Transform Calculus and its Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 29
Evaluation of Fourier Transform of various functions

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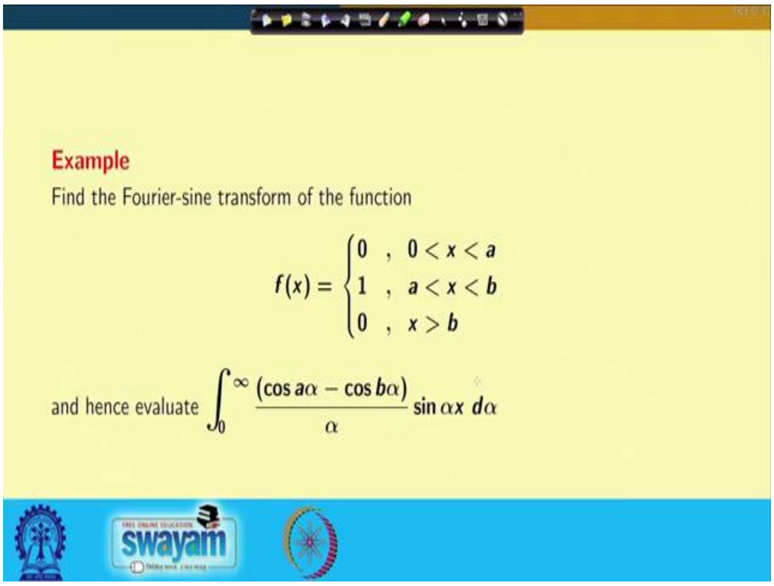

We wish to find the Fourier sine transform of $f(x)$ where $f(x)$ is defined as,

$$f(x) = \begin{cases} 0 & , \; 0 < x < a \\ 1 & , \; a < x < b \\ 0 & , \; x > b \end{cases}$$

and using this, we will calculate the value of

$$\int_0^\infty \frac{(\cos a\alpha - \cos b\alpha)}{\alpha} \sin \alpha x \;\; d\alpha$$

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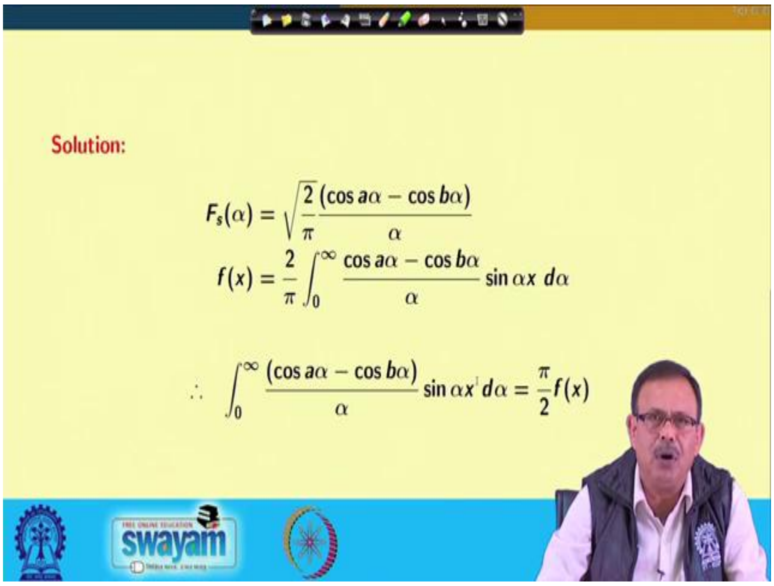

Therefore,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\ dx$$

$$= \sqrt{\frac{2}{\pi}}\int_a^b \sin\alpha x\ dx$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{1}{\alpha}\cos\alpha x\right]_a^b$$

$$= \sqrt{\frac{2}{\pi}}\frac{(\cos a\alpha - \cos b\alpha)}{\alpha}$$

and again, using inverse transform, we have,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\alpha)\sin\alpha x\ d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(\cos a\alpha - \cos b\alpha)}{\alpha}\sin\alpha x\ d\alpha$$

$$\therefore \int_0^\infty \frac{(\cos a\alpha - \cos b\alpha)}{\alpha}\sin\alpha x\ d\alpha = \frac{\pi}{2}f(x)$$

$$\therefore \int_0^{\infty} \frac{(\cos a\alpha - \cos b\alpha)}{\alpha} \sin \alpha x \; d\alpha = \begin{cases} 0 \;, & 0 < x < a \\ \frac{\pi}{2} \;, & a < x < b \\ 0 \;, & x > b \end{cases}$$

