Transform Calculus and its Applications in Differential Equations
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Lecture - 03
Shifting properties of Laplace Transform

Welcome back. Now, we are going to revise the Laplace transform of various functions in tabular form.

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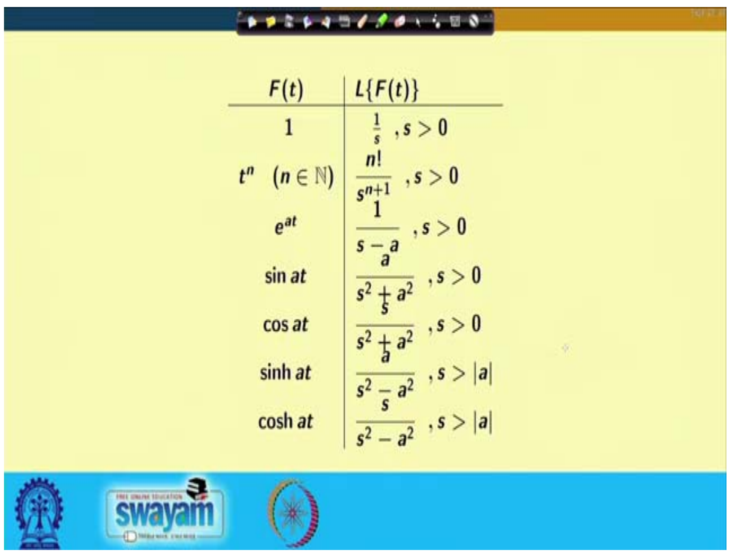

$F(t)$	$L\{F(t)\}$
1	$\frac{1}{s}$, $s > 0$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s > \lvert a \rvert$

At a glance, let us see some functions and their corresponding Laplace transforms, like: Laplace transform of 1 is $\frac{1}{s}$, where $s > 0$.

Laplace transform of t^n, where n is positive integer is $\frac{n!}{s^{n+1}}$.

Laplace transform of e^{at} is $\frac{1}{s-a}$, .

Laplace transform of $\sin at$ equals $\frac{a}{s^2+a^2}$.

Laplace transform of $\cos at$ is $\frac{s}{s^2+a^2}$.

Laplace transform of $\sinh at$ is $\frac{a}{s^2-a^2}$, where $s > |a|$.

And the Laplace transform of $\cosh at$ is equal to $\frac{s}{s^2-a^2}$, where $s > |a|$.

Now, let us see more examples to find out the Laplace transform of some complex functions, using these basic results already obtained.

The first example is we want to find out the Laplace transform of $\sin t \cos t$.

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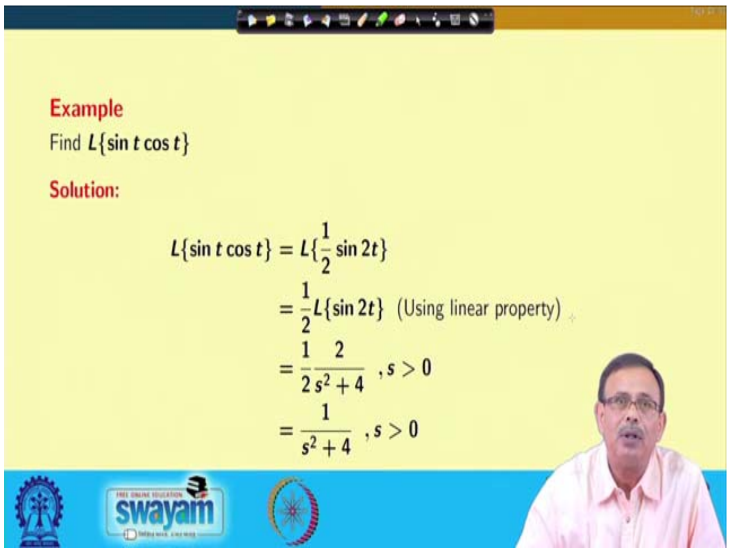

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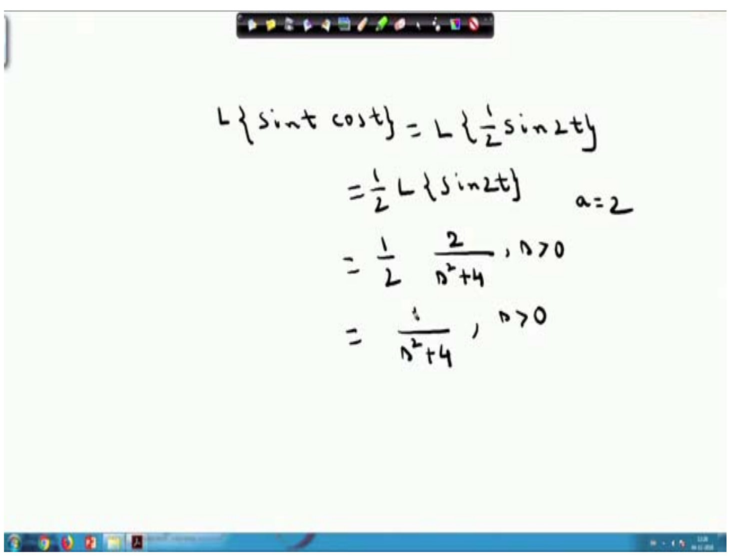

So, obviously in terms of trigonometric functions, we can write down $\sin t \cos t = \frac{1}{2}\sin 2t$. Therefore

$$L\{\sin t \cos t\} = L\left\{\frac{1}{2}\sin 2t\right\} = \frac{1}{2}L\{\sin 2t\}.$$

Now, we know $L\{\sin at\} = \frac{a}{s^2+a^2}$.

