Transform Calculus and Its Application in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

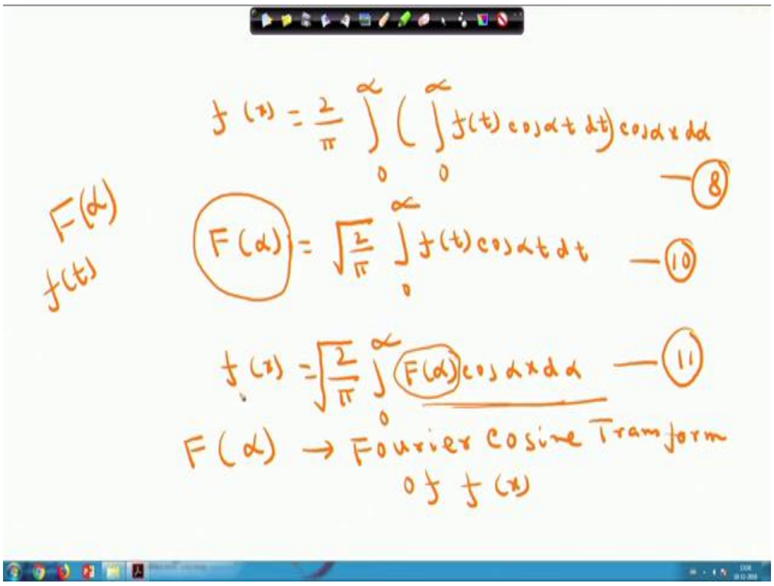
Lecture - 27
Introduction to Fourier Transform

In the last lecture, we started the Fourier integral representation, where we derived the following two equations, namely equation (8) and equation (9) (considering equation numbers from the previous lecture)

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos\alpha t\,dt\right)\cos\alpha x\,d\alpha \quad (8)$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\sin\alpha t\,dt\right)\sin\alpha x\,d\alpha \quad (9)$$

(Refer Slide Time: 00:35)



Let us see equation (8) first.

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos\alpha t\,dt\right)\cos\alpha x\,d\alpha \quad (8)$$

Let,

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \alpha t \, dt \tag{10}$$

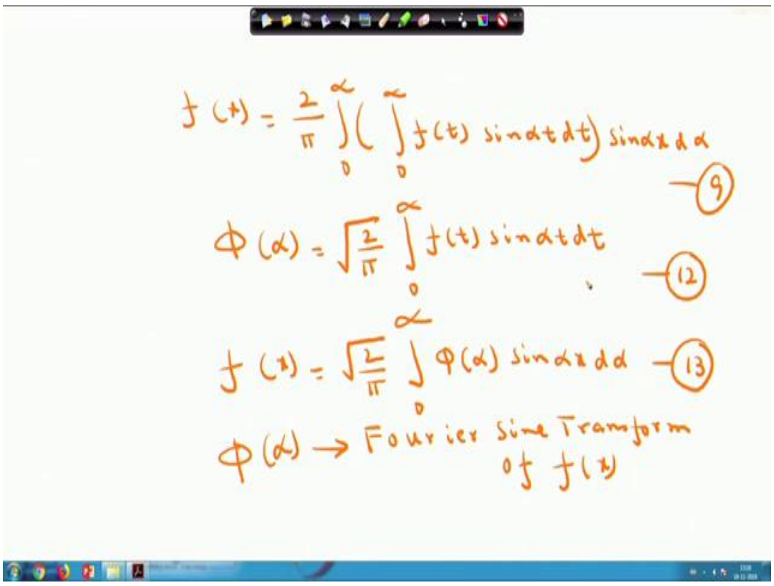
Then, equation (8) becomes

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\alpha) \cos \alpha x \, d\alpha \tag{11}$$

This $F(\alpha)$ is called the Fourier cosine transform of $f(x)$ and it is represented by the equation (10).

Suppose $F(\alpha)$ is given and $f(x)$ is an unknown function, then using equation (11), we can find $f(x)$.

