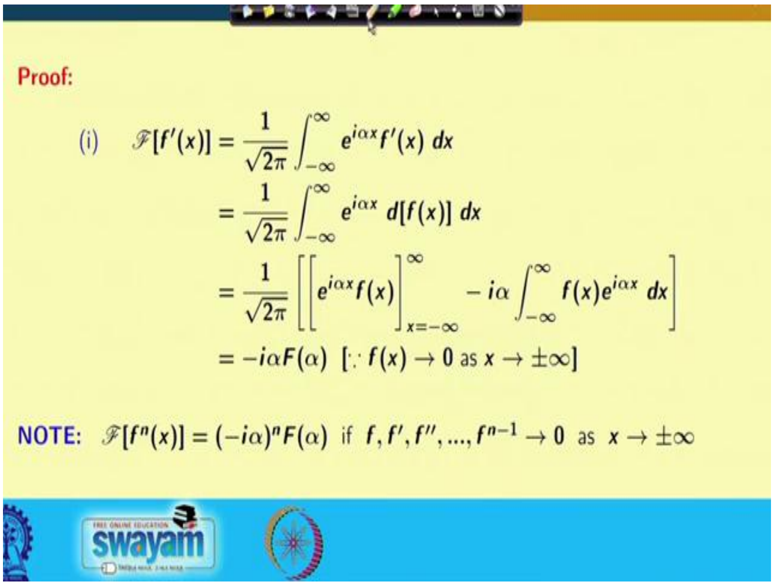

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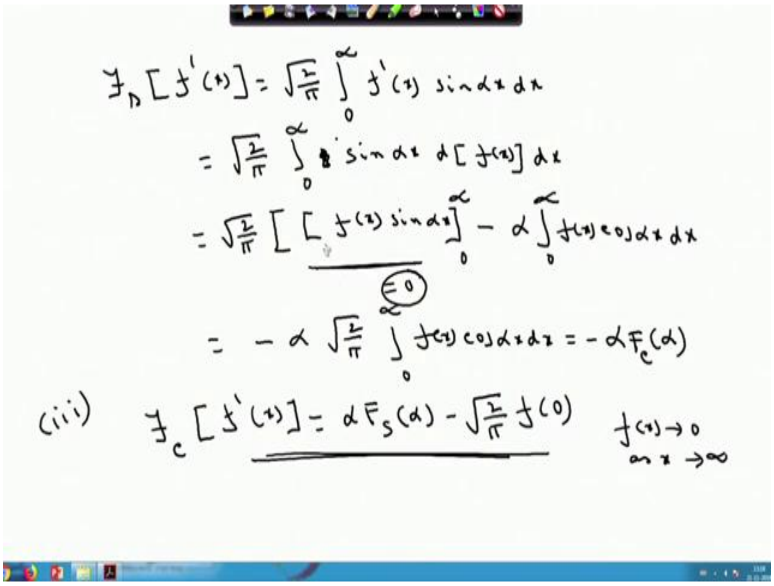

Now, let us see the second one i.e., Fourier sine transform of $f'(x)$ and we proceed in the same manner as the first one. So, we have,

$$\mathcal{F}_s[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\sin\alpha x\, dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin\alpha x \frac{d}{dx}\, [f(x)]\, dx$$

If we take $\sin\alpha x$ as the first function and $\frac{d}{dx}\,[f(x)]$ as the second function and we use integration by parts, then we have,

$$\mathcal{F}_s[f'(x)] = \sqrt{\frac{2}{\pi}}\left([\sin\alpha x\, f(x)]_0^{\infty} - \alpha\int_0^{\infty}\cos\alpha x\, f(x)\, dx\right)$$

Since $f(x) \to 0$ as $x \to \infty$, so the first term on the right hand side of the above equation will vanish and from the second term, we get nothing but Fourier cosine transform of the function $f(x)$. So, we have,

$$\mathcal{F}_s[f'(x)] = -\alpha\, F_c(\alpha)$$

So, this completes the proof. Similarly, the third one can also be proved easily.

