Transform Calculus and its applications in Differential Equations
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Lecture - 31
Change of Scale and Modulation Properties of Fourier Transform

In the last lecture, we have started the properties of Fourier transform, Fourier sine transform and Fourier cosine transform. We started with the linearity property, shifting property and then the multiplicative property, that is if the function $f(x)$ is multiplied by the exponential function e^{iax}, then what is its effect on the Fourier transform.

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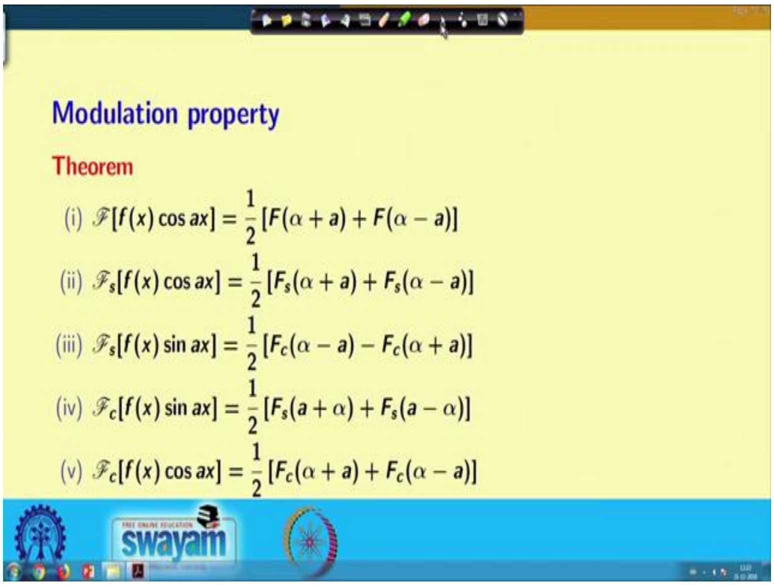

Now, let us see some other important properties. The next property is the modulation property where if we multiply $f(x)$ with $\cos ax$ or $\sin ax$, then what will be the effect.

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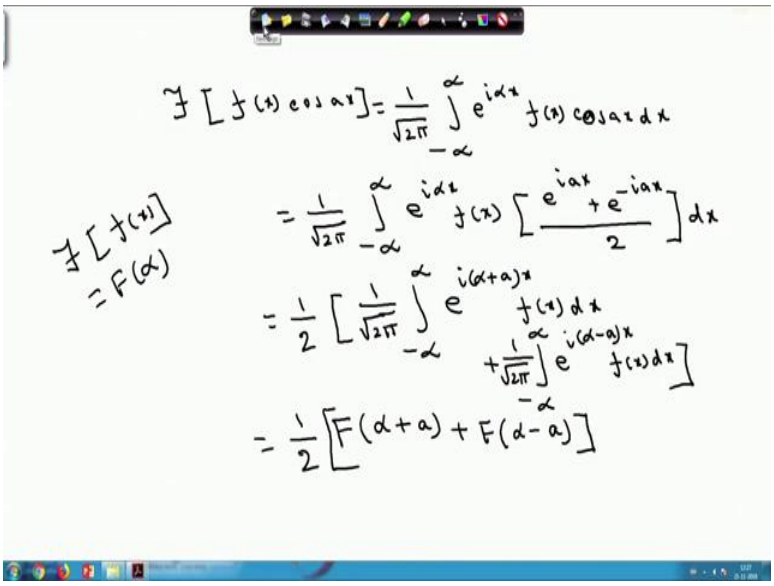

So, let us go to the proof of the first one i.e., Fourier transform of $f(x)\cos ax$ given that $\mathcal{F}[f(x)] = F(\alpha)$. So, we are starting from the left hand side i.e.,

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\ e^{i\alpha x}dx$$

If we substitute the complex form of $\cos ax$ in the above equation then, we get,

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{i\alpha x}dx$$

Now, we break it into two different integrals that is

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(\alpha+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(\alpha-a)x}dx\right]$$