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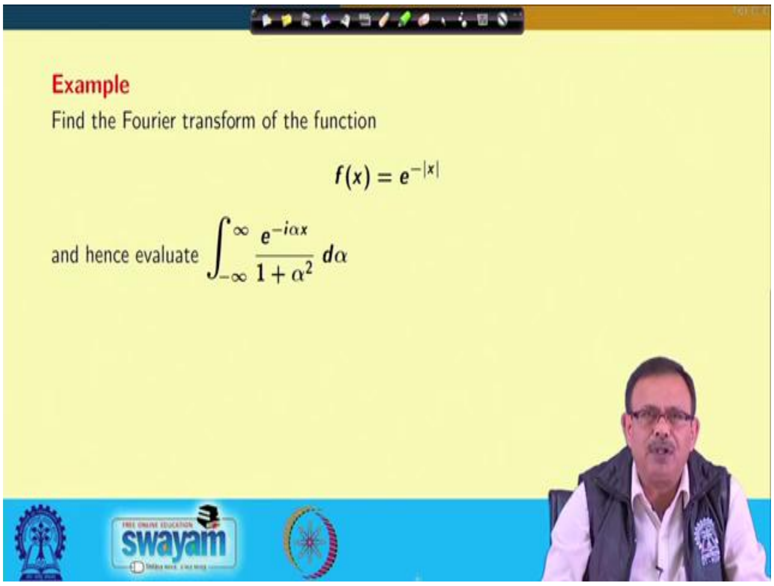

Now, we want to find out the Fourier transform of $f(x) = e^{-|x|}$ and using that, we will evaluate the value of the integral $\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+\alpha^2} d\alpha$

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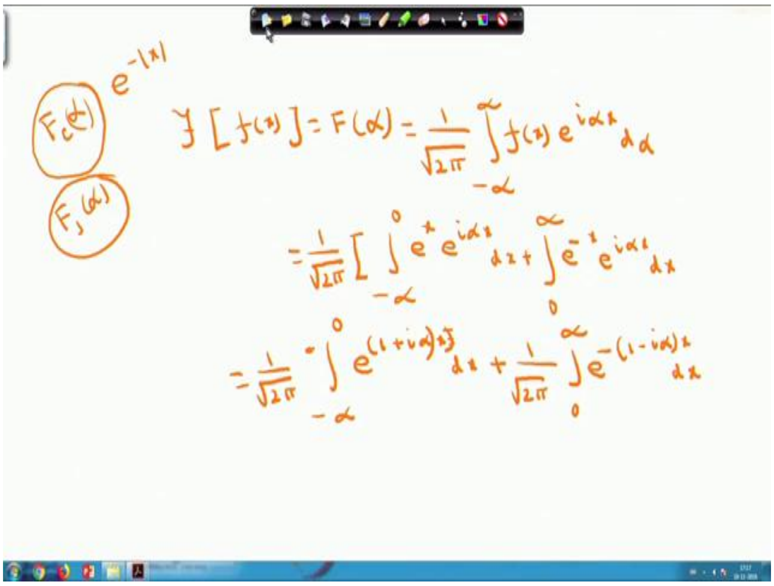

For computing Fourier transform of a function, we will use the kernel $e^{i\alpha x}$ (For assignment and Exam)

So from the definition we have,

$$\mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(1+i\alpha)x} dx + \int_{0}^{\infty} e^{-(1-i\alpha)x} dx \right]$$

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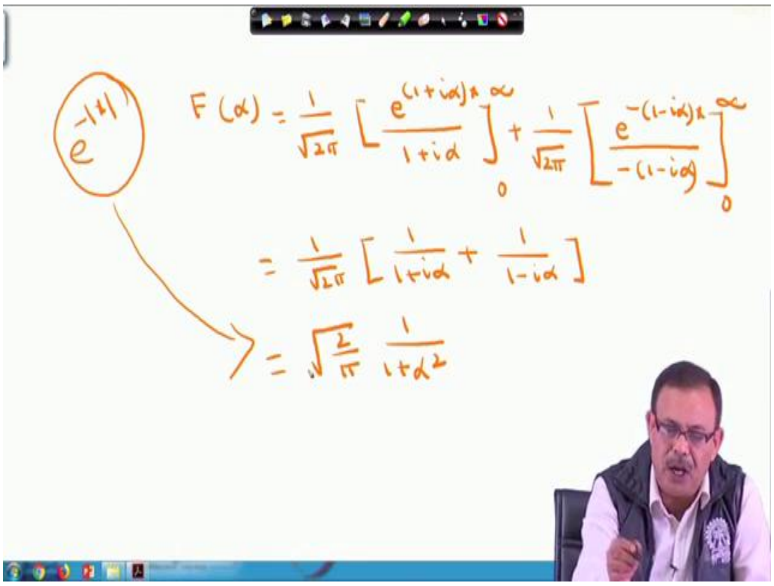

