Transform Calculus and its Applications in Differential Equations
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Lecture - 30
Linearity Property and Shifting Properties of Fourier Transform

In the last lecture, we have studied how to find out the Fourier transform or Fourier cosine transform or Fourier sine transform of a particular function. And also in the last lecture, we have seen that if the Fourier transform or Fourier cosine transform or Fourier sine transform of a function is the function itself, then we call the function as a self reciprocal function with respect to that transformation. We have also proved that Fourier transform of an even function is equal to its Fourier cosine transform.

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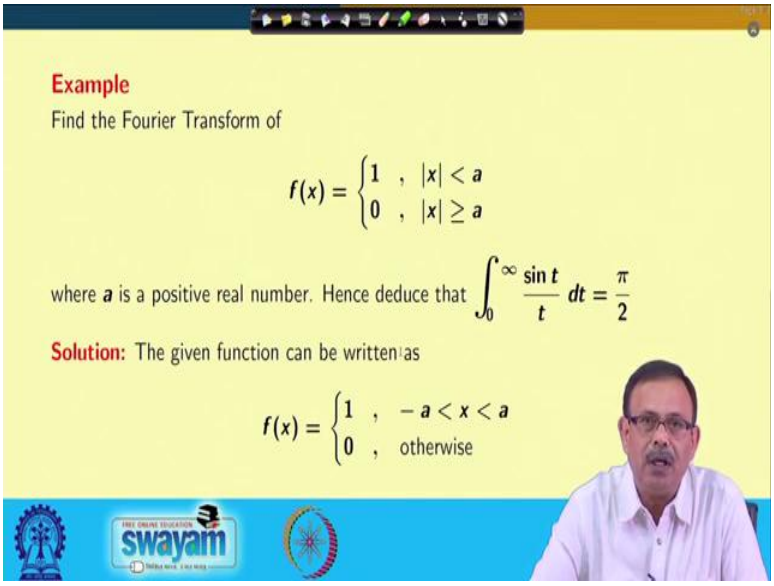

Let us find the Fourier transform of $f(x)$ where $f(x)$ is defined as

$$f(x) = \begin{cases} 1 & , \ |x| < a \\ 0 & , \ |x| \geq a \end{cases}$$

Here a is a positive real number and using this, we will show that

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

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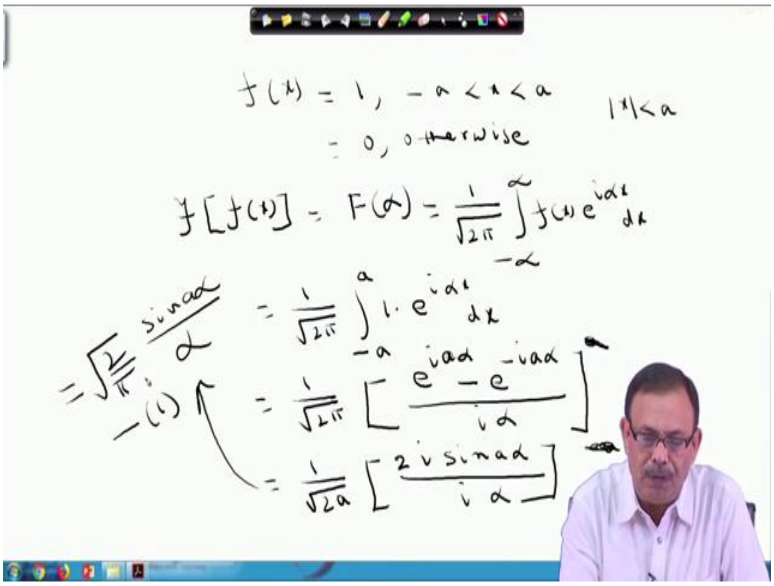

$f(x)$ can be written as,

$$f(x) = \begin{cases} 1 \;, & -a < x < a \\ 0 \;, & \text{otherwise} \end{cases}$$

So, from the definition of Fourier transform, we have,

$$\mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ia\alpha} - e^{-ia\alpha}}{i\alpha} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$$