In the first integral, if we see, then it is noticed that it represents the Fourier transform of $f(x)$ where the kernel α has been replaced by $\alpha + a$ and in the second integral, α has been replaced by $\alpha - a$. So, we can write it as

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{2}[F(\alpha + a) + F(\alpha - a)]$$

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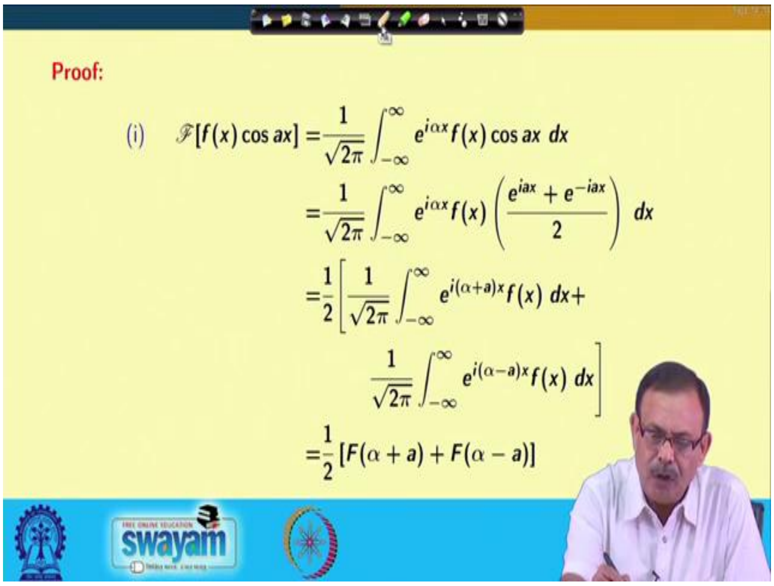

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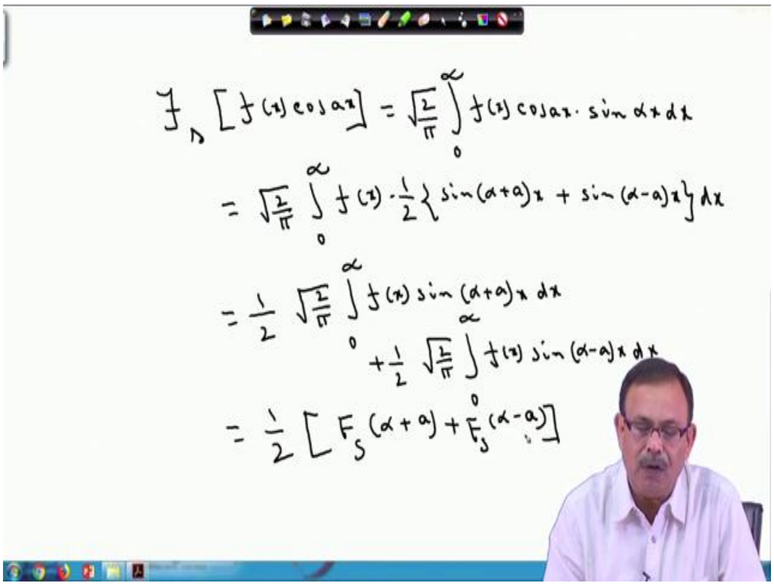

Next let us see the proof of the second one. Here, we have to find the Fourier sine transform of $f(x)\cos ax$. From definition, we get,

$$\mathcal{F}_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax \sin \alpha x\, dx$$

Now, if we put $2\sin \alpha x \cos ax = \sin(\alpha + a)x + \sin(\alpha - a)\,x$ on the right side, then we get,

$$\mathcal{F}_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{1}{2}[\sin(\alpha+a)x + \sin(\alpha-a)x\,]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha-a)x\,dx\right]$$