So, if we evaluate both the integrals, then we will obtain

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1+i\alpha)x}}{1+i\alpha} \right]_{x=-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-i\alpha)x}}{-(1-i\alpha)} \right]_{x=0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1+i\alpha} + \frac{1}{1-i\alpha} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

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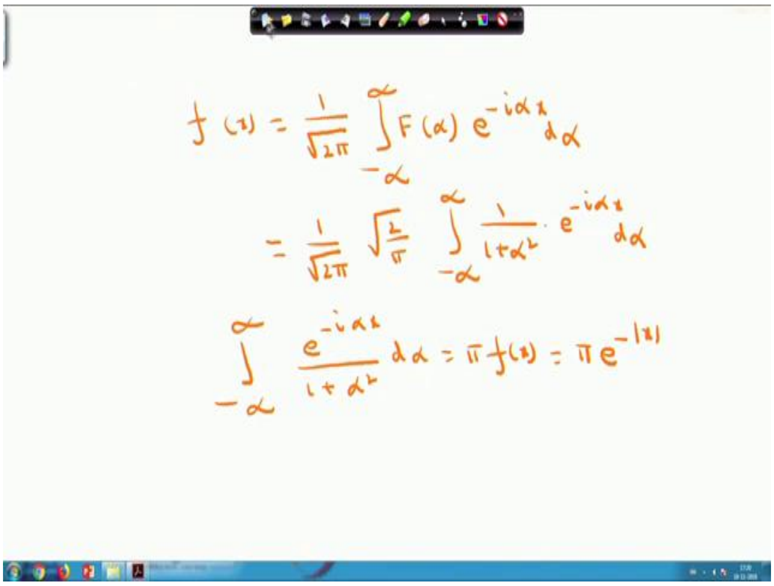

Now using inverse Fourier transform we get,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{1}{1+\alpha^2} e^{-i\alpha x} d\alpha$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+\alpha^2} d\alpha = \pi f(x) = \pi e^{-|x|}$$

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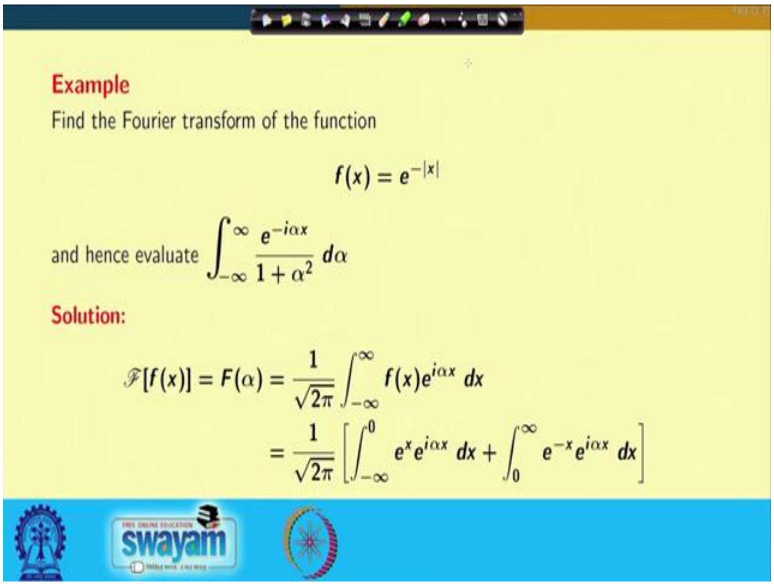

