

Transform Calculus and its Applications in Differential Equations
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Lecture - 30
Linearity Property and Shifting Properties of Fourier Transform

In the last lecture, we have studied how to find out the Fourier transform or Fourier cosine transform or Fourier sine transform of a particular function. And also in the last lecture, we have seen that if the Fourier transform or Fourier cosine transform or Fourier sine transform of a function is the function itself, then we call the function as a self reciprocal function with respect to that transformation. We have also proved that Fourier transform of an even function is equal to its Fourier cosine transform.

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Example
Find the Fourier Transform of

$$f(x) = \begin{cases} 1 & , |x| < a \\ 0 & , |x| \geq a \end{cases}$$

where a is a positive real number. Hence deduce that $\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$

Solution: The given function can be written as

$$f(x) = \begin{cases} 1 & , -a < x < a \\ 0 & , \text{otherwise} \end{cases}$$

Let us find the Fourier transform of $f(x)$ where $f(x)$ is defined as

$$f(x) = \begin{cases} 1 & , |x| < a \\ 0 & , |x| \geq a \end{cases}$$

Here a is a positive real number and using this, we will show that

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

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The image shows a handwritten derivation on a whiteboard. At the top, the function $f(x)$ is defined as $f(x) = 1, -a < x < a$ and $f(x) = 0, \text{ otherwise}$. Below this, the Fourier transform $F(\alpha)$ is calculated as $F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$. The integral is then evaluated from $-a$ to a , resulting in $\frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{i\alpha x} dx$. This is further simplified to $\frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} \right]$. Finally, it is shown that this is equal to $\frac{1}{\sqrt{2\pi}} \left[\frac{2i \sin \alpha a}{i\alpha} \right]$, which simplifies to $\sqrt{\frac{2}{\pi}} \frac{\sin \alpha a}{\alpha}$. A small inset video shows a man speaking.

$f(x)$ can be written as,

$$f(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

So, from the definition of Fourier transform, we have,

$$\begin{aligned} \mathcal{F}[f(x)] = F(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a 1 \cdot e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} \right] \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin \alpha a}{\alpha} \end{aligned}$$

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$$\begin{aligned}
 f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin \alpha a}{\alpha} \right) (\cos \alpha x - i \sin \alpha x) d\alpha \\
 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\left(\frac{\sin \alpha a}{\alpha} \right) \cos \alpha x}_{\text{even}} d\alpha - \frac{i}{\pi} \int_{-\infty}^{\infty} \underbrace{\left(\frac{\sin \alpha a}{\alpha} \right) \sin \alpha x}_{\text{odd}} d\alpha \\
 &= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin \alpha a}{\alpha} \right) \cos \alpha x d\alpha
 \end{aligned}$$

Now if we take the inverse Fourier transform of $F(\alpha)$, we will obtain,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha$$

Now already we know what is $F(\alpha)$. So if we substitute $F(\alpha)$, we will obtain,

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \alpha a}{\alpha} \right) (\cos \alpha x - i \sin \alpha x) d\alpha$$

If we break it into two parts, we will obtain,

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \alpha a}{\alpha} \right) \cos \alpha x d\alpha - \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \alpha a}{\alpha} \right) \sin \alpha x d\alpha$$

Now, $\frac{\sin \alpha a \cos \alpha x}{\alpha}$ is an even function of α whereas $\frac{\sin \alpha a \sin \alpha x}{\alpha}$ is an odd function of α .