(Refer Slide Time: 04:15)



Similarly, if we consider the equation 9,

$$f(x) = \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty f(t) \sin \alpha t \, dt \right) \sin \alpha x \, d\alpha \tag{9}$$

Let,

$$\phi(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \alpha t \, dt \tag{12}$$

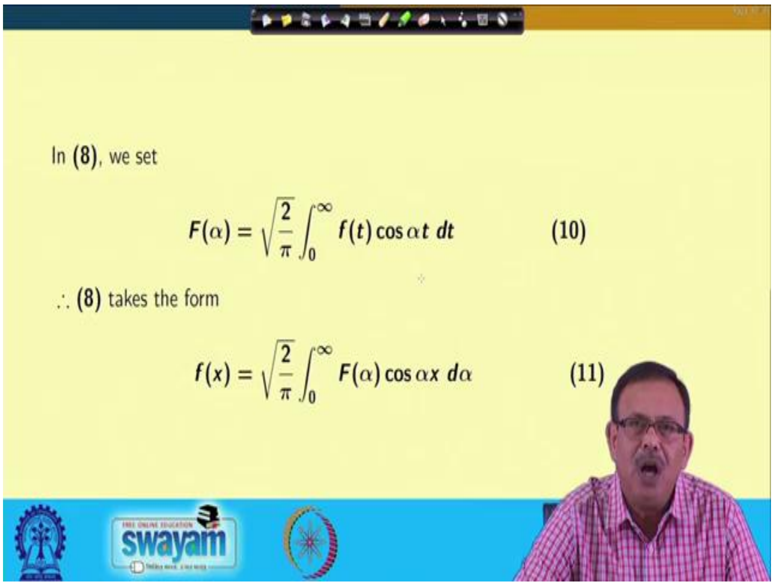
Then, equation (9) becomes

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi(\alpha) \sin \alpha x \, d\alpha \tag{13}$$

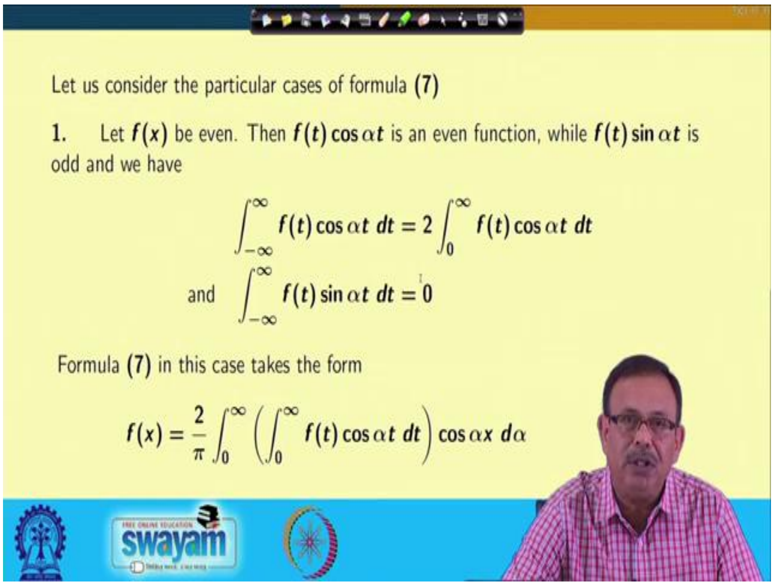
Therefore, in this case, $\phi(\alpha)$ is known as Fourier sine transform of $f(x)$.

Suppose $\phi(\alpha)$ is given and $f(x)$ is an unknown function, then using equation (13), we can find $f(x)$.

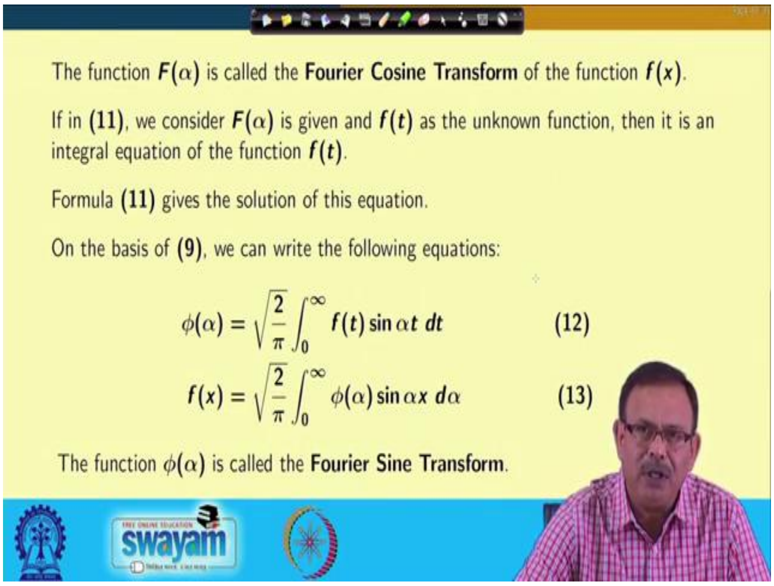
(Refer Slide Time: 06:15)