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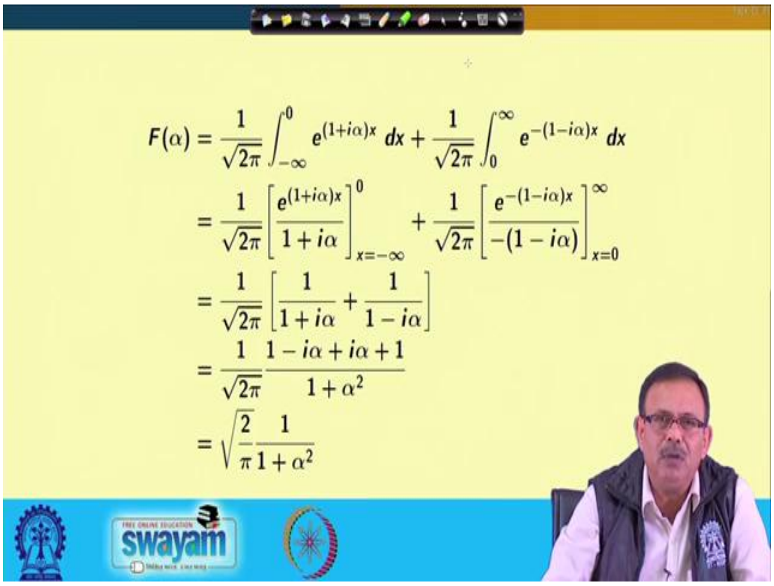

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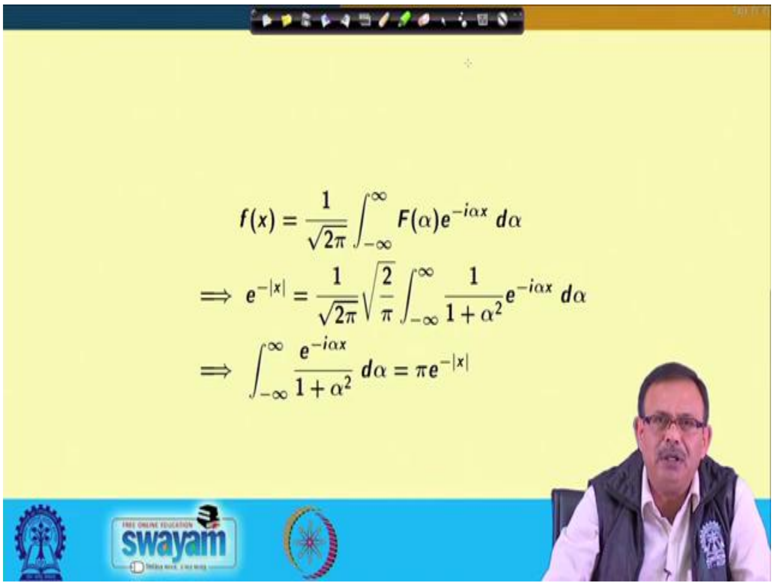

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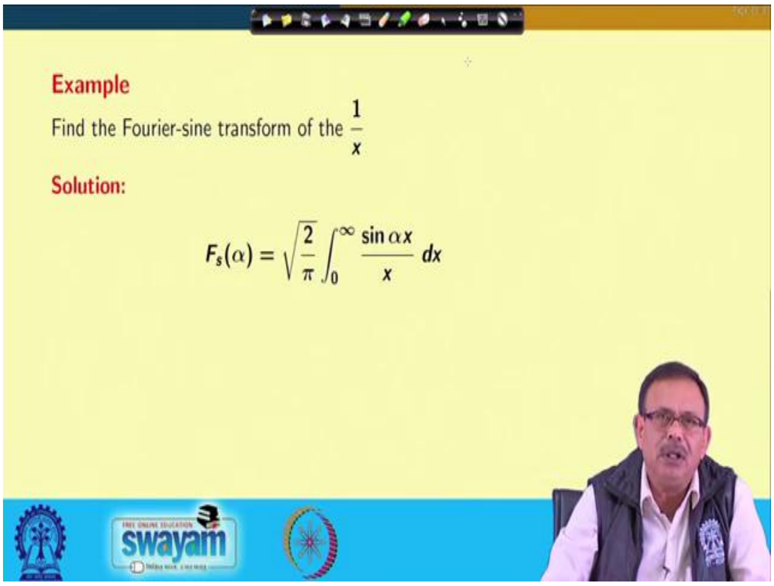

Let us take another example. Suppose we want to find the Fourier sine transform of the function $\frac{1}{x}$