Clearly, here we have $a = 2$. So, $L\{\sin 2t\} = \frac{2}{s^2+4}$, where $s > 0$.

$$\therefore L\{\sin t \ \cos t\} = \frac{1}{2}\frac{2}{s^2+4}$$

$$= \frac{1}{s^2+4}, \qquad s > 0.$$

Let us take the next example, $F(t) = \begin{cases} 0 \ , & 0 < t < 1 \\ t \ , & 1 \le t \le 2 \\ 0 \ , & t > 2 \end{cases}$

We need to evaluate $L\{F(t)\}$.

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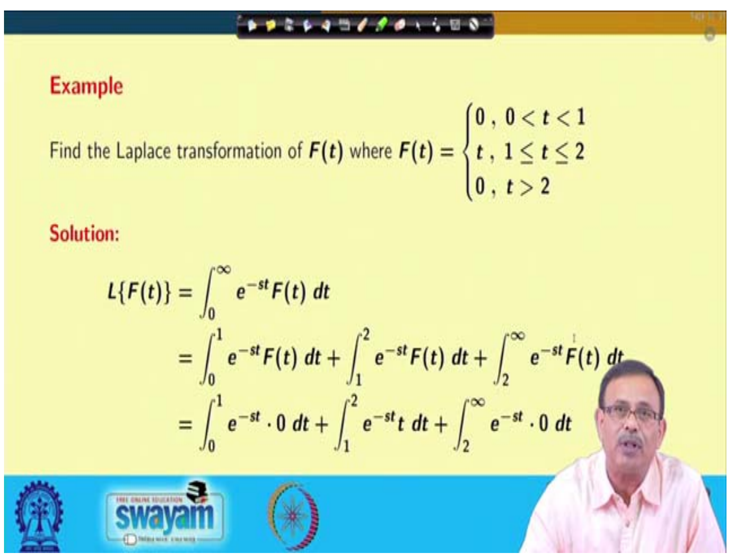

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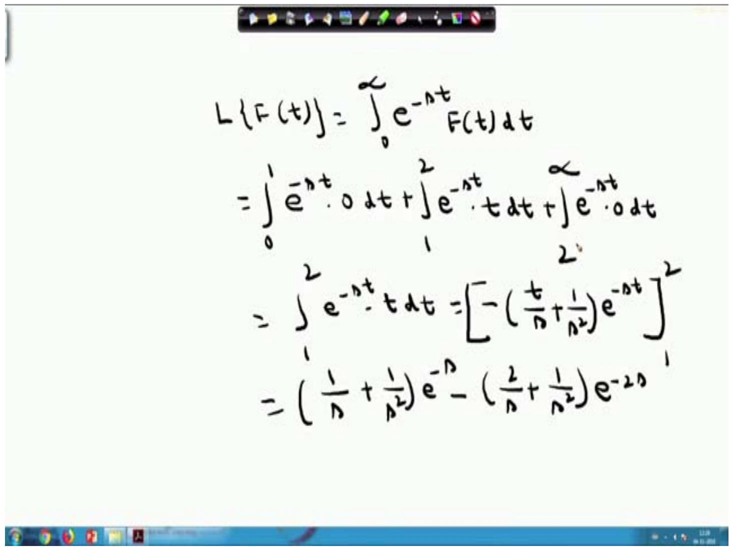

So, by the definition of Laplace transform, we have,

$$L\{F(t)\} = \int_0^\infty e^{-st}\, F(t)dt.$$

Now within this interval $[0, \infty)$, the function is defined in 3 sub-intervals, $[0,1]$, $[1,2]$ and $[2, \infty)$. So, we have to break the above integration into 3 parts as follows:

$$L\{F(t)\} = \int_0^1 e^{-st} \cdot 0dt + \int_1^2 e^{-st}\, t\, dt + \int_2^\infty e^{-st} \cdot 0\, dt$$

So, this is nothing but $\int_1^2 e^{-st}\, t\, dt$. And if we evaluate this integral using integration by parts, then it results into

$$L\{F(t)\} = \left[-\frac{t}{s}e^{-st}\right]_1^2 + \frac{1}{s}\int_1^2 e^{-st}\, dt$$

$$= \left[-\left(\frac{t}{s} + \frac{1}{s^2}\right)e^{-st}\right]_1^2$$

So, if we put the limits, we will obtain

$$L\{F(t)\} = -\left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-2s} + \left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s}.$$

So, whenever we have a function which is defined in many sub-intervals, we will simply break $[0, \infty)$ into those many intervals like we did it here, and we can solve the problem easily.

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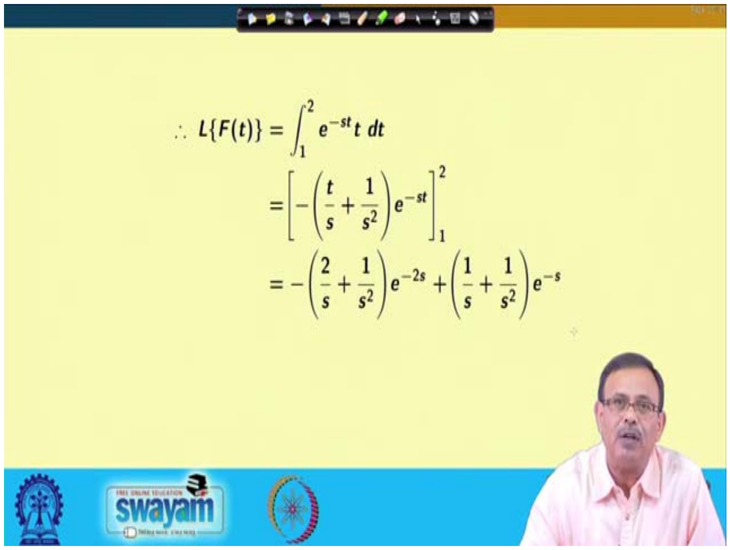

Our next example is we want to find out Laplace transform of $\sin\sqrt{t}$.