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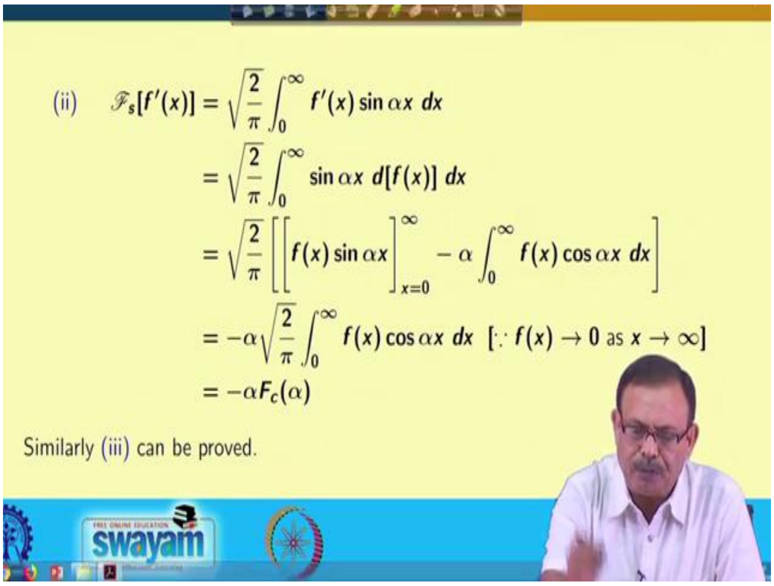

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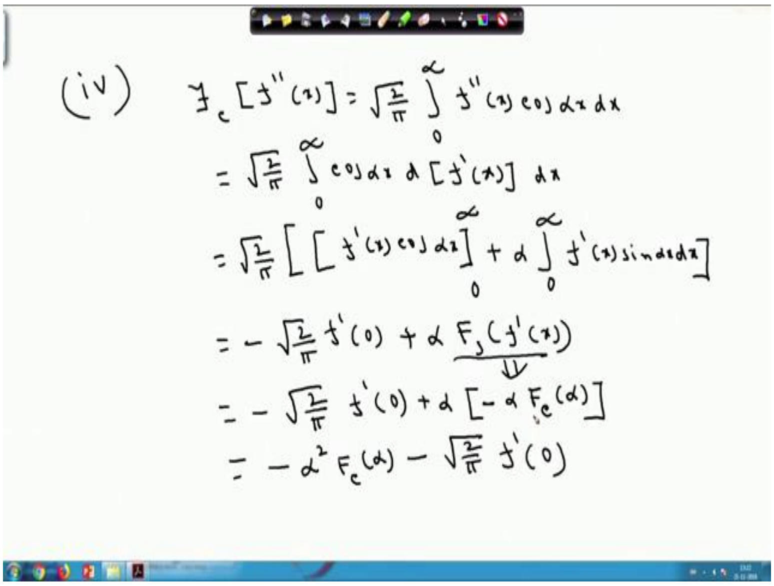

Let us see the proof of the fourth one. From the definition, we have,

$$\mathcal{F}_c[f''(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos\alpha x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos\alpha x\frac{d}{dx}\,[f'(x)]\,dx$$

Using integration by parts, we get,

$$\mathcal{F}_c[f''(x)] = \sqrt{\frac{2}{\pi}}\left([\cos\alpha x\,f'(x)]_0^\infty + \alpha\int_0^\infty \sin\alpha x\,f'(x)\,dx\right)$$

Since $f'(x) \to 0$ as $x \to \infty$, then we have

$$\mathcal{F}_c[f''(x)] = -\sqrt{\frac{2}{\pi}}f'(0) + \alpha\mathcal{F}_s[f'(x)]$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + \alpha[-\alpha F_c(\alpha)]$$

$$= -\alpha^2 F_c(\alpha) - \sqrt{\frac{2}{\pi}}f'(0)$$

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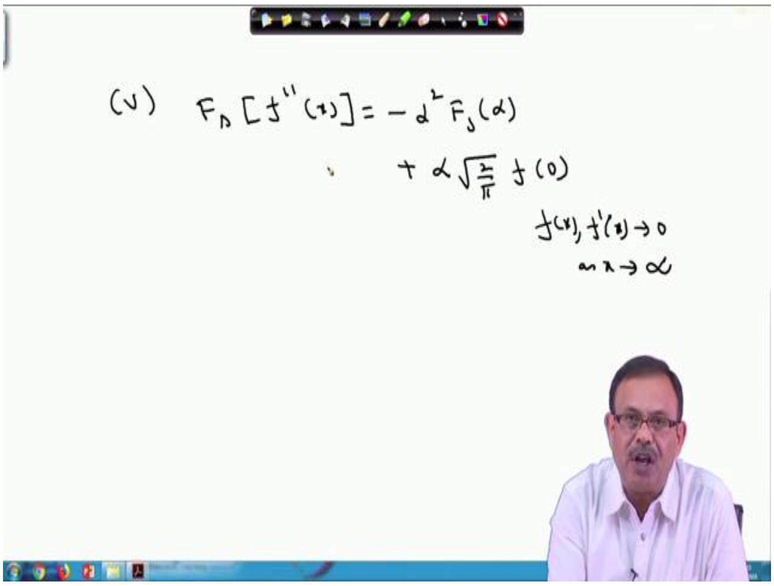

And similarly, we can prove the fifth one also.