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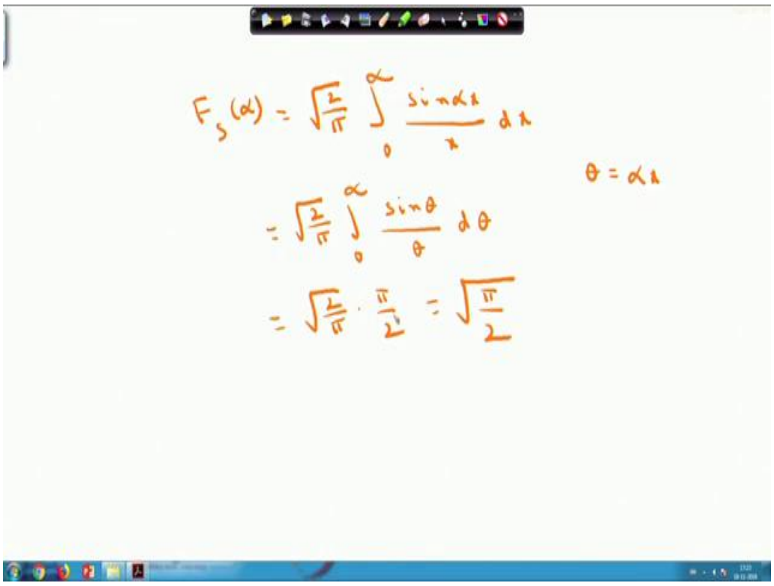

Using the definition, we have,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin \alpha x}{x} dx$$

Now substituting $\theta = \alpha x$ i.e., $dx = \frac{1}{\alpha} d\theta$, we get,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin\theta}{\theta} d\theta = \sqrt{\frac{\pi}{2}} \quad \left(\because \int_0^{\infty} \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}\right)$$

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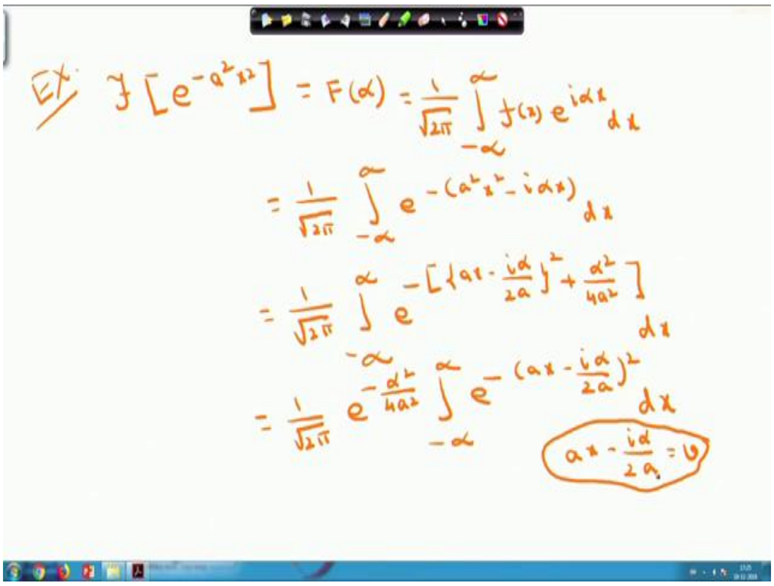

Let us take next example, suppose we want to find the Fourier transform of $e^{-a^2x^2}$

$$\therefore \mathcal{F}\left[e^{-a^2x^2}\right] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - i\alpha x)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{i\alpha}{2a}\right)^2 + \frac{\alpha^2}{4a^2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{i\alpha}{2a}\right)^2} dx$$

Substituting $ax - \frac{i\alpha}{2a} = v$ i.e., $dx = \frac{1}{a} dv$ in the above integral, we get,

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\alpha^2}{4a^2}}\, \frac{1}{a} \int_{-\infty}^{\infty} e^{-v^2} dv$$

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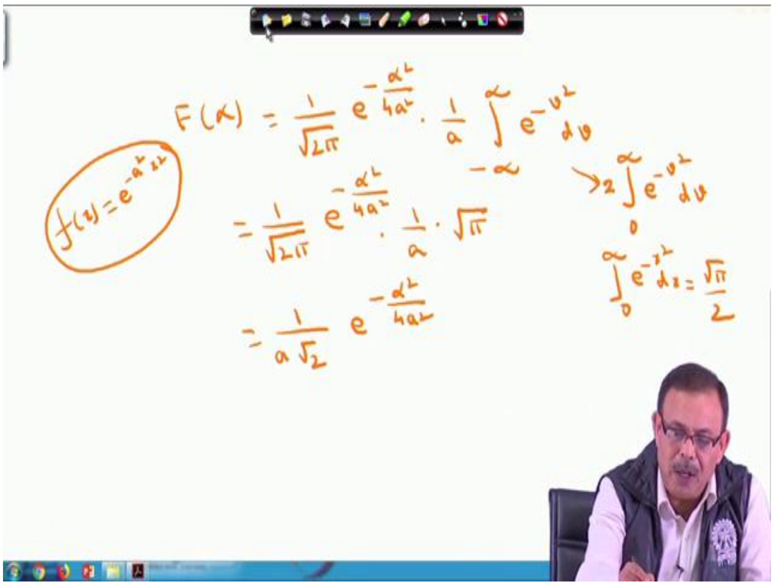