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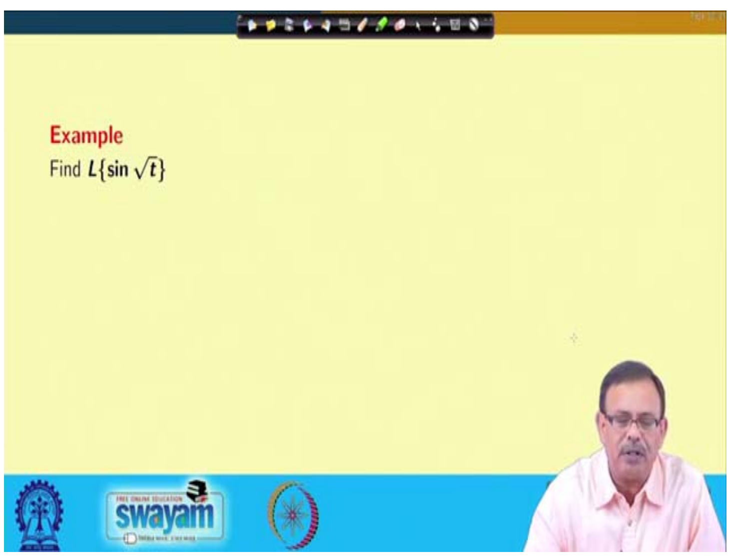

We have already discussed the Laplace transform of $\sin t$, but now it is $\sin\sqrt{t}$.

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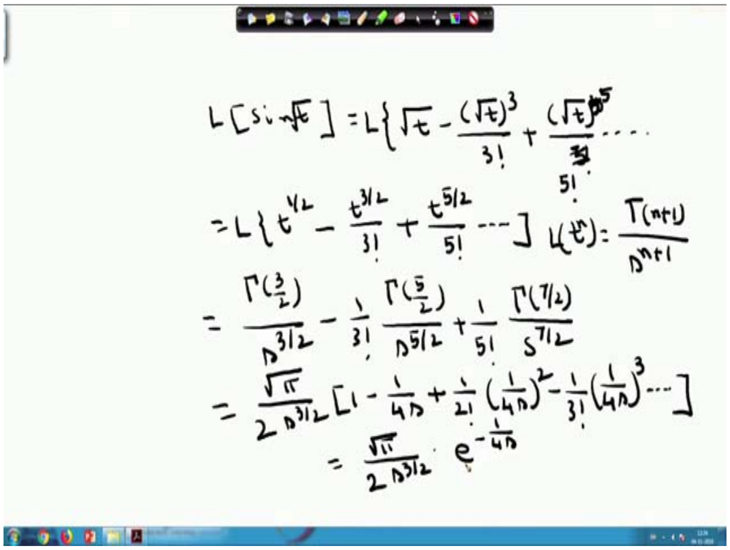

We can simply expand $\sin\sqrt{t}$ in Taylor series. After expansion, we will obtain

$$\sin\sqrt{t} = \sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - \cdots$$

which will continue to infinite number of terms.

So, now we have

$$L\{\sin\sqrt{t}\} = L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right\}$$

Laplace transform of each of these terms is known to us. Therefore, we can write it as:

$$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \cdots$$

using the linearity property.

Evaluating the individual Laplace transforms, we have,

$$L\{\sin\sqrt{t}\} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}} - \cdots.$$

NOTE: $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$.

$$\Rightarrow L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

And the expression within the third bracket is nothing but the series expansion of $e^{-\frac{1}{4s}}$. Therefore,

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}}.$$

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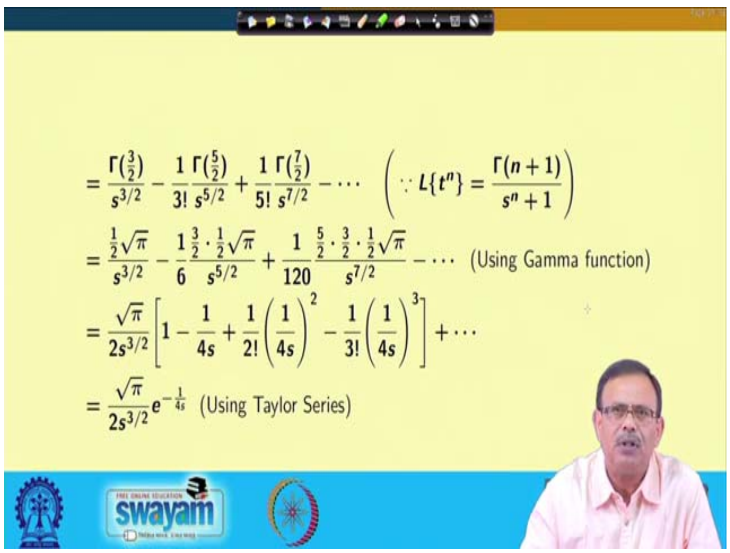

So, we see that if the function is complicated, it becomes difficult sometimes to find out the Laplace transform. So in order to make it simple, we will study certain properties of Laplace transform with which, very easily, we can find out the desired results.

We start with the First Translation (or Shifting) Property.