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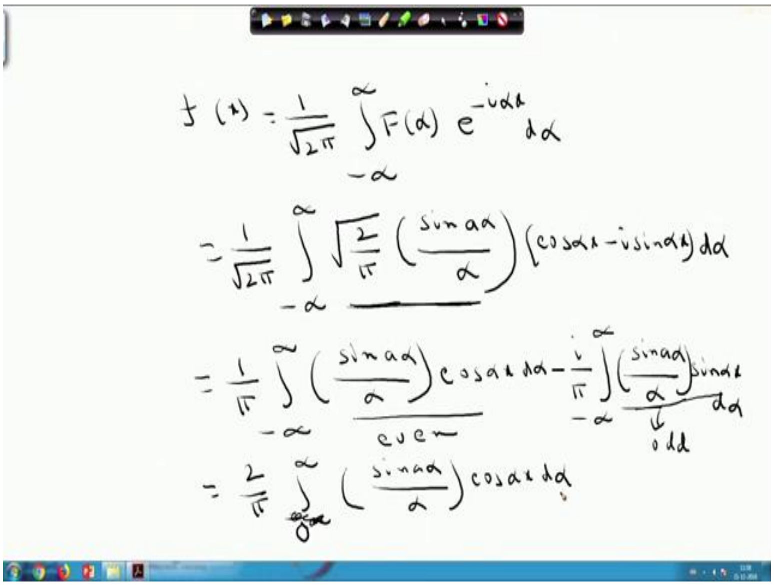

Now if we take the inverse Fourier transform of $F(\alpha)$, we will obtain,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x} d\alpha$$

Now already we know what is $F(\alpha)$. So if we substitute $F(\alpha)$, we will obtain,

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a\alpha}{\alpha}\right) (\cos \alpha x - i \sin \alpha x) d\alpha$$

If we break it into two parts, we will obtain,

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a\alpha}{\alpha}\right) \cos \alpha x \, d\alpha - \frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin a\alpha}{\alpha}\right) \sin \alpha x \, d\alpha$$

Now, $\frac{\sin a\alpha \;\cos \alpha x}{\alpha}$ is an even function of α whereas $\frac{\sin a\alpha \;\sin \alpha x}{\alpha}$ is an odd function of α. So, above integral reduces to,

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin a\alpha}{\alpha}\right) \cos \alpha x \, d\alpha$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin a\alpha}{\alpha}\right) \cos \alpha x \, d\alpha = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

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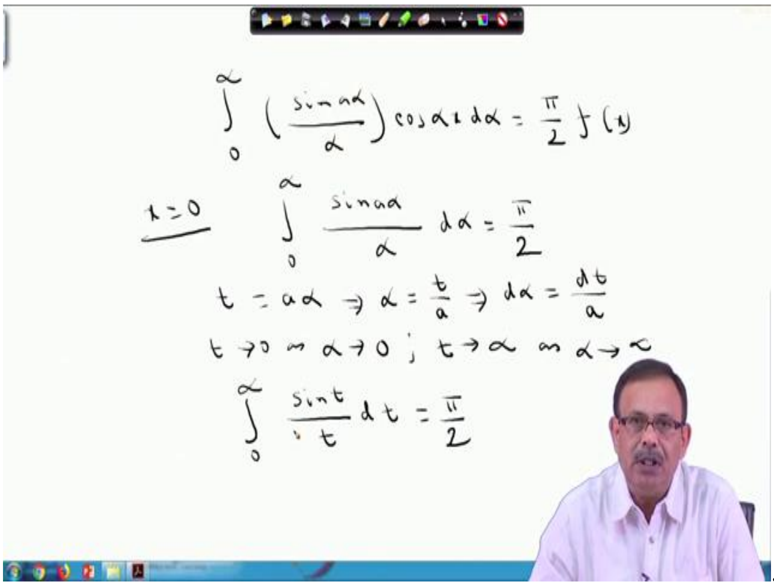

Now, if we put $x = 0$, then we get,

$$\int_0^\infty \frac{\sin a\alpha}{\alpha} d\alpha = \frac{\pi}{2} \qquad (\because f(0) = 1)$$

Let us substitute $t = a\alpha$ in the above integral. Then $d\alpha = \frac{dt}{a}$. Also, $t \to 0$ as $\alpha \to 0$ and $t \to \infty$ as $\alpha \to \infty$.

$$\therefore \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

This completes the solution to the given problem.