Since e^{-v^2} is an even function, so we have,

$$F(\alpha) = \frac{2}{\sqrt{2\pi}}\, e^{-\frac{\alpha^2}{4a^2}}\, \frac{1}{a}\int_0^{\infty} e^{-v^2} dv$$

$$= \frac{1}{a\sqrt{2}}\, e^{-\frac{\alpha^2}{4a^2}} \qquad \left(\because \int_0^{\infty} e^{-v^2} dv = \frac{\sqrt{\pi}}{2}\right)$$

Thus we have obtained the Fourier transform of $e^{-a^2x^2}$ as

$$F(\alpha) = \frac{1}{a\sqrt{2}}\, e^{-\frac{\alpha^2}{4a^2}}$$

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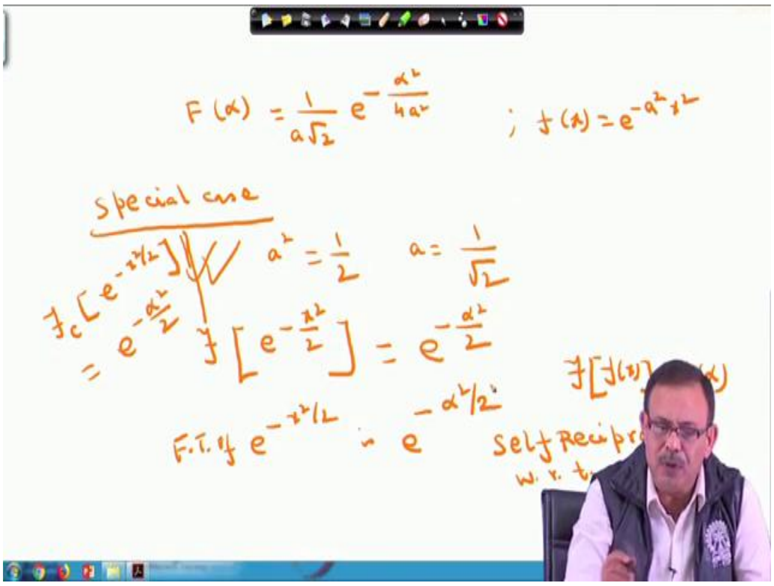

Now, let us see a special case from this problem. Let us take, $a = \frac{1}{\sqrt{2}}$. Then we have,

$$\mathcal{F}\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{\alpha^2}{2}}$$

Therefore, Fourier transform of $e^{-\frac{x^2}{2}}$ is nothing but the function itself. These types of functions are called self-reciprocal with respect to the given transformation. Here, $e^{-\frac{x^2}{2}}$ is self-reciprocal with respect to Fourier transform.

This function is self-reciprocal with respect to Fourier cosine transform also i.e.,

$$\mathcal{F}_c\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{\alpha^2}{2}}$$

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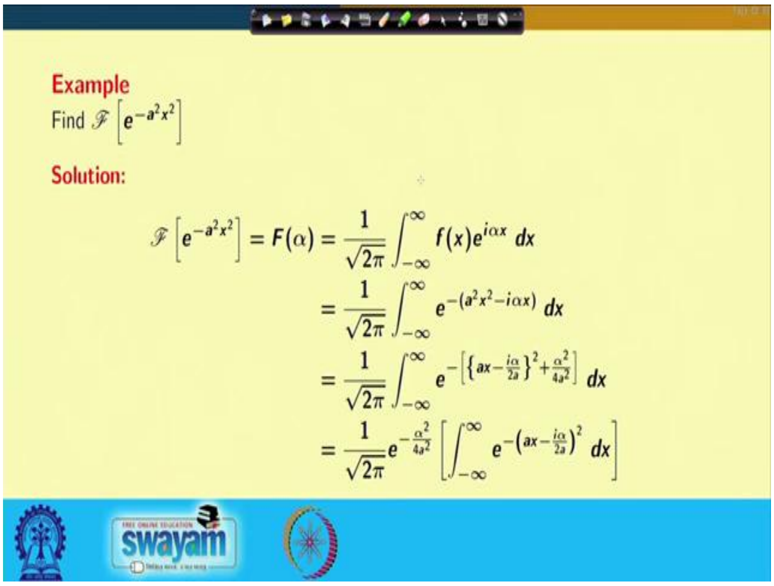