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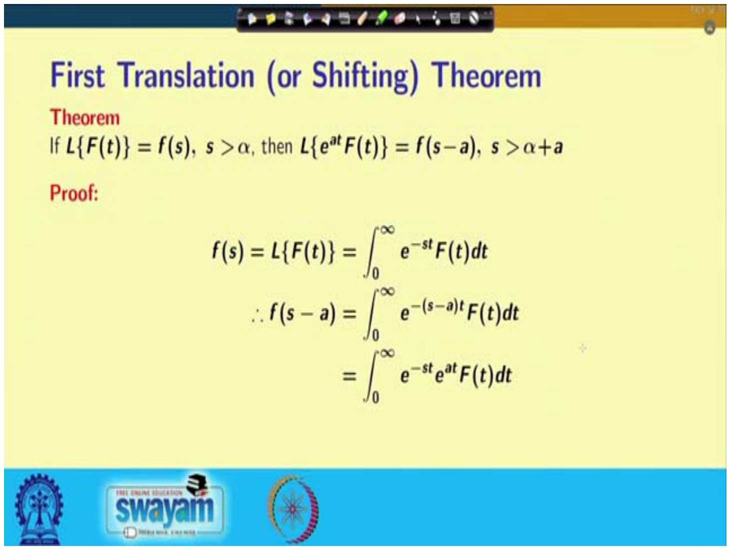

If $L\{F(t)\} = f(s)$ for $s > \alpha$, then $L\{e^{at}F(t)\} = f(s-a),\ s > \alpha + a$. Let us see, how we can derive this one.

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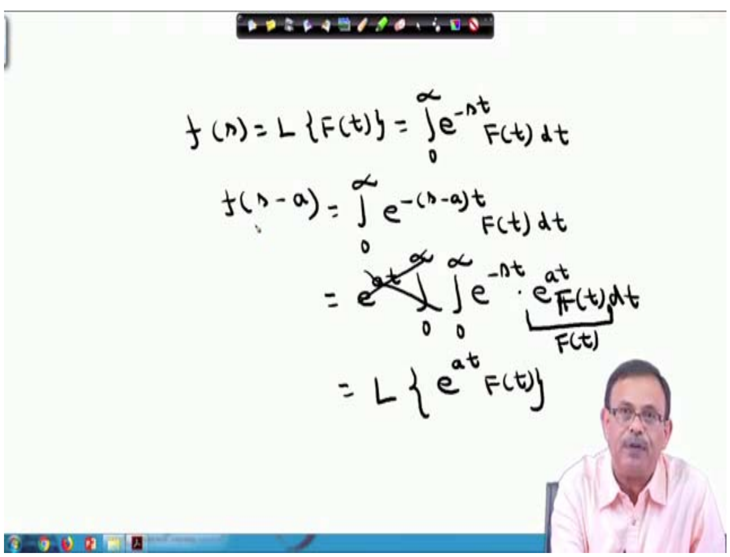

From the definition of Laplace transform, we can write,

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt.$$

Here, if s is replaced by $(s-a)$, we have,

$$f(s-a)=\int_0^\infty e^{-(s-a)t}F(t)dt$$

Now, we can break the integrand into two parts as

$$f(s-a)=\int_0^\infty e^{at}.e^{-st}F(t)dt=\int_0^\infty e^{-st}\,[e^{at}F(t)]dt$$

This we can write down, $L\{e^{at}F(t)\}$ by definition of Laplace Transform of $e^{at}F(t)$.

$$\therefore f(s-a)=\int_0^\infty e^{-st}\,[e^{at}F(t)]dt$$

$$=L\{e^{at}F(t)\}.$$

This proves the theorem.

Next is the second translation or shifting theorem.

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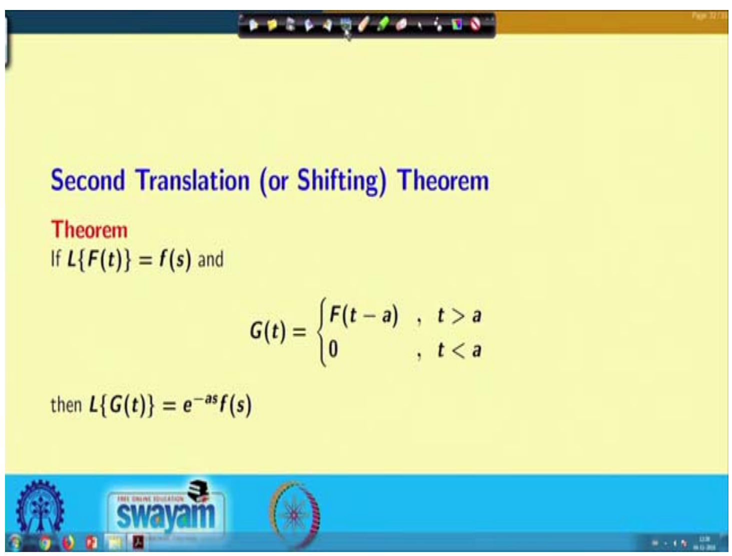

If $L\{F(t)\}=f(s)$, and if we consider a new function $G(t)=\begin{cases}F(t-a)\ , & t>a\\ \quad 0 \quad , & t<a\end{cases}$. Then $L\{G(t)\}=e^{-as}f(s)$.