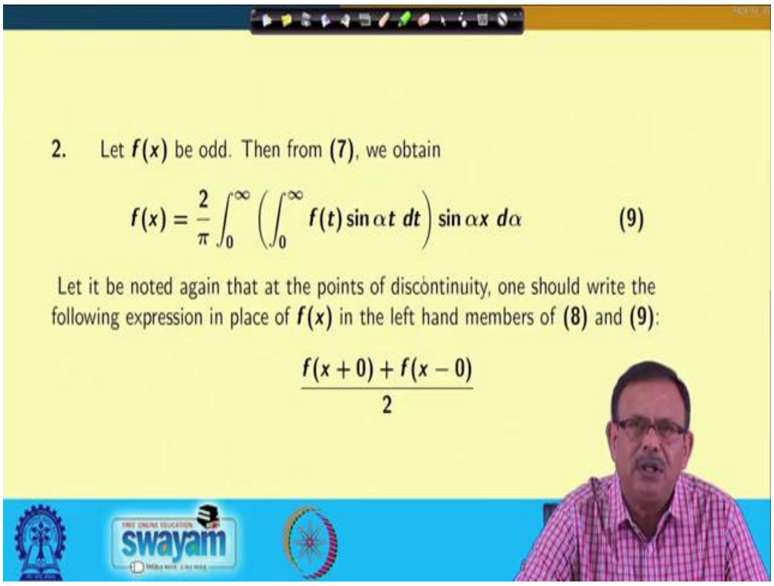
(Refer Slide Time: 06:23)



(Refer Slide Time: 06:57)



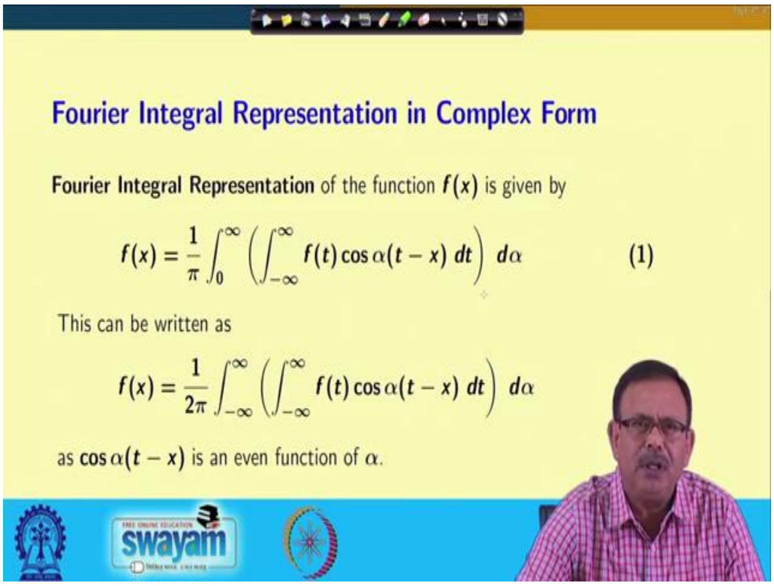
(Refer Slide Time: 07:49)