$$= \frac{1}{2}[F_s(\alpha+a) + F_s(\alpha-a)]$$

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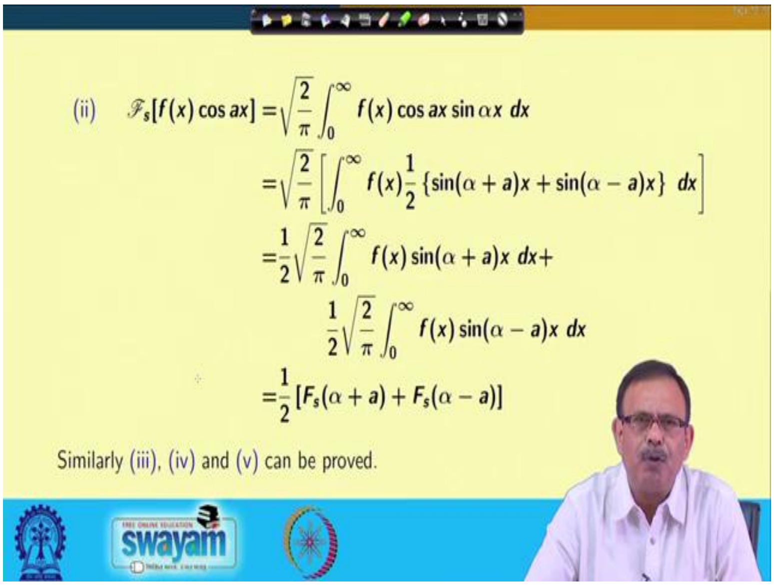

Proceeding in a similar manner, (iii), (iv) and (v) can also be proved very easily.

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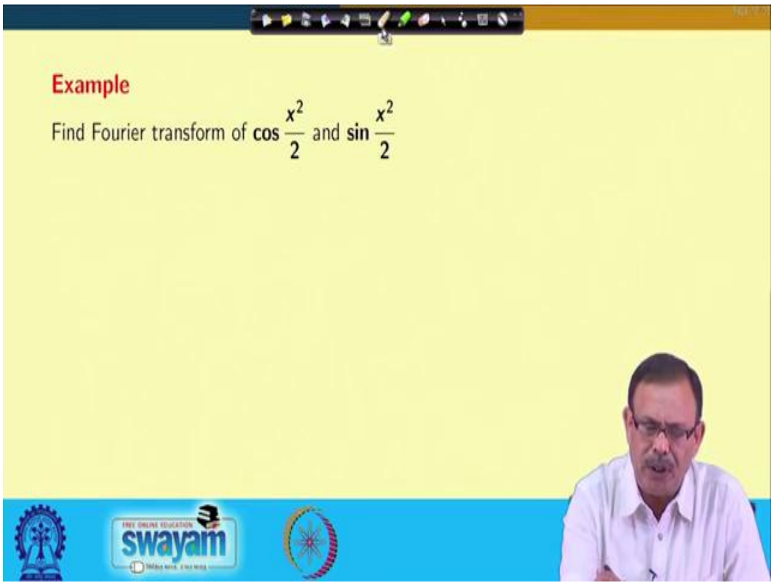

Now, let us consider an example. Suppose, we want to find out the Fourier transform of $\cos\frac{x^2}{2}$ and $\sin\frac{x^2}{2}$. Let us start with something different i.e., although we want to find out the Fourier transform of $\cos\frac{x^2}{2}$ and $\sin\frac{x^2}{2}$, we can start with the Fourier transform of $e^{-a^2x^2}$.

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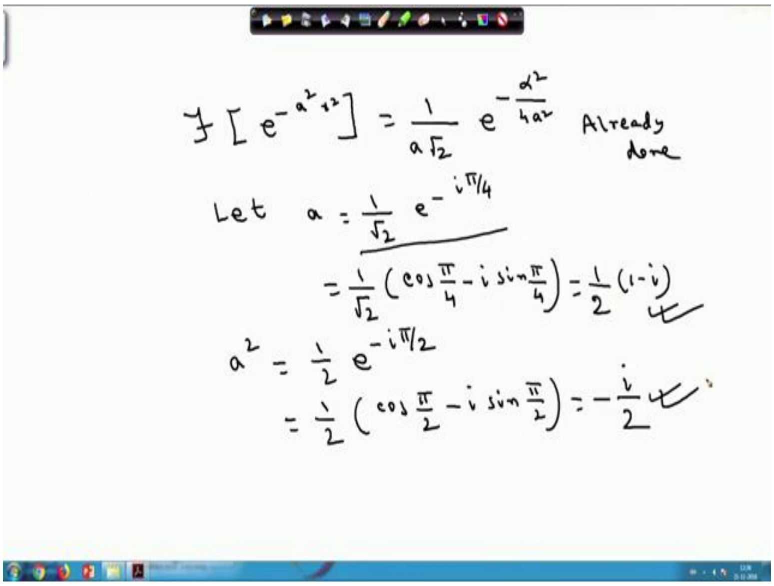