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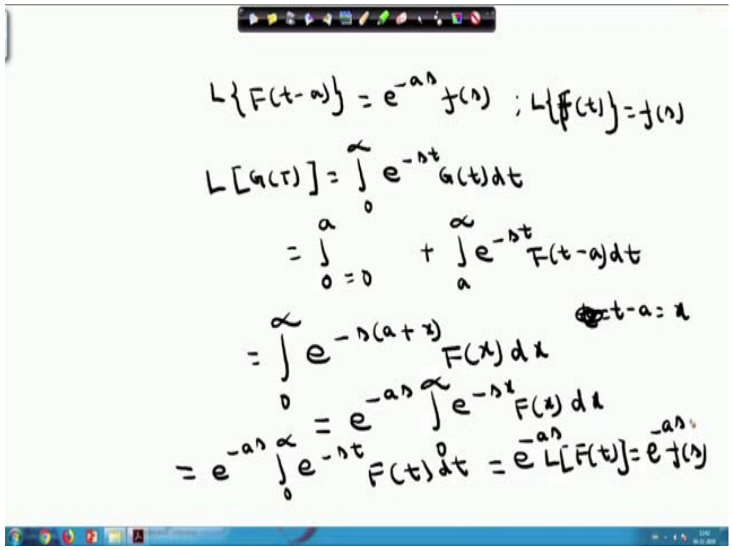

Now, we start the proof from the definition

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t) dt.$$

This we have to break into two parts according to the definition of $G(t)$ as follows:

$$L\{G(t)\} = \int_0^a e^{-st}.0\, dt + \int_a^\infty e^{-st} F(t-a) dt$$

$$= \int_a^\infty e^{-st} F(t-a) dt$$

We put $t - a = x$ so that $dt = dx$. Limits of integration will be from 0 to ∞, because at $t = a, x = 0$ and at $t = \infty, x = \infty$

$$\Rightarrow L\{G(t)\} = e^{-as} \int_0^\infty e^{-sx} F(x) dx$$

If we wish we can change the parameter x to t of the integrand, or in other sense this integral we can write as:

$$L\{G(t)\} = e^{-as} \int_0^\infty e^{-st} F(t) dt.$$

So the integrand is nothing but $L\{F(t)\} = f(s)$.

Therefore, we have,

$$L\{G(t)\} = e^{-as} f(s)$$

where $G(t)$ has been defined earlier. This completes the proof of this one.

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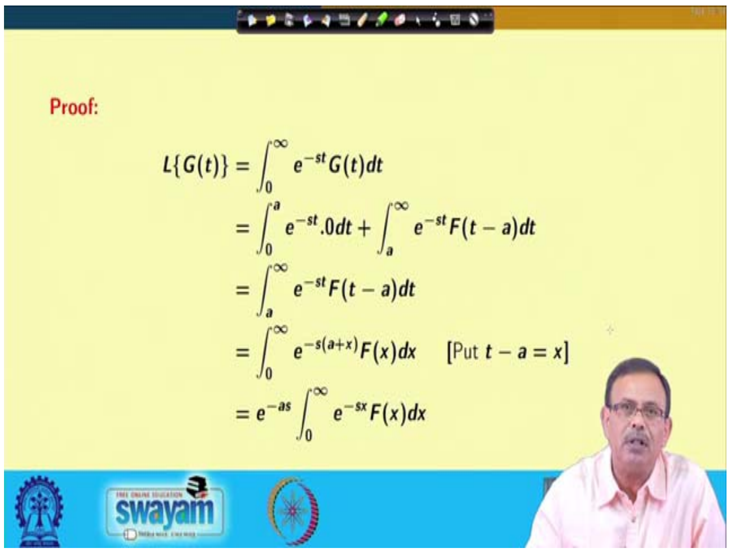

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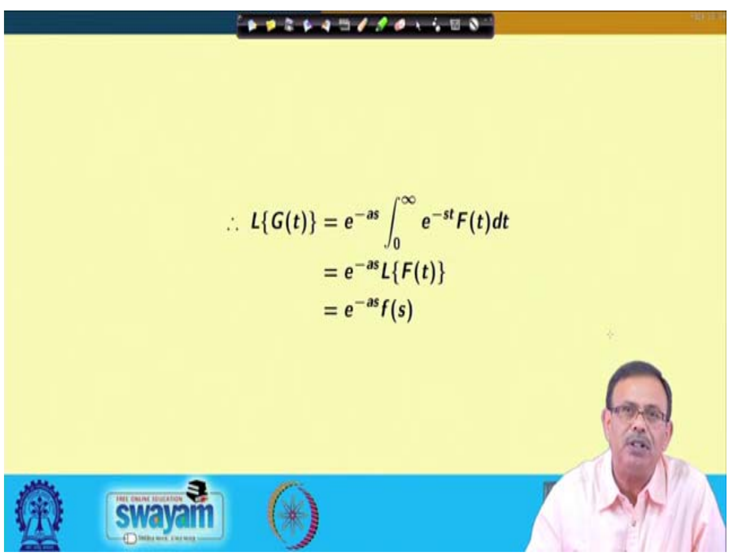

Next is change of scale property.

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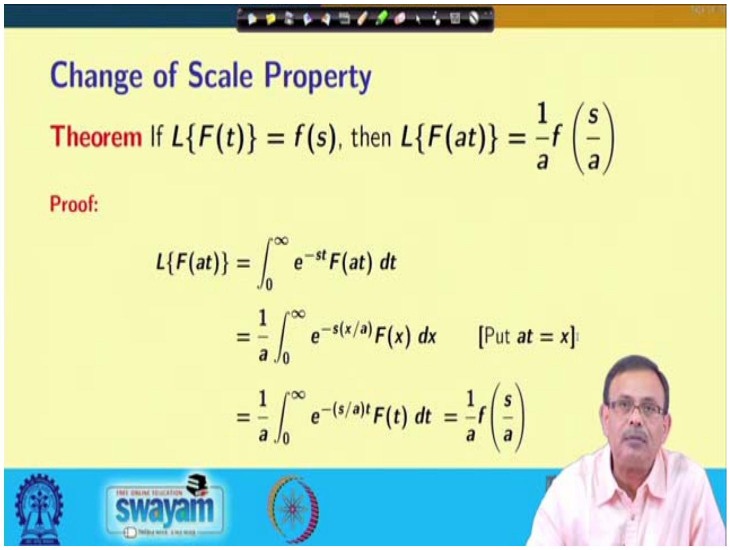

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$ where the parameter t is changed to at.

