

Transform Calculus and Its Application in Differential Equations
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Lecture - 27
Introduction to Fourier Transform

In the last lecture, we started the Fourier integral representation, where we derived the following two equations, namely equation (8) and equation (9) (considering equation numbers from the previous lecture)

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cos at \, dt \right) \cos ax \, da \quad (8)$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \sin at \, dt \right) \sin ax \, da \quad (9)$$

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$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cos at \, dt \right) \cos ax \, da \quad (8)$
 $F(a) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos at \, dt \quad (9)$
 $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(a) \cos ax \, da \quad (10)$
 $F(a) \rightarrow \text{Fourier cosine Transform of } f(x)$

Let us see equation (8) first.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cos at \, dt \right) \cos ax \, da \quad (8)$$

Let,

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos at dt \quad (10)$$

Then, equation (8) becomes

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\alpha) \cos \alpha x d\alpha \quad (11)$$

This $F(\alpha)$ is called the Fourier cosine transform of $f(x)$ and it is represented by the equation (10).

Suppose $F(\alpha)$ is given and $f(x)$ is an unknown function, then using equation (11), we can find $f(x)$.

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$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \sin at dt \right) \sin ax d\alpha \quad (9)$$

$$\phi(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin at dt \quad (12)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \phi(\alpha) \sin ax d\alpha \quad (13)$$

$\phi(\alpha) \rightarrow$ Fourier sine Transform of $f(x)$

Similarly, if we consider the equation 9,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \sin at dt \right) \sin ax d\alpha \quad (9)$$

Let,

$$\phi(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin at dt \quad (12)$$

Then, equation (9) becomes

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \phi(\alpha) \sin \alpha x \, d\alpha \quad (13)$$

Therefore, in this case, $\phi(\alpha)$ is known as Fourier sine transform of $f(x)$.

Suppose $\phi(\alpha)$ is given and $f(x)$ is an unknown function, then using equation (13), we can find $f(x)$.

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In (8), we set

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \alpha t \, dt \quad (10)$$

\therefore (8) takes the form

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\alpha) \cos \alpha x \, d\alpha \quad (11)$$

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Let us consider the particular cases of formula (7)

- Let $f(x)$ be even. Then $f(t) \cos \alpha t$ is an even function, while $f(t) \sin \alpha t$ is odd and we have

$$\int_{-\infty}^{\infty} f(t) \cos \alpha t \, dt = 2 \int_0^{\infty} f(t) \cos \alpha t \, dt$$

and

$$\int_{-\infty}^{\infty} f(t) \sin \alpha t \, dt = 0$$

Formula (7) in this case takes the form

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cos \alpha t \, dt \right) \cos \alpha x \, d\alpha$$

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The function $F(\alpha)$ is called the **Fourier Cosine Transform** of the function $f(x)$.

If in (11), we consider $F(\alpha)$ is given and $f(t)$ as the unknown function, then it is an integral equation of the function $f(t)$.

Formula (11) gives the solution of this equation.

On the basis of (9), we can write the following equations:

$$\phi(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \alpha t \, dt \quad (12)$$
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \phi(\alpha) \sin \alpha x \, d\alpha \quad (13)$$

The function $\phi(\alpha)$ is called the **Fourier Sine Transform**.

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2. Let $f(x)$ be odd. Then from (7), we obtain

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \sin \alpha t \, dt \right) \sin \alpha x \, d\alpha \quad (9)$$

Let it be noted again that at the points of discontinuity, one should write the following expression in place of $f(x)$ in the left hand members of (8) and (9):

$$\frac{f(x+0) + f(x-0)}{2}$$

For Fourier sine transform, the kernel is $\sin at$ whereas, for Fourier cosine transform, the kernel is $\cos at$, similar to Laplace transform where it was e^{-st} . Next we want to study the Fourier integral representation in the complex form.

Equations from now on will be renumbered, starting from (1).

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Fourier Integral Representation in Complex Form

Fourier Integral Representation of the function $f(x)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad (1)$$

This can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha$$

as $\cos \alpha(t-x)$ is an even function of α .