We have already evaluated the Fourier transform of $e^{-a^2x^2}$ in the previous lectures. Using that result, we have,

$$\mathcal{F}\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-\frac{\alpha^2}{4a^2}}$$

Now, let us take $a = \frac{1}{\sqrt{2}} e^{-\frac{i\pi}{4}}$. This can be written as

$$a = \frac{1}{\sqrt{2}} e^{-\frac{i\pi}{4}} = \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] = \frac{1}{2}(1-i)$$

So, from here we can write down

$$a^2 = \frac{1}{2} e^{-\frac{i\pi}{2}} = \frac{1}{2}\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right] = -\frac{i}{2}$$

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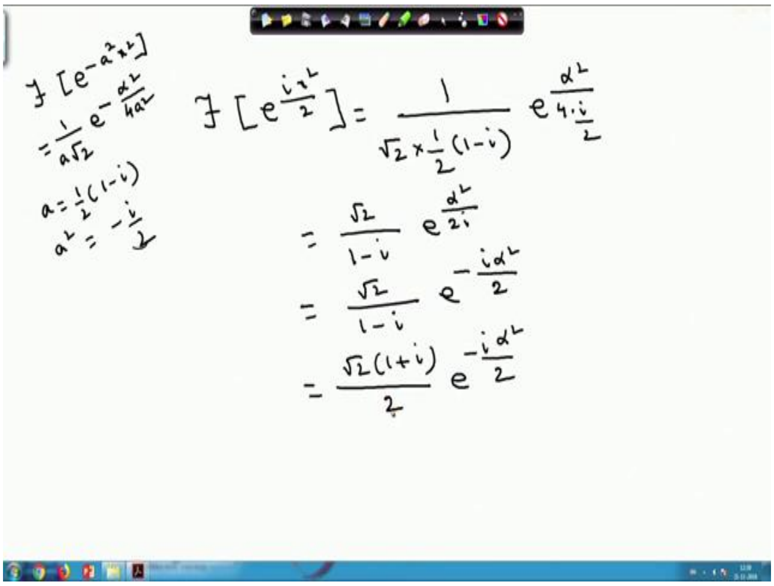

Using the values of a and a^2 in the Fourier transform of $e^{-a^2x^2}$, we have,

$$\mathcal{F}\left[e^{\frac{ix^2}{2}}\right] = \frac{1}{\sqrt{2}\times\frac{1}{2}(1-i)} e^{\frac{\alpha^2}{2i}} = \frac{(1+i)}{\sqrt{2}} e^{-\frac{i\alpha^2}{2}}$$

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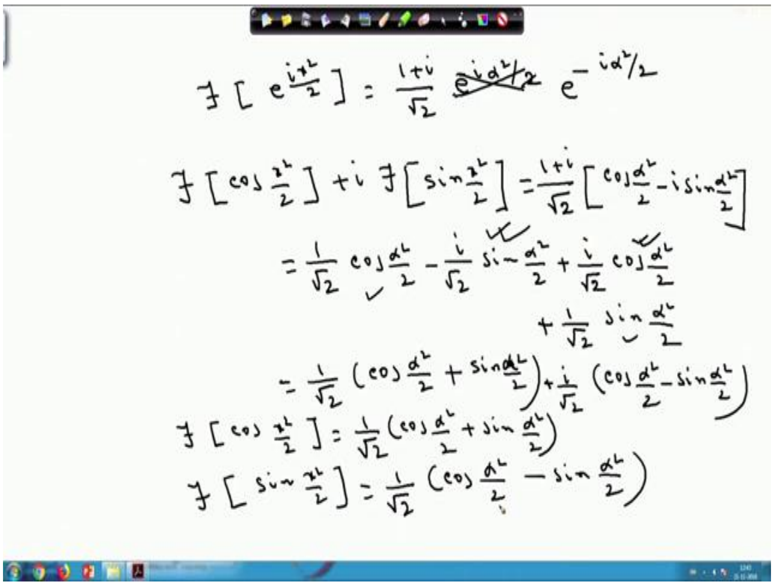