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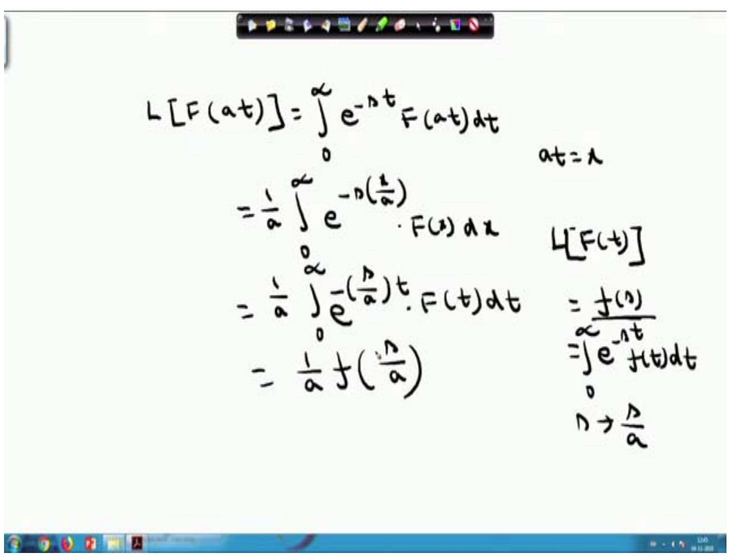

For the proof of this one, we will start with

$$L\{F(at)\} = \int_0^\infty e^{-st} F(at) dt.$$

We put in the integral $at = x$ so that $dt = \frac{1}{a}dx$ and the limits of integration will remain unchanged. Then

$$L\{F(at)\} = \frac{1}{a}\int_0^\infty e^{-(s/a)x}F(x)dx$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t}F(t)dt$$

$$= \frac{1}{a}f\left(\frac{s}{a}\right).$$

So, the question arises “what is the use of these properties?” These properties will actually help us to find out the Laplace transform of various complicated functions. Let us see, how we can use these properties in solving complicated problems.

First let us take this example. We want to find out $L\{t^3e^{-3t}\}$.

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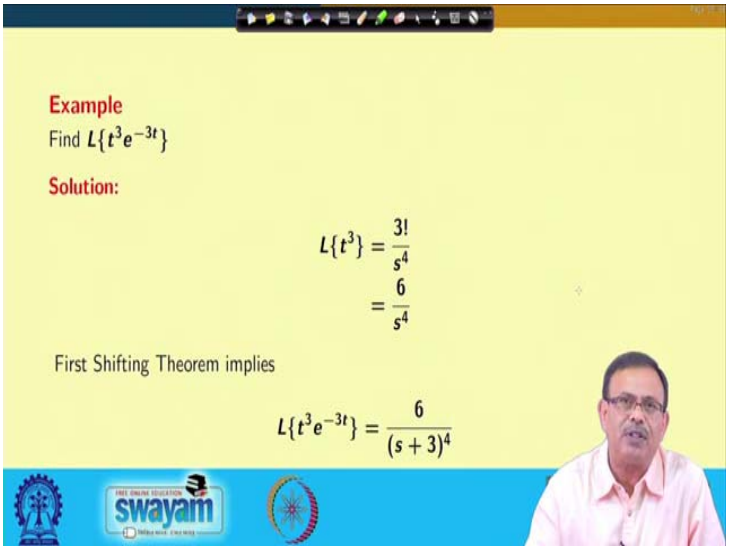

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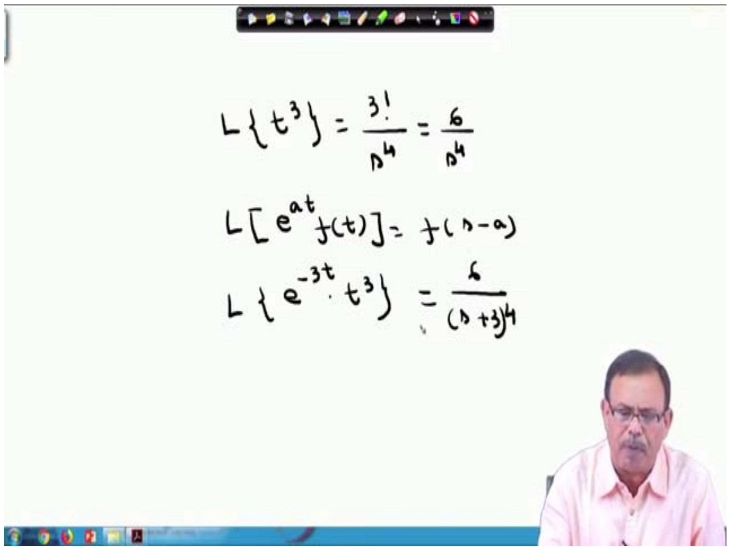

So, we know $L\{t^3\} = \frac{\Gamma(4)}{s^4} = \frac{3!}{s^4} = \frac{6}{s^4} = f(s)$ (say). Also from the first shifting theorem, we know that,

$$L\{e^{at} F(t)\} = f(s-a)$$

where $L\{F(t)\} = f(s)$.

$$\therefore L\{t^3 e^{-3t}\} = \frac{6}{(s+3)^4}.$$

Here our problem was to find out the Laplace transform of $t^3 e^{-3t}$. If we had to use the direct method, we would have to evaluate one complicated integral. But with a very simple calculation, using the first translation property, we are able to find out the Laplace transform of such complicated function as well. This is the use of the properties which we have discussed. Let us see some more examples.