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$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad \text{--- (1)}$$
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad \text{--- (2)}$$

~~cos~~ $\cos \alpha(t-x)$ is even func. of α

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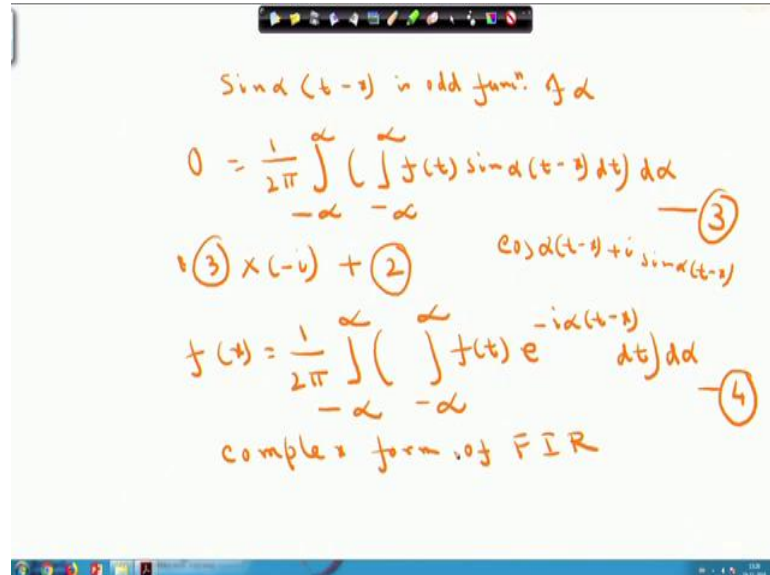
From Fourier integral representation of the function $f(x)$, we have

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad (1)$$

Since $\cos \alpha(t-x)$ is an even function of α , therefore, using the property of definite integral, we have,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad (2)$$

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Since $\sin \alpha(t-x)$ is an odd function of α , so we must have,

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \sin \alpha(t-x) dt \right) d\alpha \quad (3)$$

Therefore, $(2) - i \times (3)$ implies

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\alpha(t-x)} dt \right) d\alpha \quad (4)$$

This is the complex form of Fourier integral representation of $f(x)$ as given by equation (4).

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Fourier Transform

Kaise (5)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{i\alpha(x-t)} dt \right) d\alpha \quad (5)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \right) d\alpha \quad (6)$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \quad (7)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \quad (8)$$

From equation (4) we get,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{i\alpha(x-t)} dt \right) d\alpha \quad (5)$$

Equation (5) can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \right) d\alpha \quad (6)$$

In equation (6), we are denoting

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \quad (7)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \quad (8)$$

If equation (7) exists, then we call it as the Fourier transform of the function $f(t)$ where the kernel we are assuming as $e^{-i\alpha t}$ and denoted as

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \quad (9)$$

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$f [f(t)] = F(\alpha)$
 $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \quad (9)$
 $F(\alpha) \quad (8) \quad f(x) = --$
 Inverse F.T.
 $f(x) = f^{-1}[F(\alpha)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \quad (10)$

If equation (8) exists, then this is called the inverse Fourier transform of $F(\alpha)$ and denoted as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \quad (10)$$

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Also since $\sin \alpha(t - x)$ is an odd function of α , we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \sin \alpha(t - x) dt \right) d\alpha \quad (3)$$

Multiplying (3) by $-i$ and adding to (2),

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\alpha(t-x)} dt \right) d\alpha \quad (4)$$

which is the complex form of FIR

i.e., $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{i\alpha(x-t)} dt \right) d\alpha$

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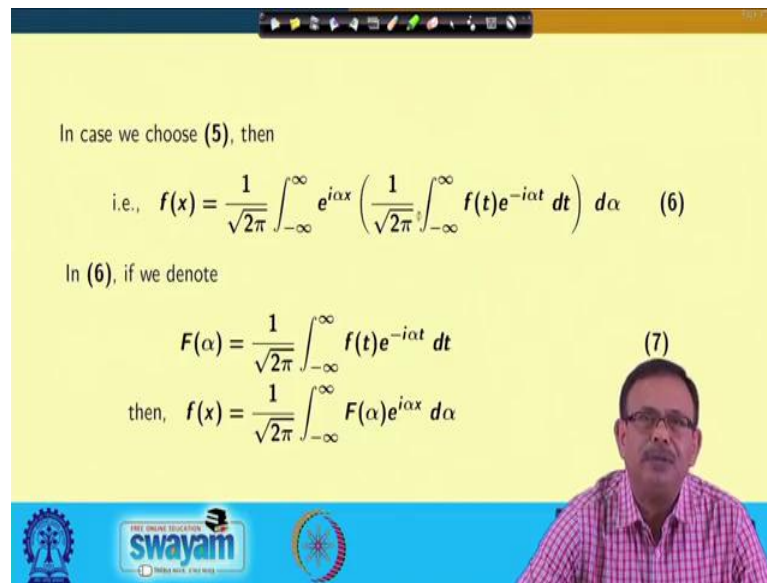
In case we choose (5), then

$$\text{i.e., } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \right) d\alpha \quad (6)$$