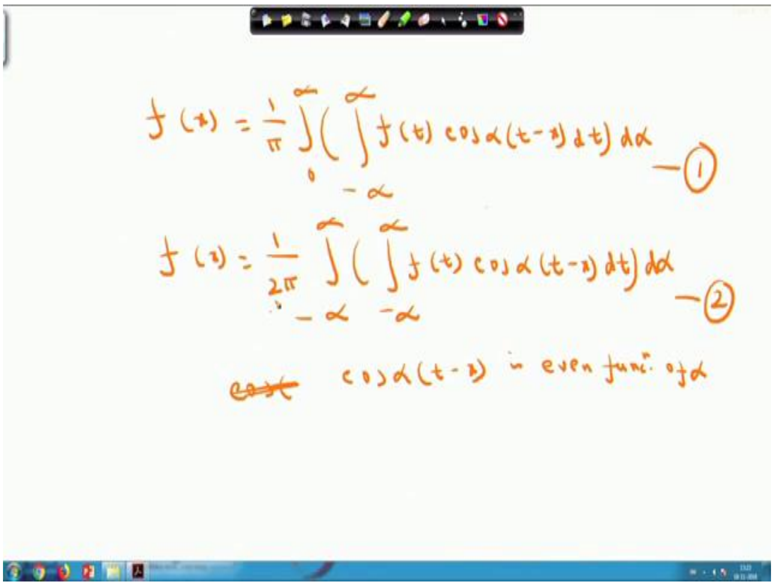
For Fourier sine transform, the kernel is $\sin \alpha t$ whereas, for Fourier cosine transform, the kernel is $\cos \alpha t$, similar to Laplace transform where it was e^{-st}. Next we want to study the Fourier integral representation in the complex form.

Equations from now on will be renumbered, starting from (1).

(Refer Slide Time: 08:57)



(Refer Slide Time: 09:15)



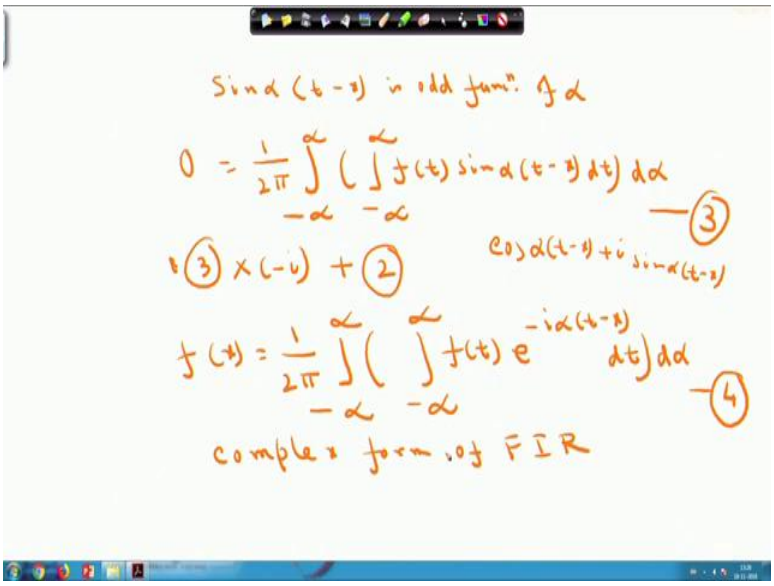
From Fourier integral representation of the function $f(x)$, we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dt\right)d\alpha \tag{1}$$

Since $\cos\alpha(t-x)$ is an even function of α, therefore, using the property of definite integral, we have,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right)d\alpha \tag{2}$$

(Refer Slide Time: 11:11)



Since $\sin\alpha(t-x)$ is an odd function of α, so we must have,

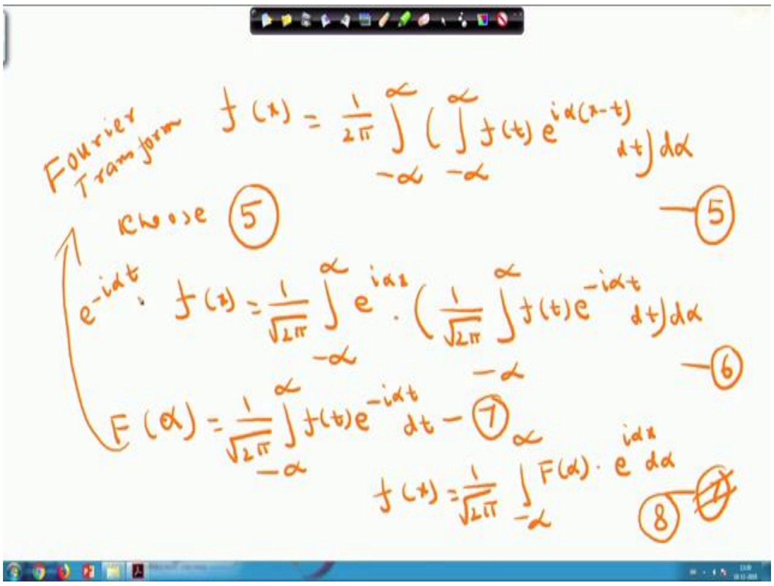
$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\right)d\alpha \tag{3}$$

Therefore, $(2) - i \times (3)$ implies

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\alpha(t-x)}\,dt\right)d\alpha \tag{4}$$

This is the complex form of Fourier integral representation of $f(x)$ as given by equation (4).