Let us see another complicated problem, to find $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$.

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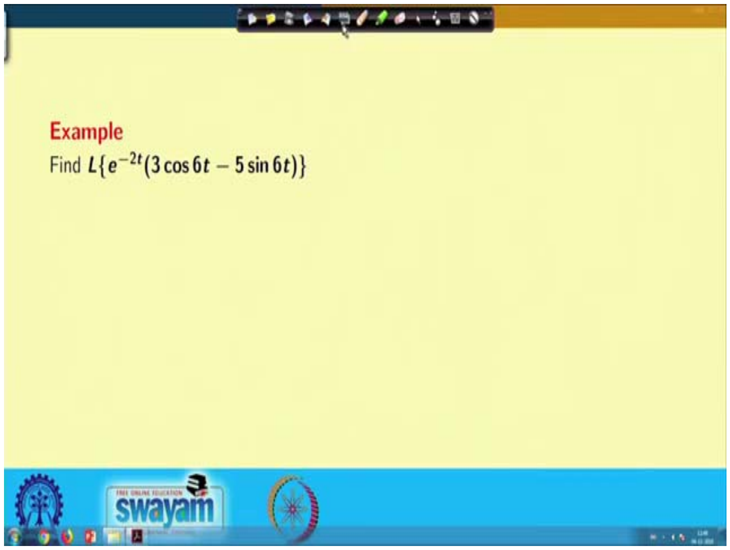

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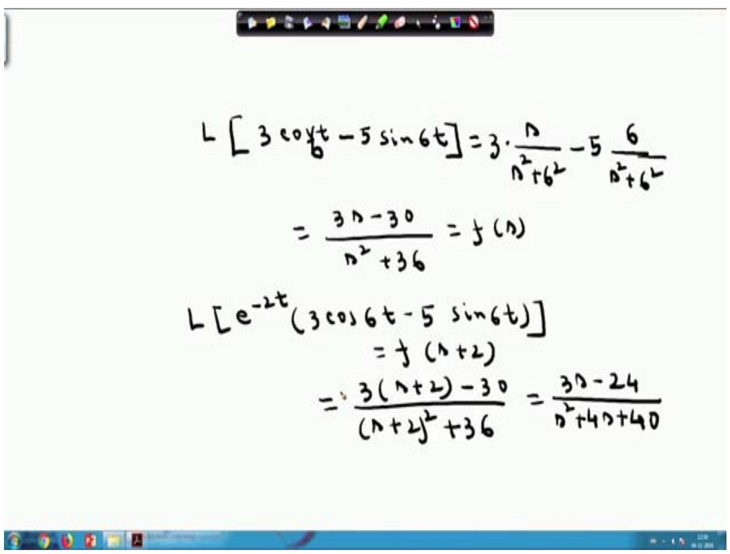

So, we will start from $L\{3\cos 6t - 5\sin 6t\}$. We know that

$$L\{\cos at\} = \frac{s}{s^2+a^2} \quad \text{and} \quad L\{\sin at\} = \frac{a}{s^2+a^2}.$$

So, we can find out the Laplace transform of $(3\cos 6t - 5\sin 6t)$ very easily.

$$L\{3\cos 6t - 5\sin 6t\} = 3L\{\cos 6t\} - 5L\{\sin 6t\}$$

$$\Rightarrow L\{3\cos 6t - 5\sin 6t\} = 3 \cdot \frac{s}{s^2+36} - 5 \cdot \frac{6}{s^2+36}$$

$$= \frac{3s-30}{s^2+36}$$

$$= f(s) \text{ (say).}$$

Therefore, $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = f(s+2)$, using First shifting theorem.

$$\therefore L\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = f(s+2)$$

$$= \frac{3(s+2)-30}{(s+2)^2+36}$$

$$= \frac{3s-24}{s^2+4s+40}.$$

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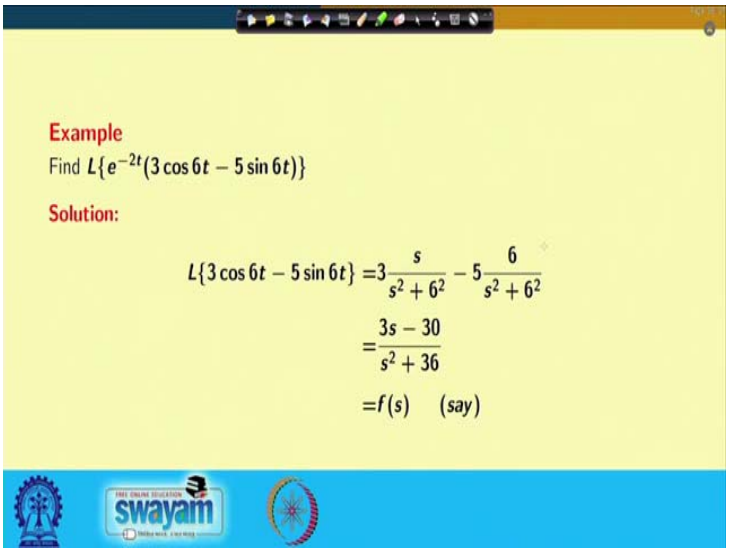