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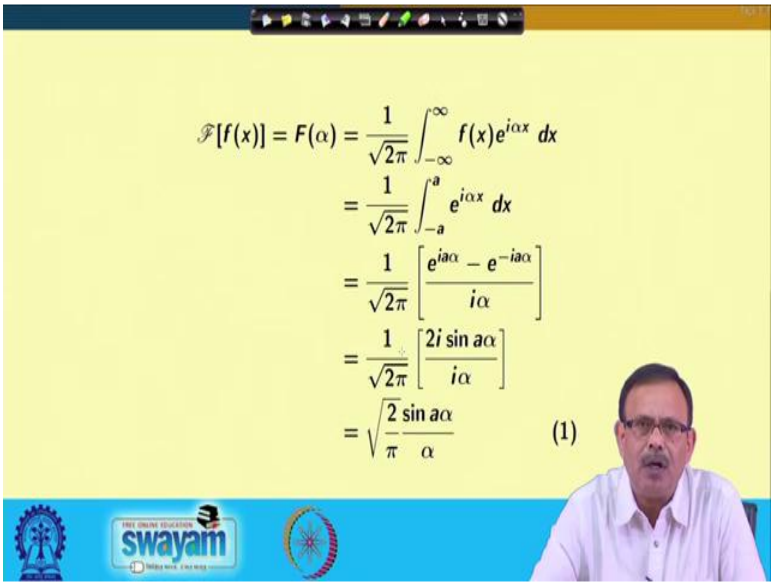

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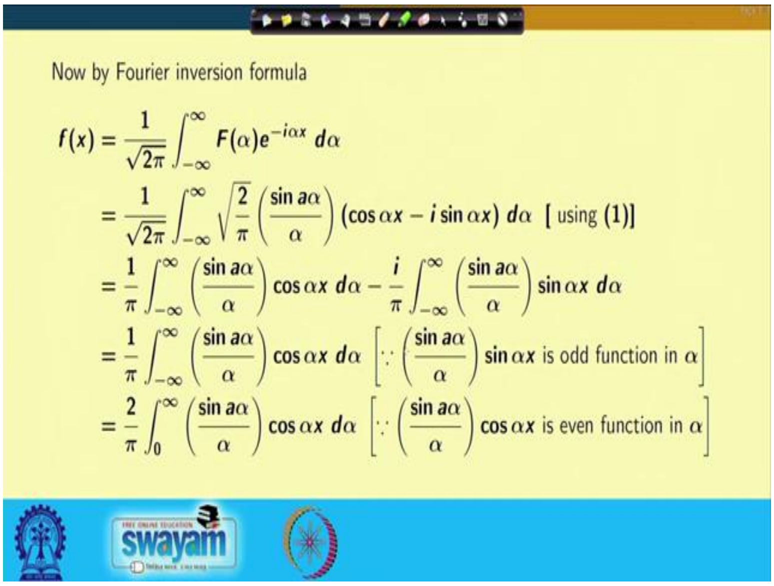

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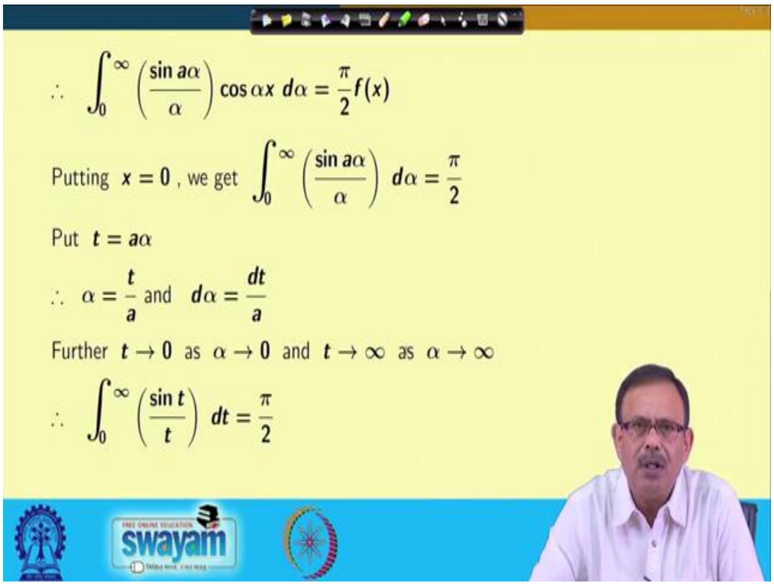