So, above integral reduces to,

$$\begin{aligned}
 f(x) &= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin \alpha a}{\alpha} \right) \cos \alpha x d\alpha \\
 \Rightarrow \int_0^{\infty} \left(\frac{\sin \alpha a}{\alpha} \right) \cos \alpha x d\alpha &= \frac{\pi}{2} f(x) \\
 &= \begin{cases} \frac{\pi}{2} , & -a < x < a \\ 0 , & \text{otherwise} \end{cases}
 \end{aligned}$$

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$$\int_0^{\infty} \left(\frac{\sin a\alpha}{a} \right) \cos x \, d\alpha = \frac{\pi}{2} f(x)$$
$$\underline{x=0} \quad \int_0^{\infty} \frac{\sin a\alpha}{a} \, d\alpha = \frac{\pi}{2}$$
$$t = a\alpha \Rightarrow \alpha = \frac{t}{a} \Rightarrow d\alpha = \frac{dt}{a}$$
$$t \rightarrow 0 \Rightarrow \alpha \rightarrow 0; \quad t \rightarrow \infty \Rightarrow \alpha \rightarrow \infty$$
$$\int_0^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}$$

Now, if we put $x = 0$, then we get,

$$\int_0^{\infty} \frac{\sin a\alpha}{a} \, d\alpha = \frac{\pi}{2} \quad (\because f(0) = 1)$$

Let us substitute $t = a\alpha$ in the above integral. Then $d\alpha = \frac{dt}{a}$. Also, $t \rightarrow 0$ as $\alpha \rightarrow 0$ and $t \rightarrow \infty$ as $\alpha \rightarrow \infty$.

$$\therefore \int_0^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}$$

This completes the solution to the given problem.

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The slide displays the following derivation for the Fourier transform of a rectangular pulse function $f(x)$ of width $2a$ centered at the origin:

$$\begin{aligned}\mathcal{F}[f(x)] = F(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x} - e^{-i\alpha x}}{i\alpha} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{2i \sin a\alpha}{i\alpha} \right] \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha} \quad (1)\end{aligned}$$

The slide also features a video feed of a presenter in the bottom right corner and logos for 'swayam' and 'All India Institute of Space Technology' at the bottom.

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The slide shows the derivation of the Fourier inversion formula for the rectangular pulse function:

Now by Fourier inversion formula

$$\begin{aligned}f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x} d\alpha \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin a\alpha}{\alpha} \right) (\cos \alpha x - i \sin \alpha x) d\alpha \quad [\text{using (1)}] \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a\alpha}{\alpha} \right) \cos \alpha x d\alpha - \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a\alpha}{\alpha} \right) \sin \alpha x d\alpha \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a\alpha}{\alpha} \right) \cos \alpha x d\alpha \quad \left[\because \left(\frac{\sin a\alpha}{\alpha} \right) \sin \alpha x \text{ is odd function in } \alpha \right] \\ &= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin a\alpha}{\alpha} \right) \cos \alpha x d\alpha \quad \left[\because \left(\frac{\sin a\alpha}{\alpha} \right) \cos \alpha x \text{ is even function in } \alpha \right]\end{aligned}$$

The slide also features a video feed of a presenter in the bottom right corner and logos for 'swayam' and 'All India Institute of Space Technology' at the bottom.

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$$\therefore \int_0^\infty \left(\frac{\sin a\alpha}{\alpha}\right) \cos \alpha x \, d\alpha = \frac{\pi}{2} f(x)$$

Putting $x = 0$, we get
$$\int_0^\infty \left(\frac{\sin a\alpha}{\alpha}\right) \, d\alpha = \frac{\pi}{2}$$

Put $t = a\alpha$

$$\therefore \alpha = \frac{t}{a} \text{ and } d\alpha = \frac{dt}{a}$$

Further $t \rightarrow 0$ as $\alpha \rightarrow 0$ and $t \rightarrow \infty$ as $\alpha \rightarrow \infty$

$$\therefore \int_0^\infty \left(\frac{\sin t}{t}\right) \, dt = \frac{\pi}{2}$$

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Properties of Fourier Transforms:

Theorem
Fourier Transform, Fourier-sine Transform and Fourier-cosine Transform are linear i.e.,

(i) $\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$

(ii) $\mathcal{F}_s[af(x) + bg(x)] = a\mathcal{F}_s[f(x)] + b\mathcal{F}_s[g(x)]$

(iii) $\mathcal{F}_c[af(x) + bg(x)] = a\mathcal{F}_c[f(x)] + b\mathcal{F}_c[g(x)]$

where a, b are real numbers

Let us now discuss certain properties of Fourier transform.