Again $e^{\frac{ix^2}{2}} = \cos\frac{x^2}{2} + i\sin\frac{x^2}{2}$.

$$\therefore \mathcal{F}\left[\cos\frac{x^2}{2}\right] + i\,\mathcal{F}\left[\sin\frac{x^2}{2}\right] = \frac{(1+i)}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} - i\sin\frac{\alpha^2}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} + \sin\frac{\alpha^2}{2}\right] + \frac{i}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} - \sin\frac{\alpha^2}{2}\right]$$

Now comparing the real and imaginary parts from both sides, we get,

$$\mathcal{F}\left[\cos\frac{x^2}{2}\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} + \sin\frac{\alpha^2}{2}\right]$$

$$\mathcal{F}\left[\sin\frac{x^2}{2}\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} - \sin\frac{\alpha^2}{2}\right]$$

which provides the required result.

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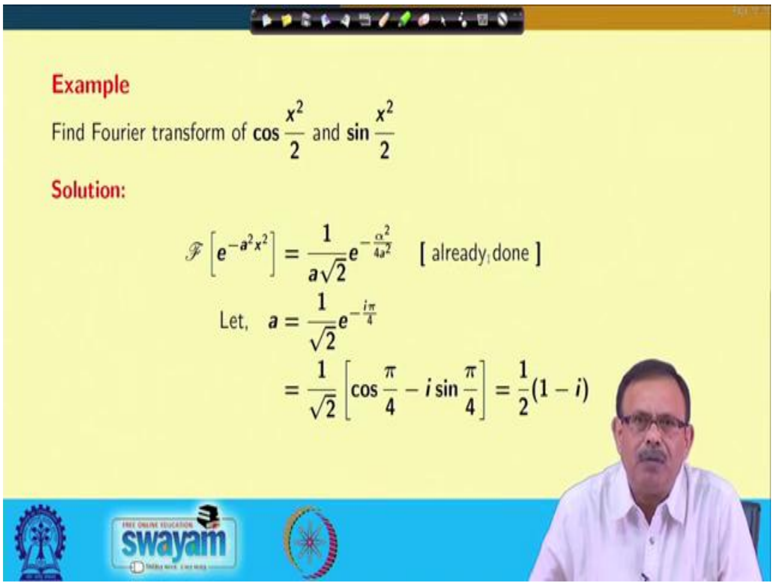

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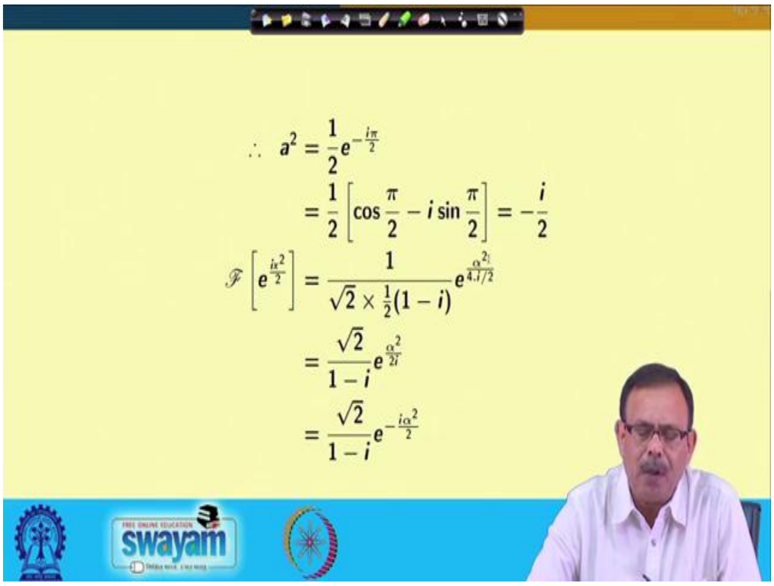