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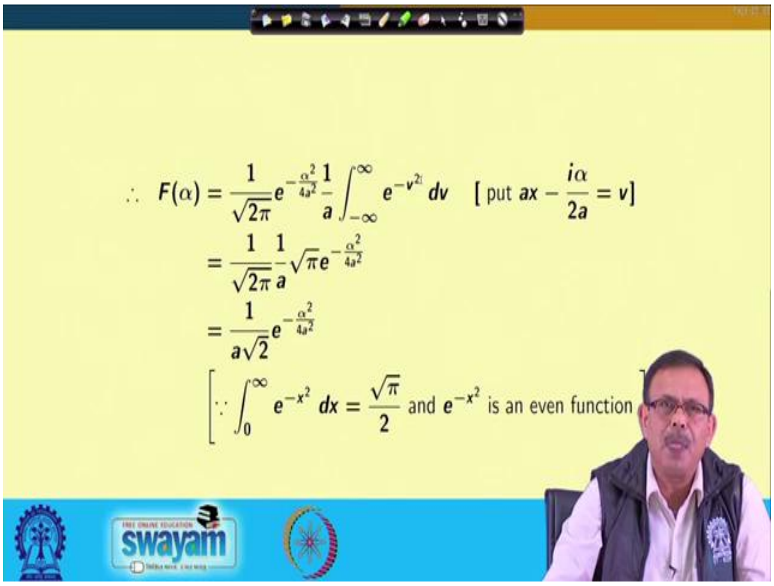

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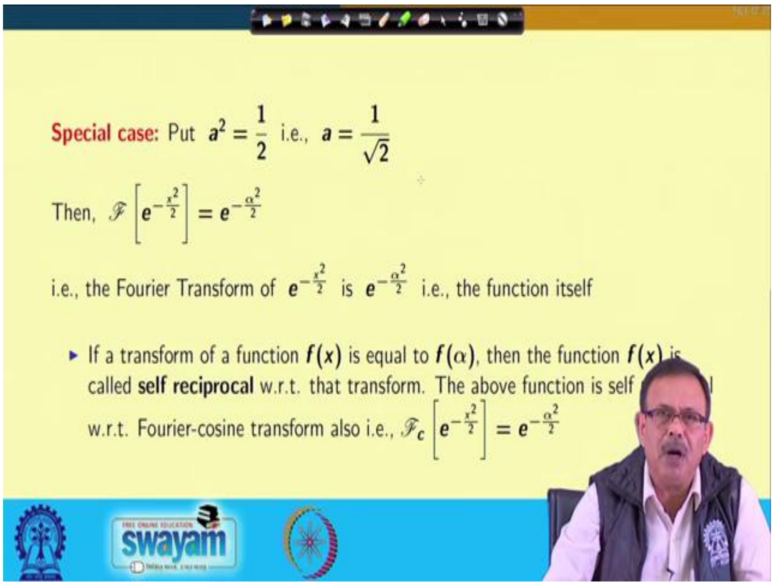

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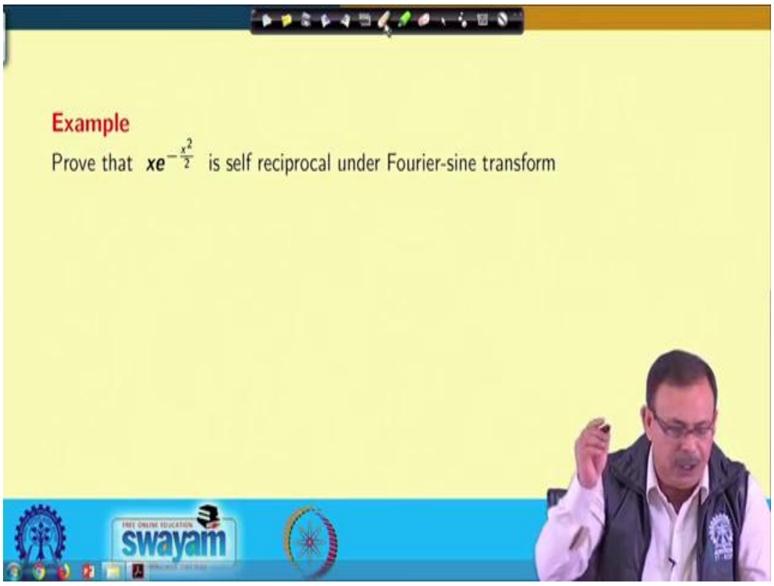

Now, let us see this problem, where we have to show that $xe^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier sine transform.