Fourier transform, Fourier sine transform and Fourier cosine transform all are linear, i.e.,

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$$

$$\mathcal{F}_s[af(x) + bg(x)] = a\mathcal{F}_s[f(x)] + b\mathcal{F}_s[g(x)]$$

$$\mathcal{F}_c[af(x) + bg(x)] = a\mathcal{F}_c[f(x)] + b\mathcal{F}_c[g(x)]$$

where a, b are real numbers.

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The image shows a handwritten derivation of the linearity property of the Fourier transform. The equations are written on a whiteboard and read as follows:

$$\begin{aligned}\mathcal{F}[af(x) + bg(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{iax} dx \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iax} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{iax} dx \\ &= a \mathcal{F}[f(x)] + b \mathcal{F}[g(x)]\end{aligned}$$

The derivation is presented in three lines. The first line shows the definition of the Fourier transform of a sum. The second line separates the sum into two integrals. The third line identifies each integral as the Fourier transform of the respective function, scaled by the constants a and b. A person's head and shoulders are visible in the bottom right corner of the whiteboard frame.

Let us see the first proof. By definition, we have,

$$\mathcal{F}[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{iax} dx$$

If we break it into two parts, then we have,

$$\begin{aligned}\mathcal{F}[af(x) + bg(x)] &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iax} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{iax} dx \\ &= a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]\end{aligned}$$

Therefore, Fourier transform is linear. Similarly, we can show that, Fourier sine transform and Fourier cosine transform are also linear.

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Proof:

$$\begin{aligned} \text{(i)} \quad \mathcal{F}[af(x) + bg(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]e^{i\alpha x} dx \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\alpha x} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{i\alpha x} dx \\ &= a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)] \end{aligned}$$

Similarly (ii) and (iii) can be proved.

The slide also features a small video inset of a man speaking in the bottom right corner and logos for Swamyam and other institutions at the bottom.

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Shifting Theorem

Theorem
 $\mathcal{F}[f(x - a)] = e^{ia\alpha} F(\alpha)$ where $F(\alpha) = \mathcal{F}[f(x)]$

Proof:

$$\begin{aligned} \mathcal{F}[f(x - a)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - a)e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\alpha(a+t)} dt \quad [\text{put } t = x - a] \\ &= \frac{e^{ia\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\alpha t} dt = e^{ia\alpha} F(\alpha) \end{aligned}$$

The slide also features logos for Swamyam and other institutions at the bottom.

Now we discuss about the Shifting theorem for Fourier transform. The theorem states that, if $\mathcal{F}[f(x)] = F(\alpha)$ then $\mathcal{F}[f(x - a)] = e^{ia\alpha} F(\alpha)$ where a is a constant.

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$$\begin{aligned}\mathcal{F}[f(x-a)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha(a+t)} dt \quad (t = x-a) \\ &= \frac{e^{i\alpha a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \\ &= e^{i\alpha a} F(\alpha)\end{aligned}$$

From the definition,

$$\mathcal{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{i\alpha x} dx$$

If we put $x - a = t$ in the right side of the above equation, then we have

$$\begin{aligned}\mathcal{F}[f(x-a)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha(a+t)} dt \\ &= \frac{1}{\sqrt{2\pi}} e^{i\alpha a} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \\ &= e^{i\alpha a} F(\alpha)\end{aligned}$$

So, this completes the proof of shifting theorem.

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Theorem
 $\mathcal{F}[e^{iax}f(x)] = F(\alpha + a)$

Proof:

$$\begin{aligned}\mathcal{F}[e^{iax}f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax}f(x)e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i(\alpha+a)x} dx \\ &= F(\alpha + a)\end{aligned}$$

The slide also features the Swamy logo and a video feed of a presenter in the bottom right corner.

Now we show that, $\mathcal{F}[e^{iax}f(x)] = F(\alpha + a)$

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$$\begin{aligned}\mathcal{F}[e^{iax}f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax}f(x) \cdot e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(\alpha+a)x} dx \\ &= F(\alpha + a)\end{aligned}$$

The whiteboard also features a video feed of a presenter in the bottom right corner.