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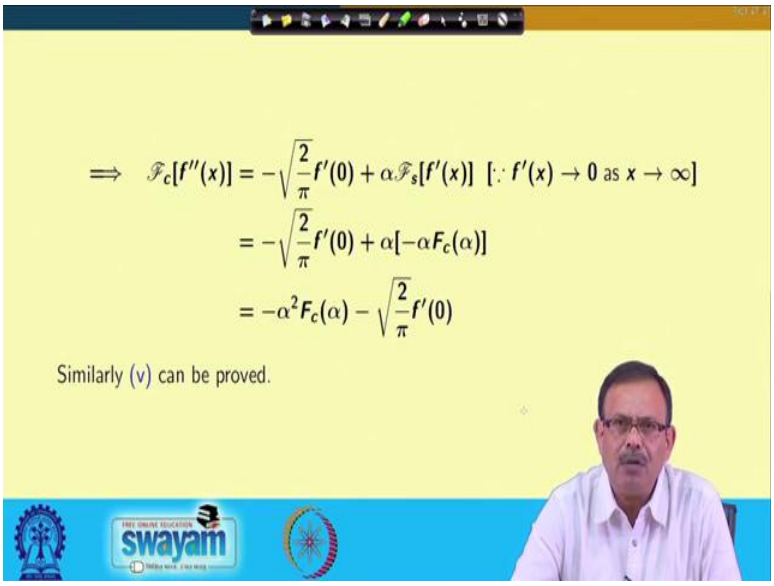

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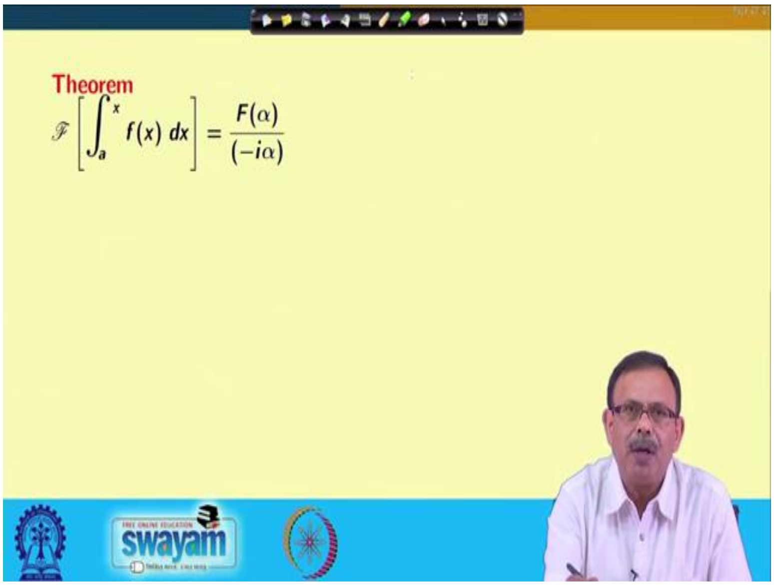

The earlier theorem was on the differentiation of the Fourier transform of a function. If we know the Fourier transform of a function, then we have studied that what should be the Fourier transform of the differentiation of that function. Now, we will discuss that if we know the Fourier transform of a function and if we integrate the function, then what would be the Fourier transform of the new function.