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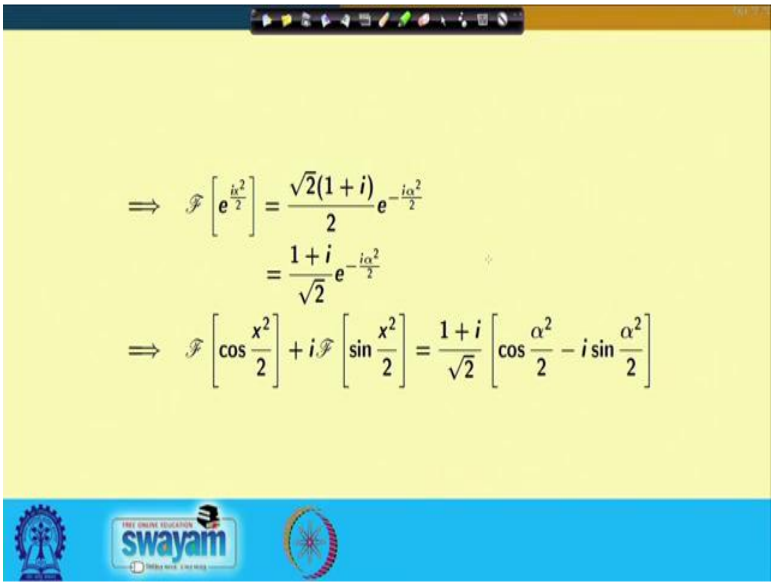

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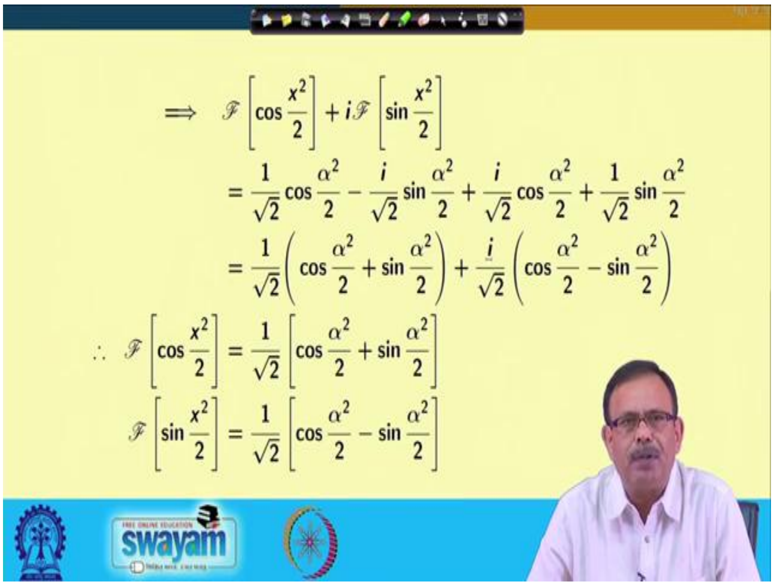

So, in this particular problem, we could have separately obtained the value of Fourier transform of $\cos\frac{x^2}{2}$ and Fourier transform of $\sin\frac{x^2}{2}$ but instead of that, we have started with a function whose Fourier transform is known to us. And then, by choosing a particular value of the parameter a, we have obtained the desired result

This was possible only because we know that e^{ix} always can be expanded in terms of $\cos x$ and $\sin x$.