From definition,

$$\begin{aligned}\mathcal{F}[e^{iax}f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax}f(x)e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i(\alpha+a)x} dx \\ &= F(\alpha + a)\end{aligned}$$

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Change of Scale property

Theorem
For any non-zero real number a ,

- (i) $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$
- (ii) $\mathcal{F}_s[f(ax)] = \frac{1}{a} F_s\left(\frac{\alpha}{a}\right), a > 0$
- (iii) $\mathcal{F}_c[f(ax)] = \frac{1}{a} F_c\left(\frac{\alpha}{a}\right), a > 0$

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If a is a non-zero real number, then from the Change of Scale property, we have,

$$(i) \mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$$

$$(ii) \mathcal{F}_s[f(ax)] = \frac{1}{a} F_s\left(\frac{\alpha}{a}\right), a > 0$$

$$(iii) \mathcal{F}_c[f(ax)] = \frac{1}{a} F_c\left(\frac{\alpha}{a}\right), a > 0$$

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(i) Let $a > 0$

$$\begin{aligned} \mathcal{F}[f(ax)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\frac{\alpha}{a}t} \cdot \frac{dt}{a} \\ &= \frac{1}{a} F\left(\frac{\alpha}{a}\right) \end{aligned}$$

$t = ax$
 $e^{i\frac{\alpha}{a}t}$

For $a > 0$, we have,

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{i\alpha x} dx \quad (1)$$

If we put $ax = t$ in the right side of (1), then we have,

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\frac{i\alpha t}{a}} \frac{dt}{a} = \frac{1}{a} F\left(\frac{\alpha}{a}\right)$$

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Let $a < 0$

$$\begin{aligned} \mathcal{F}[f(ax)] &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\frac{i\alpha t}{a}} \frac{dt}{a} \\ &= -\frac{1}{a} F\left(\frac{\alpha}{a}\right) \\ \therefore \mathcal{F}[f(ax)] &= \frac{1}{|a|} F\left(\frac{\alpha}{a}\right) \end{aligned}$$

For $a < 0$, we have, if we put $ax = t$ in the right side of (1), then $t \rightarrow \infty$ as $x \rightarrow -\infty$ and $t \rightarrow -\infty$ as $x \rightarrow \infty$. Therefore,

$$\begin{aligned} \mathcal{F}[f(ax)] &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t) e^{\frac{i\alpha t}{a}} \frac{dt}{a} \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\frac{i\alpha t}{a}} \frac{dt}{a} \\ &= -\frac{1}{a} F\left(\frac{\alpha}{a}\right) \end{aligned}$$

Therefore, $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$.

Similarly (ii) and (iii) can be proved.

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The image shows a presentation slide with a yellow background. At the top, there is a toolbar with various icons. The title 'Change of Scale property' is written in blue. Below it, the word 'Theorem' is in red. The text 'For any non-zero real number a ,' is in black. Three mathematical formulas are listed: (i) $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$ with an arrow pointing right; (ii) $\mathcal{F}_s[f(ax)] = \frac{1}{a} F_s\left(\frac{\alpha}{a}\right), a > 0$; and (iii) $\mathcal{F}_c[f(ax)] = \frac{1}{a} F_c\left(\frac{\alpha}{a}\right), a > 0$. A large right-facing curly bracket groups the last two formulas. In the bottom right corner, a man with glasses and a white shirt is visible. At the bottom of the slide, there are logos for 'swayam' and 'THE ONLINE EDUCATION'.

Change of Scale property

Theorem
For any non-zero real number a ,

(i) $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right) \longrightarrow$

(ii) $\mathcal{F}_s[f(ax)] = \frac{1}{a} F_s\left(\frac{\alpha}{a}\right), a > 0$

(iii) $\mathcal{F}_c[f(ax)] = \frac{1}{a} F_c\left(\frac{\alpha}{a}\right), a > 0$

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Thank you.