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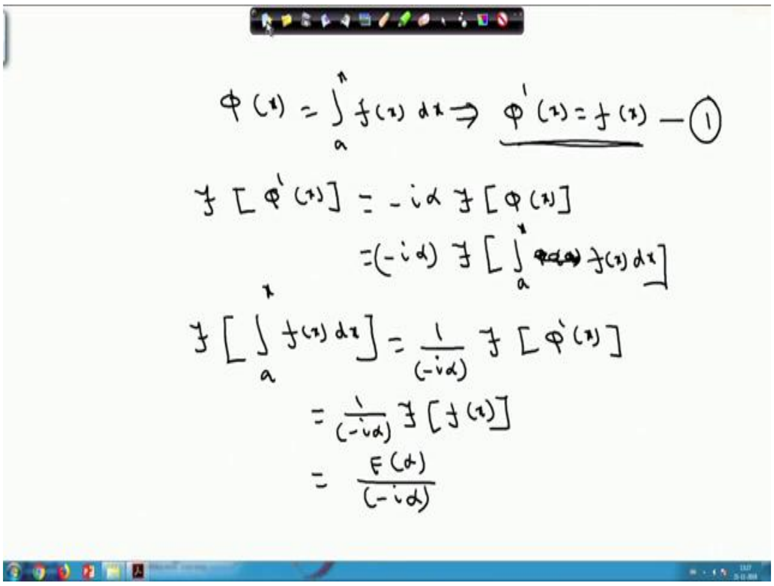

Let,

$$\phi(x) = \int_a^x f(x)dx$$

$$\therefore \phi'(x) = f(x) \tag{1}$$

Using the previous properties, we can write down

$$\mathcal{F}[\phi'(x)] = -i\alpha\mathcal{F}[\phi(x)]$$
$$= -i\alpha\mathcal{F}\left[\int_a^x f(x)dx\right]$$

$$\therefore \mathcal{F}\left[\int_a^x f(x)dx\right] = -\frac{\mathcal{F}[\phi'(x)]}{i\alpha}$$
$$= -\frac{\mathcal{F}[f(x)]}{i\alpha}$$
$$= -\frac{F(\alpha)}{i\alpha}$$

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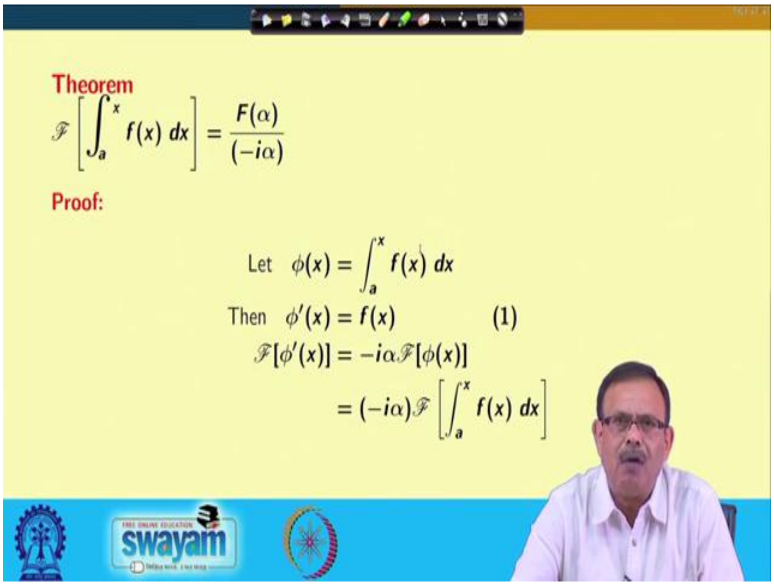

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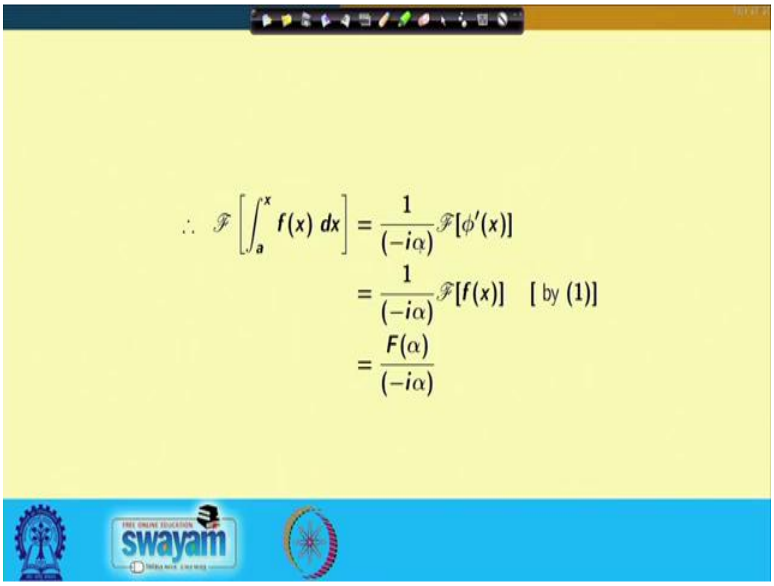

This completes the proof of the theorem. Thank you.