In (6), if we denote

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \quad (7)$$

then, $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha$

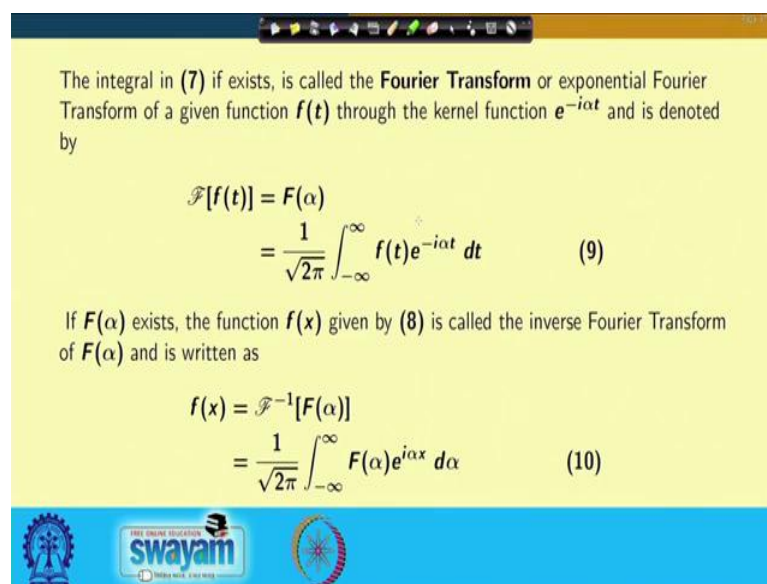


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The integral in (7) if exists, is called the **Fourier Transform** or exponential Fourier Transform of a given function $f(t)$ through the kernel function $e^{-i\alpha t}$ and is denoted by

$$\mathcal{F}[f(t)] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\alpha t} dt \quad (9)$$

If $F(\alpha)$ exists, the function $f(x)$ given by (8) is called the inverse Fourier Transform of $F(\alpha)$ and is written as

$$f(x) = \mathcal{F}^{-1}[F(\alpha)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \quad (10)$$


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In (4), $\alpha = -\omega$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{i\omega(t-x)} dt \right) (-d\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega(x-t)} dt \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\alpha(x-t)} dt \right) d\alpha \quad (4a)$$

In (4), if we put $\alpha = -\omega$, then,

$$f(x) = \frac{1}{2\pi} \int_{\infty}^{-\infty} \left(\int_{-\infty}^{\infty} f(t) e^{i\omega(t-x)} dt \right) (-d\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega(x-t)} dt \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\alpha(x-t)} dt \right) d\alpha \quad (4a)$$

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consider (4a)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \right) d\alpha \quad (11)$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \quad (12)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha \quad (13)$$

So, instead of (4), if we consider (4a), then we will get some other form,

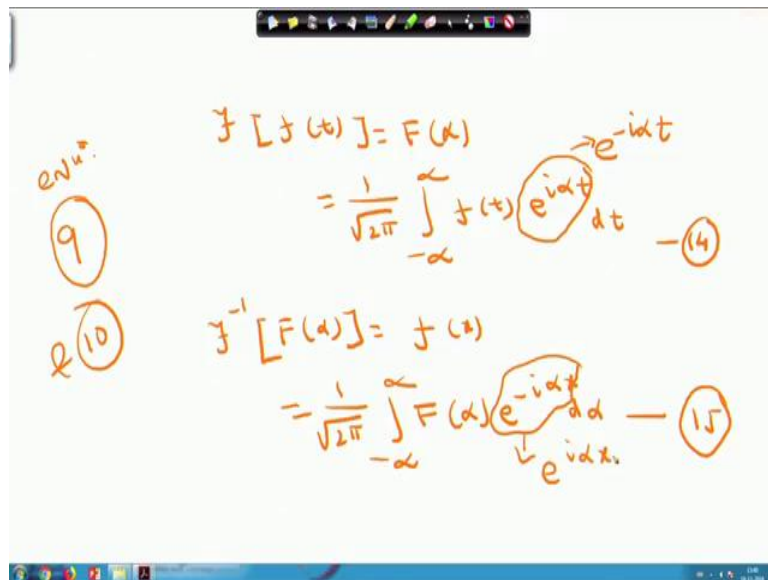
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \right) d\alpha \quad (11)$$

In equation (11), say we are denoting

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \quad (12)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha \quad (13)$$

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If equation (12) exists, then we call it as the Fourier transform of the function $f(t)$ where the kernel is assumed to be $e^{i\alpha t}$ and denoted as

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt \quad (14)$$

If equation (13) exists, then this is called the inverse Fourier transform of $F(\alpha)$ and denoted as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha \quad (15)$$

So, in both cases, we will get the Fourier transform and please note that in the books, somewhere, they have used the kernel as $e^{-i\alpha t}$, somewhere they have used $e^{i\alpha t}$. Both are correct, so, either can be used as kernel and we can obtain the Fourier transform of the function accordingly. Thank you.