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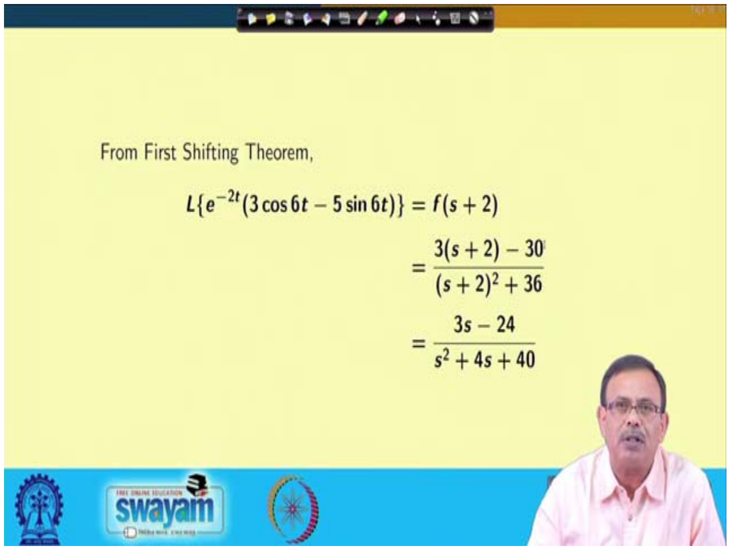

Let us see, another problem $L\{e^t \sin^2 t\}$.

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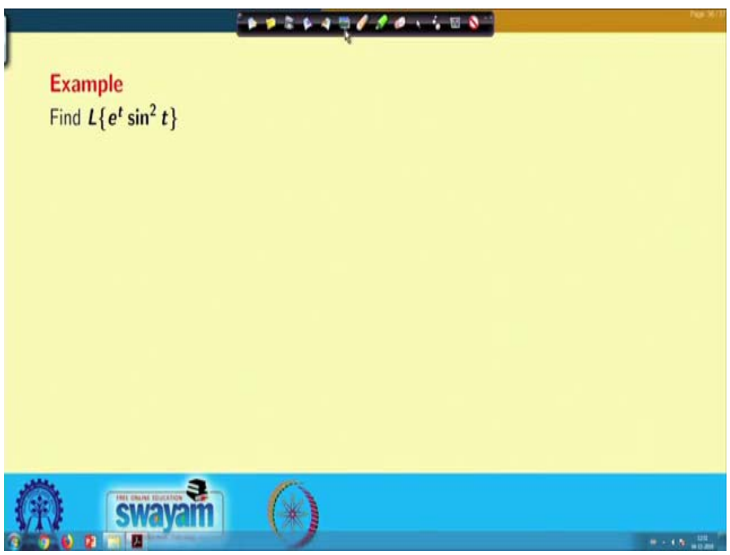

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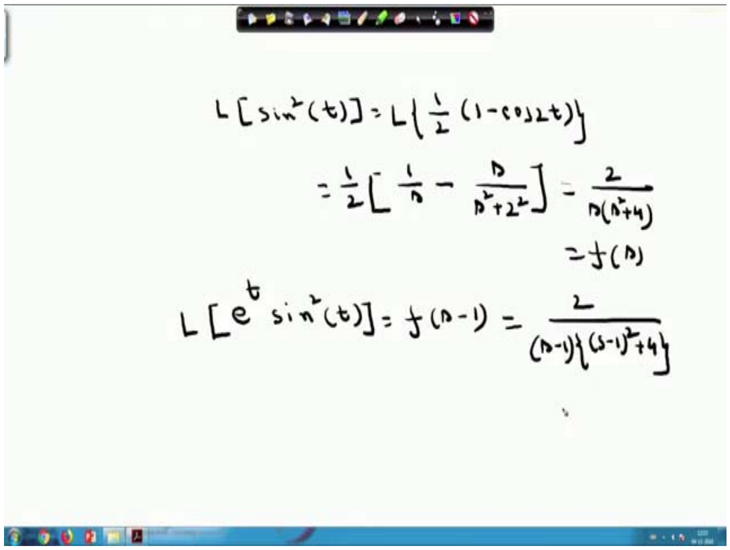

Since we know Laplace transform of $e^{at}F(t)$, so if we know Laplace transform of $F(t)$, from there we can easily calculate Laplace transform of $e^{at}F(t)$ using First translation property. Therefore, for this problem we assume $F(t) = \sin^2 t$.

So, first we will try to find out the Laplace transform of $\sin^2 t$. For that, using trigonometry we can obtain it as $\frac{1}{2}(1 - \cos 2t)$ i.e.,

$$L\{\sin^2 t\} = L\left\{\frac{1}{2}(1 - \cos 2t)\right\}$$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 2t\}.$$

We know the Laplace transform of 1, and we also know the Laplace transform of $\cos 2t$, so we can find out easily:

$$L\{\sin^2 t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2 + 4}\right] = f(s) \quad \text{(say)}.$$

Therefore,

$$L\{e^t \sin^2 t\} = f(s - 1)$$

$$= \frac{1}{2}\left[\frac{1}{s - 1} - \frac{s - 1}{(s - 1)^2 + 4}\right]$$

$$\Rightarrow L\{e^t \sin^2 t\} = \frac{2}{(s-1)[(s-1)^2+4]}.$$

In the next classes, we will go through some more properties on Laplace transforms and discuss some more examples. Thank you.