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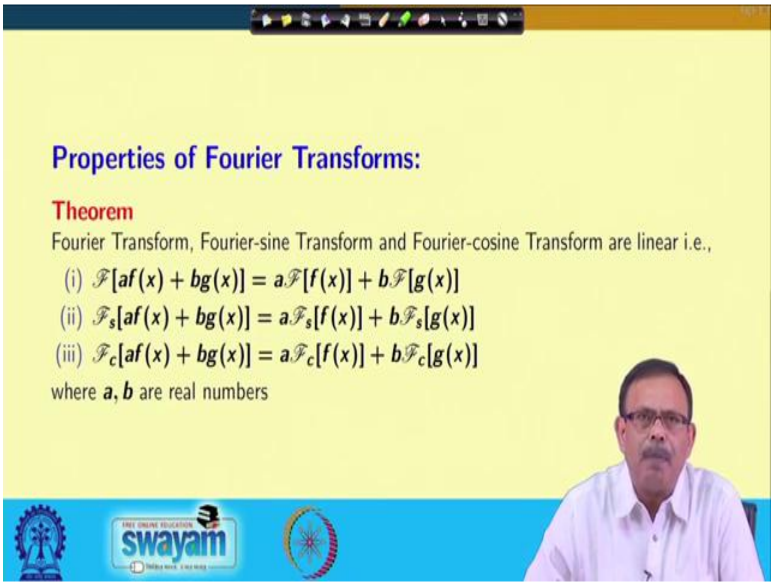

Let us now discuss certain properties of Fourier transform.

Fourier transform, Fourier sine transform and Fourier cosine transform all are linear, i.e.,

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$$
$$\mathcal{F}_s[af(x) + bg(x)] = a\mathcal{F}_s[f(x)] + b\mathcal{F}_s[g(x)]$$
$$\mathcal{F}_c[af(x) + bg(x)] = a\mathcal{F}_c[f(x)] + b\mathcal{F}_c[g(x)]$$

where a, b are real numbers.

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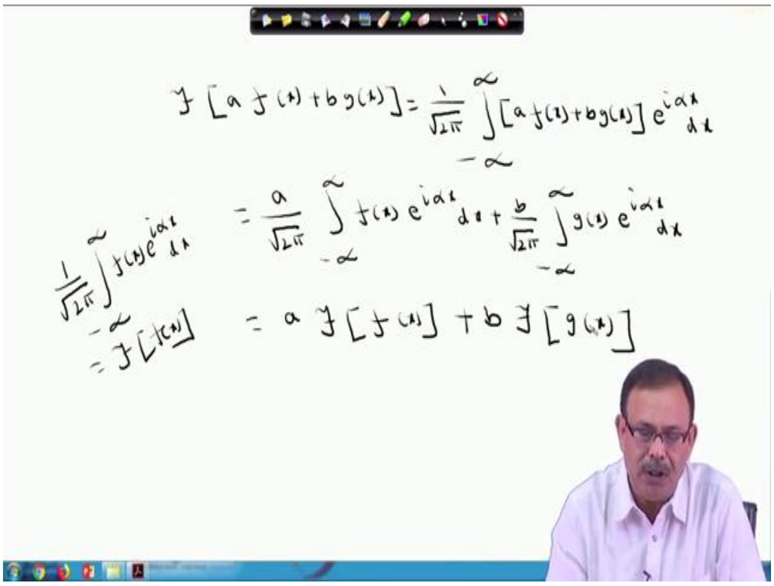

Let us see the first proof. By definition, we have,

$$\mathcal{F}[af(x)+bg(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{i\alpha x}dx$$

If we break it into two parts, then we have,

$$\mathcal{F}[af(x)+bg(x)]=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{i\alpha x}dx+\frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{i\alpha x}dx$$
$$=a\mathcal{F}[f(x)]+b\mathcal{F}[g(x)]$$

Therefore, Fourier transform is linear. Similarly, we can show that, Fourier sine transform and Fourier cosine transform are also linear.