(Refer Slide Time: 13:45)



From equation (4) we get,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{i\alpha(x-t)}\,dt\right)d\alpha \tag{5}$$

Equation (5) can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}\,dt\right)d\alpha \tag{6}$$

In equation (6), we are denoting

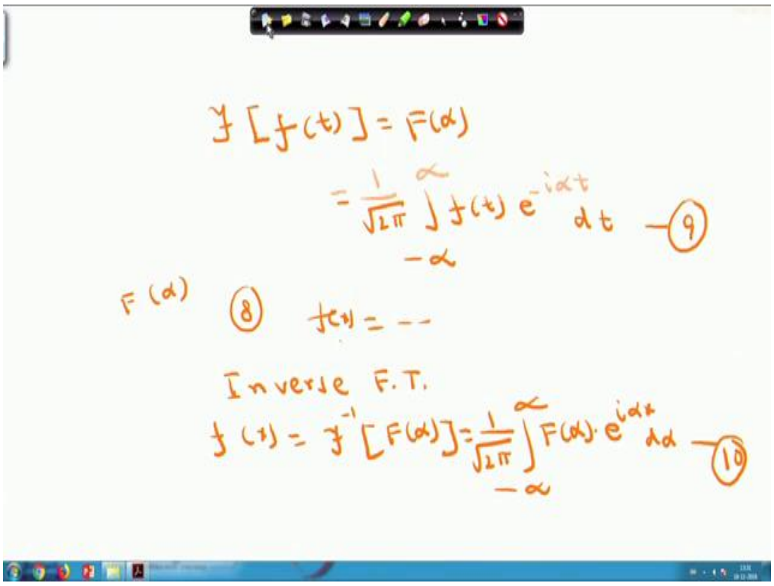
$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}\,dt \tag{7}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}\,d\alpha \tag{8}$$

If equation (7) exists, then we call it as the Fourier transform of the function $f(t)$ where the kernel we are assuming as $e^{-i\alpha t}$ and denoted as

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}\,dt \tag{9}$$

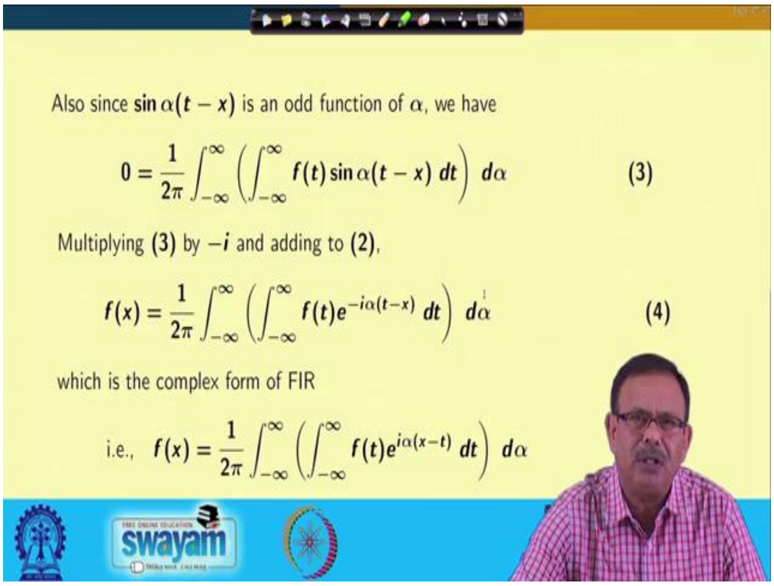
(Refer Slide Time: 17:35)