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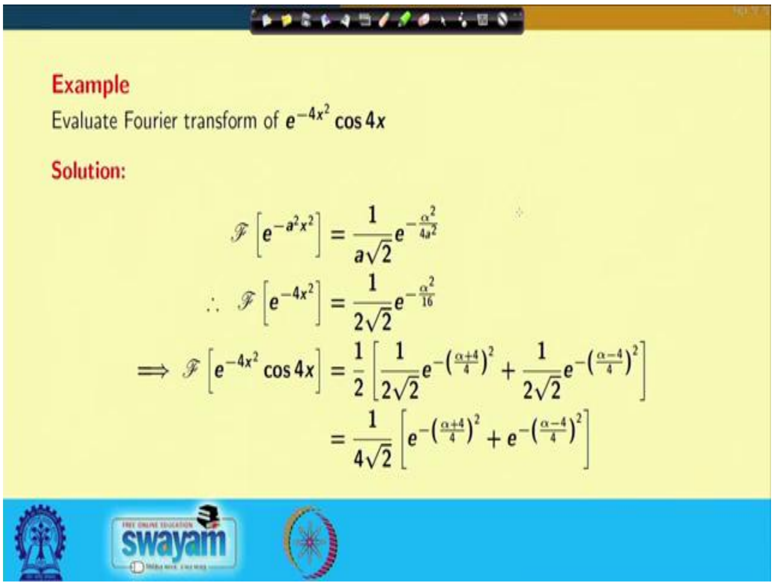

Now, let us see another example. We need to evaluate the Fourier transform of $e^{-4x^2}\cos 4x$.

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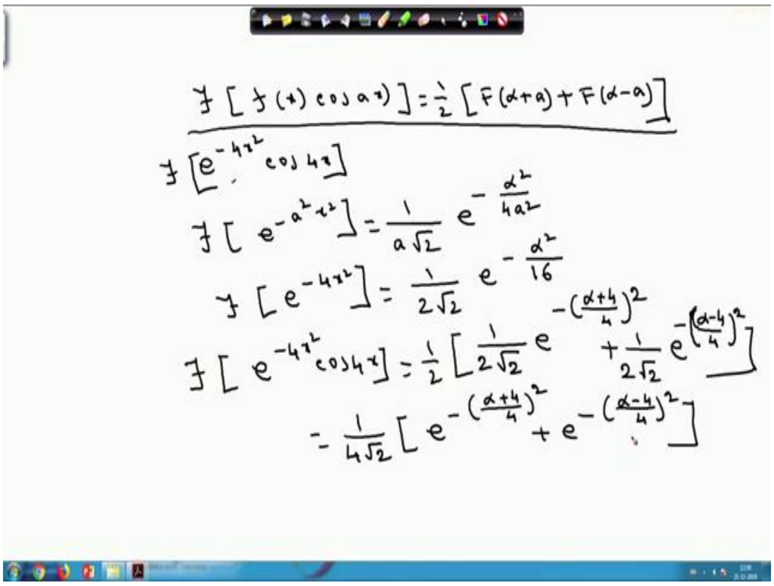

Now, by the modulation property, we know, Fourier transform of $f(x)\cos ax$ is

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{2}[F(\alpha + a) + F(\alpha - a)]$$

We have studied this property earlier in this lecture.

Again we know that,

$$\mathcal{F}\left[e^{-b^2x^2}\right] = \frac{1}{b\sqrt{2}} e^{-\frac{\alpha^2}{4b^2}}$$

Putting $b = 2$, we have,

$$\therefore \mathcal{F}\left[e^{-4x^2}\right] = \frac{1}{2\sqrt{2}} e^{-\frac{\alpha^2}{16}} = F(\alpha)$$

So, using the modulation property, now we have for $a = 4$,

$$\mathcal{F}\left[e^{-4x^2} \cos 4x\right] = \frac{1}{2}\left[\frac{1}{2\sqrt{2}} e^{-\left(\frac{\alpha+4}{4}\right)^2} + \frac{1}{2\sqrt{2}} e^{-\left(\frac{\alpha-4}{4}\right)^2}\right]$$
$$= \frac{1}{4\sqrt{2}}\left[e^{-\left(\frac{\alpha+4}{4}\right)^2} + e^{-\left(\frac{\alpha-4}{4}\right)^2}\right]$$

So, if we use the properties of Fourier transform and using the Fourier transform of certain known functions, we can very easily find out the Fourier transform of some complicated functions also, as we have observed in this example.

Thank you.