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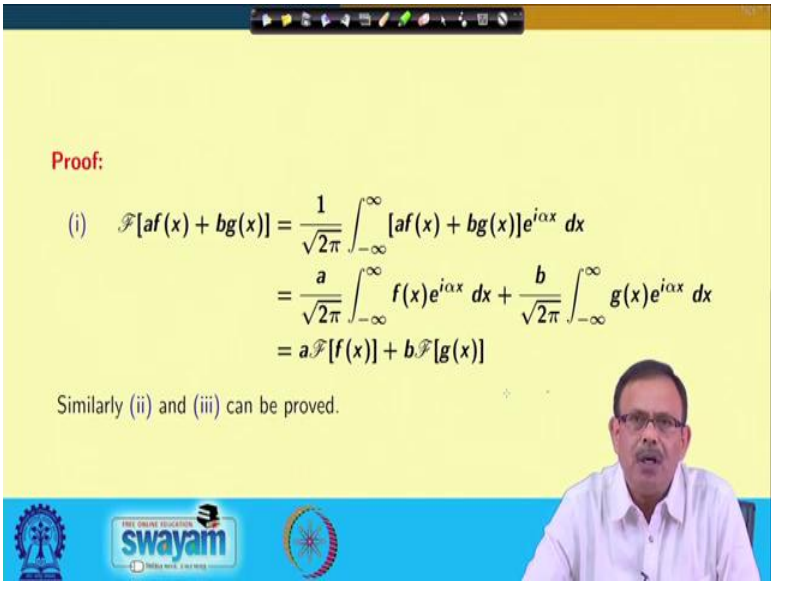

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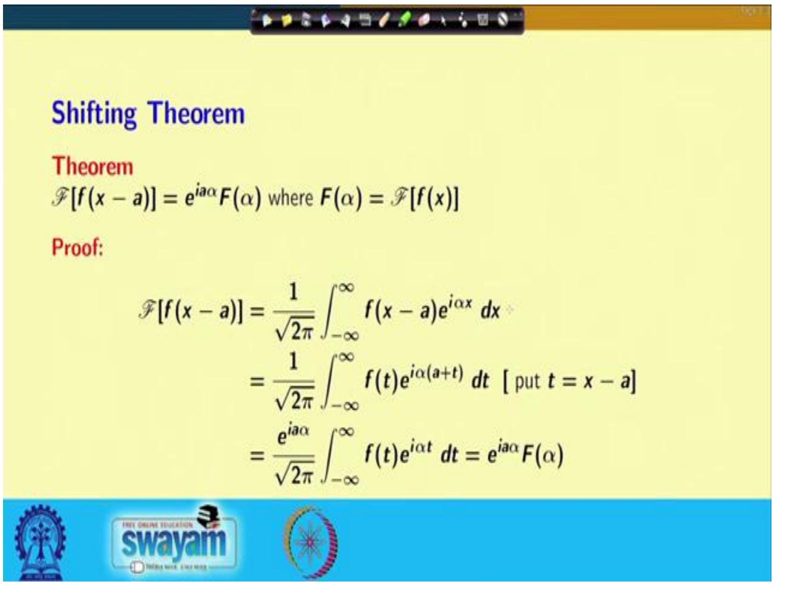

Now we discuss about the Shifting theorem for Fourier transform. The theorem states that, if $\mathcal{F}[f(x)] = F(\alpha)$ then $\mathcal{F}[f(x-a)] = e^{ia\alpha}F(\alpha)$ where a is a constant.