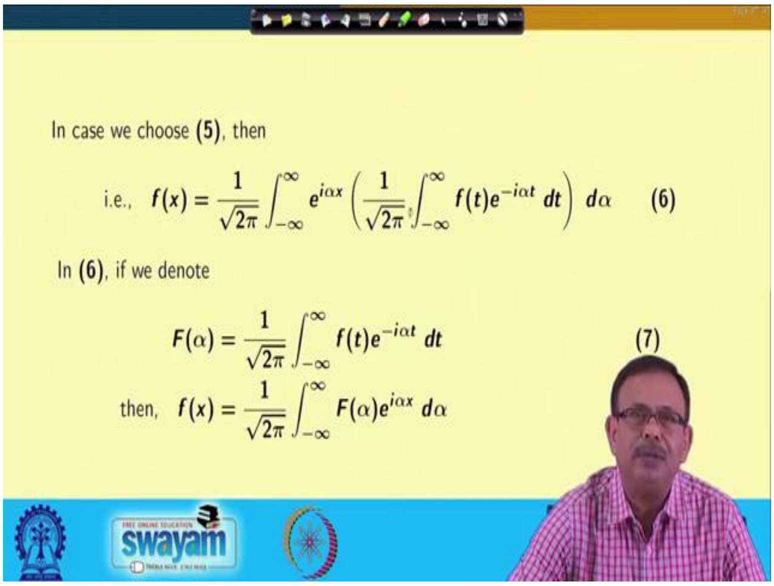
If equation (8) exists, then this is called the inverse Fourier transform of $F(\alpha)$ and denoted as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} \, d\alpha \tag{10}$$

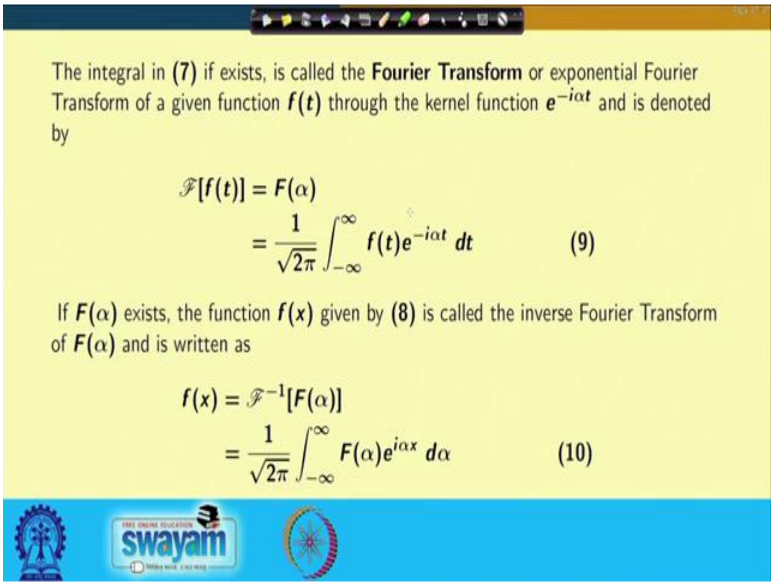
(Refer Slide Time: 19:41)



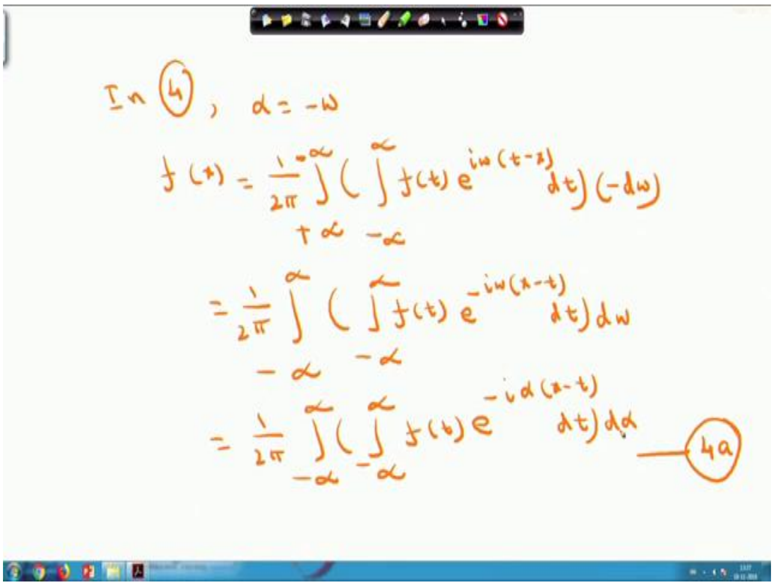
(Refer Slide Time: 20:37)



(Refer Slide Time: 21:51)