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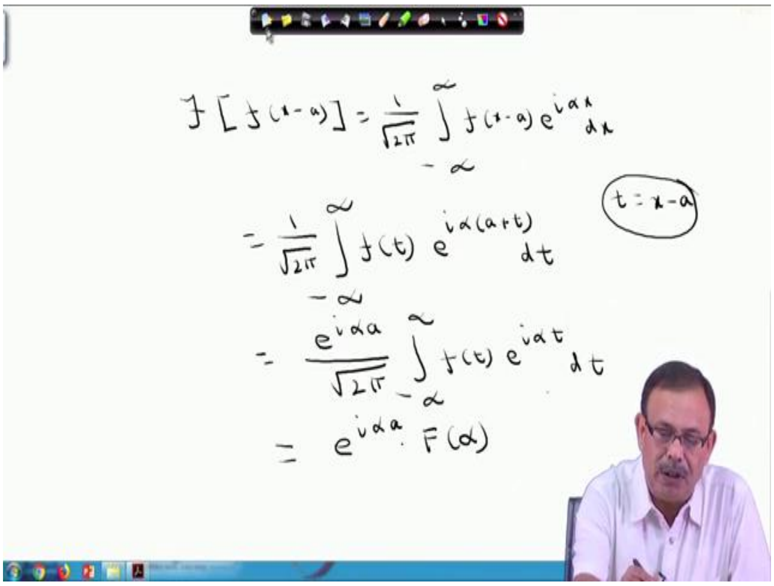

From the definition,

$$\mathcal{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{i\alpha x}dx$$

If we put $x - a = t$ in the right side of the above equation, then we have

$$\begin{aligned}\mathcal{F}[f(x-a)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\alpha(a+t)}dt \\ &= \frac{1}{\sqrt{2\pi}}e^{ia\alpha}\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt \\ &= e^{ia\alpha}F(\alpha)\end{aligned}$$

So, this completes the proof of shifting theorem.

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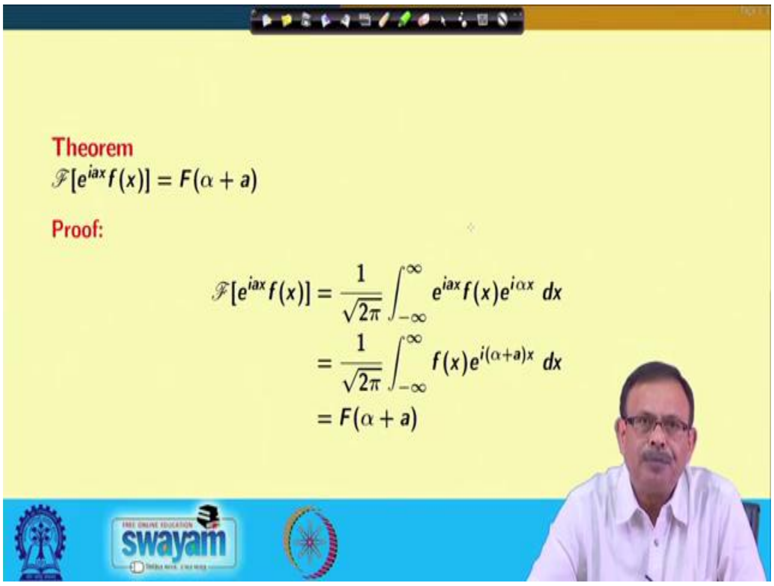

Now we show that, $\mathcal{F}\left[e^{iax}f(x)\right] = F(\alpha + a)$

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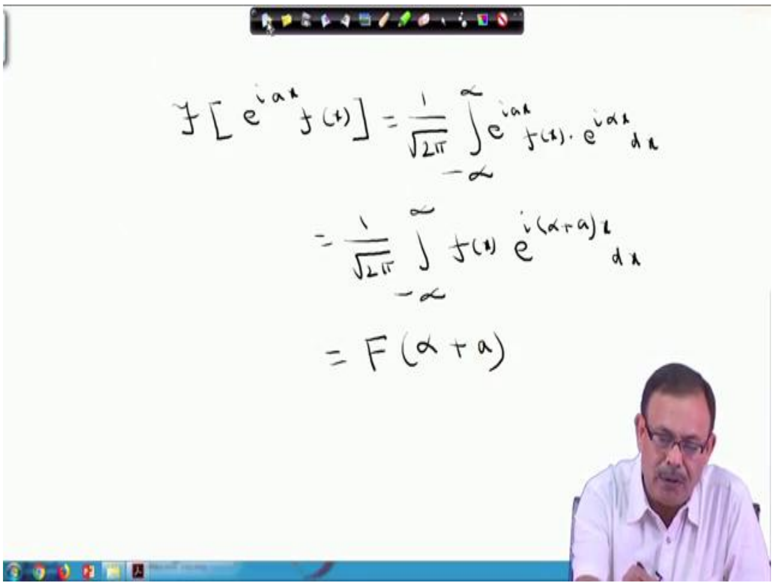