(Refer Slide Time: 22:51)



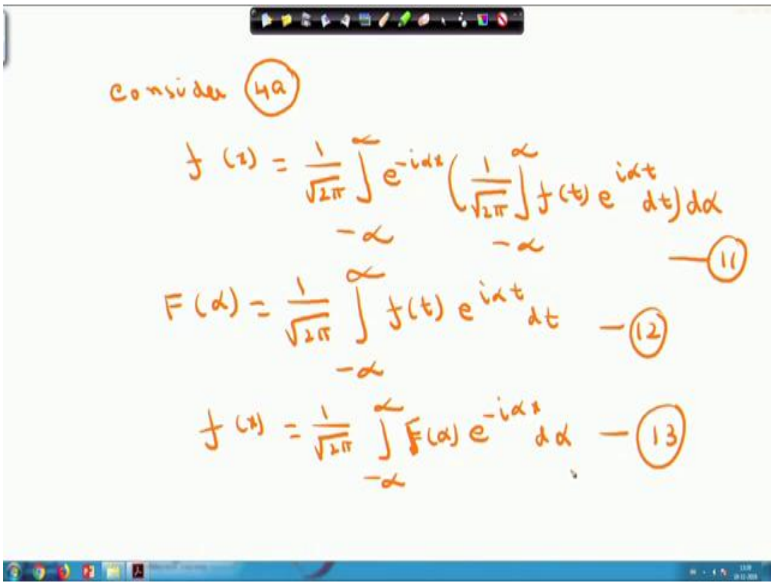
In (4), if we put $\alpha = -\omega$, then,

$$f(x) = \frac{1}{2\pi}\int_{\infty}^{-\infty}\left(\int_{-\infty}^{\infty} f(t)e^{i\omega(t-x)}\,dt\right)(-d\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega(x-t)}\,dt\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\alpha(x-t)}\,dt\right)d\alpha \qquad (4a)$$

(Refer Slide Time: 25:07)



So, instead of (4), if we consider $(4a)$, then we will get some other form,

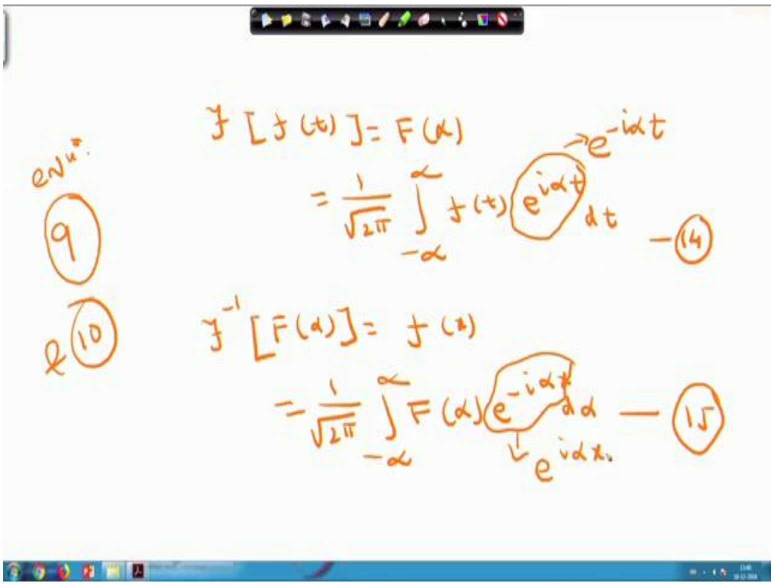
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t}\, dt \right) d\alpha \tag{11}$$

In equation (11), say we are denoting

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t}\, dt \tag{12}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x}\, d\alpha \tag{13}$$

(Refer Slide Time: 26:53)



If equation (12) exists, then we call it as the Fourier transform of the function $f(t)$ where the kernel is assumed to be $e^{i\alpha t}$ and denoted as

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t}\, dt \tag{14}$$

If equation (13) exists, then this is called the inverse Fourier transform of $F(\alpha)$ and denoted as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x}\, d\alpha \tag{15}$$

So, in both cases, we will get the Fourier transform and please note that in the books, somewhere, they have used the kernel as $e^{-i\alpha t}$, somewhere they have used $e^{i\alpha t}$. Both are correct, so, either can be used as kernel and we can obtain the Fourier transform of the function accordingly. Thank you.