From definition,

$$\mathcal{F}\left[e^{iax}f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{i\alpha x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(a+\alpha)x}dx$$

$$= F(\alpha + a)$$

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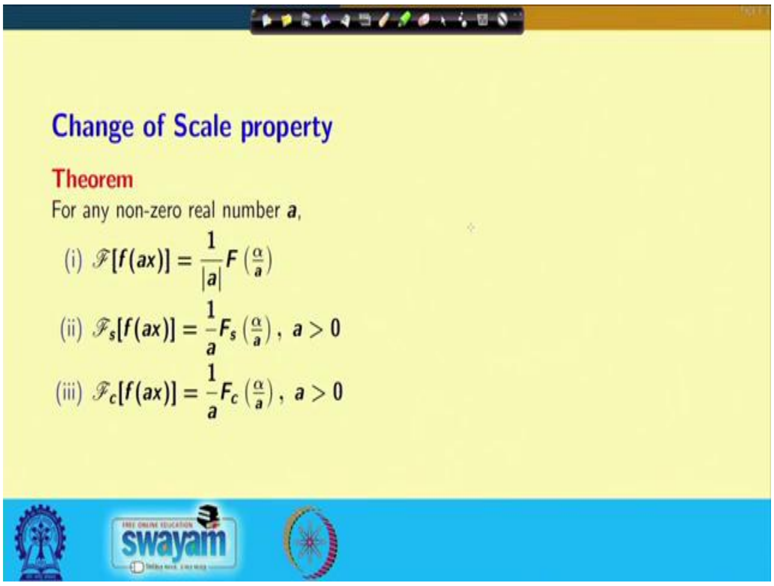

If a is a non-zero real number, then from the Change of Scale property, we have,

$$(i)\ \mathcal{F}[f(ax)] = \frac{1}{|a|}F\left(\frac{\alpha}{a}\right)$$

$$(ii)\ \mathcal{F}_s[f(ax)] = \frac{1}{a}F_s\left(\frac{\alpha}{a}\right)\ \ ,\ \ a > 0$$

$$(iii)\ \mathcal{F}_c[f(ax)] = \frac{1}{a}F_c\left(\frac{\alpha}{a}\right)\ \ ,\ \ a > 0$$

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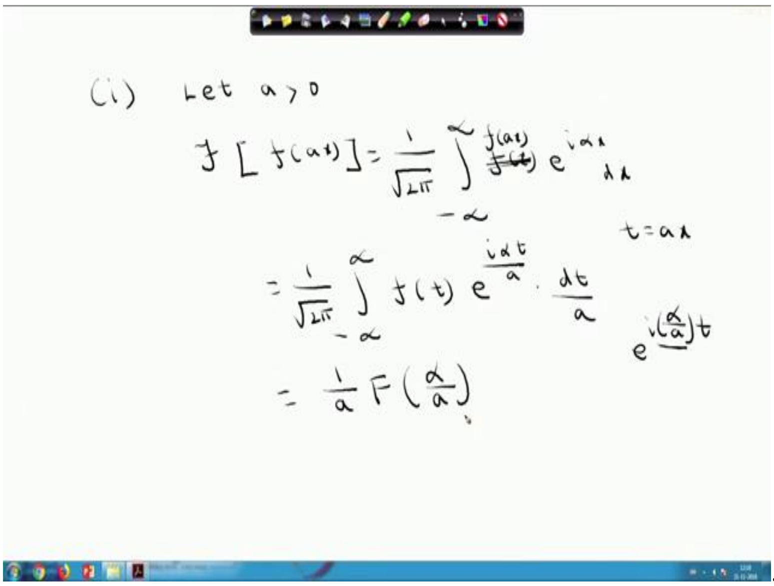

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For $a > 0$, we have,

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{i\alpha x} dx \tag{1}$$

If we put $ax = t$ in the right side of (1), then we have,

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\frac{i\alpha t}{a}} \frac{dt}{a} = \frac{1}{a} F\left(\frac{\alpha}{a}\right)$$

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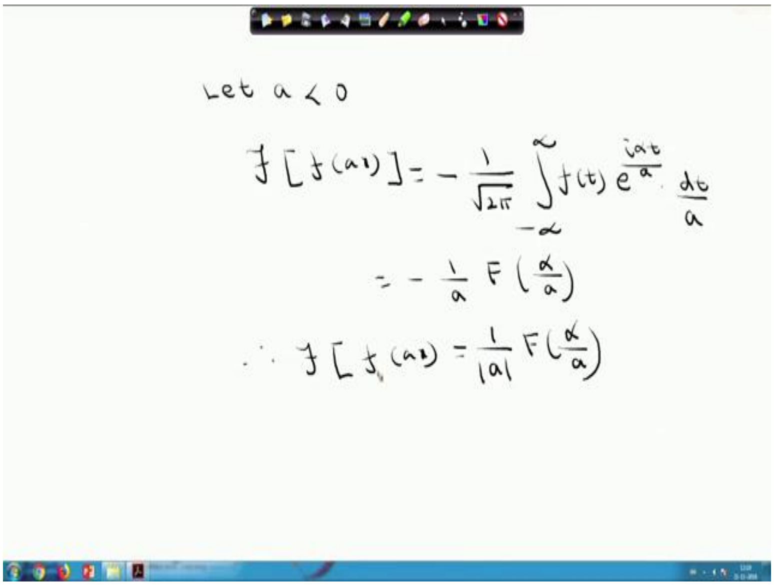

For $a < 0$, we have, if we put $ax = t$ in the right side of (1), then $t \to \infty$ as $x \to -\infty$ and $t \to -\infty$ as $x \to \infty$. Therefore,

$$\begin{aligned}\mathcal{F}[f(ax)] &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t) e^{\frac{i\alpha t}{a}} \frac{dt}{a} \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\frac{i\alpha t}{a}} \frac{dt}{a} \\ &= -\frac{1}{a} F\left(\frac{\alpha}{a}\right)\end{aligned}$$

Therefore, $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$.

Similarly (ii) and (iii) can be proved.

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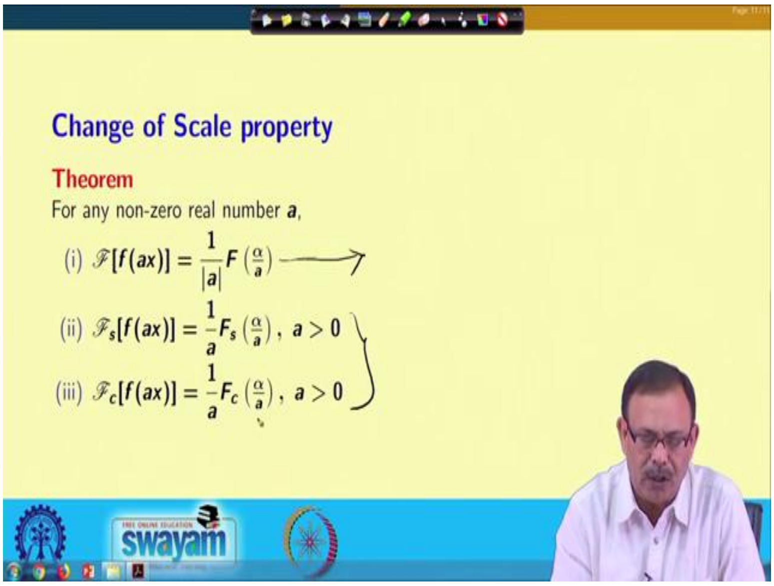

Thank you.