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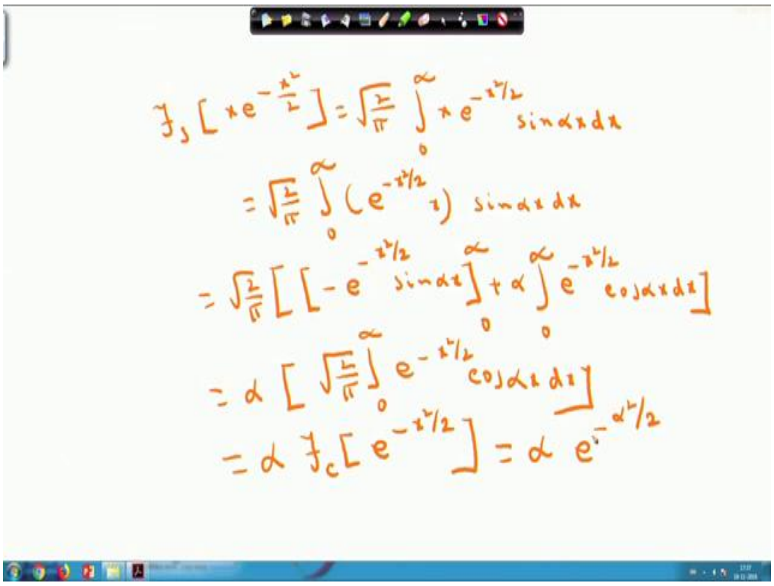

Let us see how to prove it.

$$\mathcal{F}_s\left[xe^{-\frac{x^2}{2}}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(xe^{-\frac{x^2}{2}}\right)\sin \alpha x\ \ dx$$

Using integration by parts, we have,

$$\mathcal{F}_s\left[xe^{-\frac{x^2}{2}}\right] = \sqrt{\frac{2}{\pi}}\left(\left[-e^{-\frac{x^2}{2}}\sin \alpha x\right]_0^\infty + \alpha\int_0^\infty e^{-\frac{x^2}{2}}\cos \alpha x\ \ dx\right)$$

Please note that, once we put the limiting values, value of the first integral will vanish.

So, we have,

$$\mathcal{F}_s\left[xe^{-\frac{x^2}{2}}\right] = \alpha\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{x^2}{2}}\cos \alpha x\ \ dx = \alpha\mathcal{F}_c\left[e^{-\frac{x^2}{2}}\right] = \alpha e^{-\frac{\alpha^2}{2}}$$

Therefore, this function is self reciprocal with respect to Fourier sine transform.

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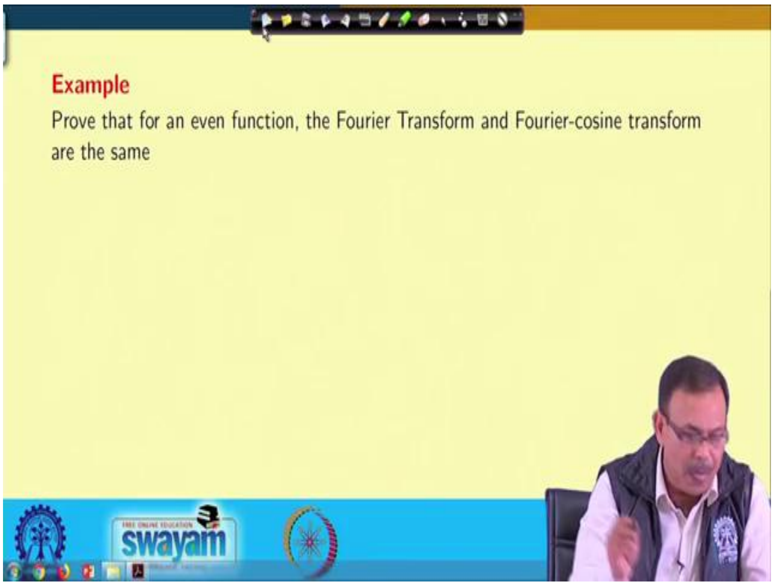

Next, we want to prove that for an even function, the Fourier transform and the Fourier cosine transform are always same.

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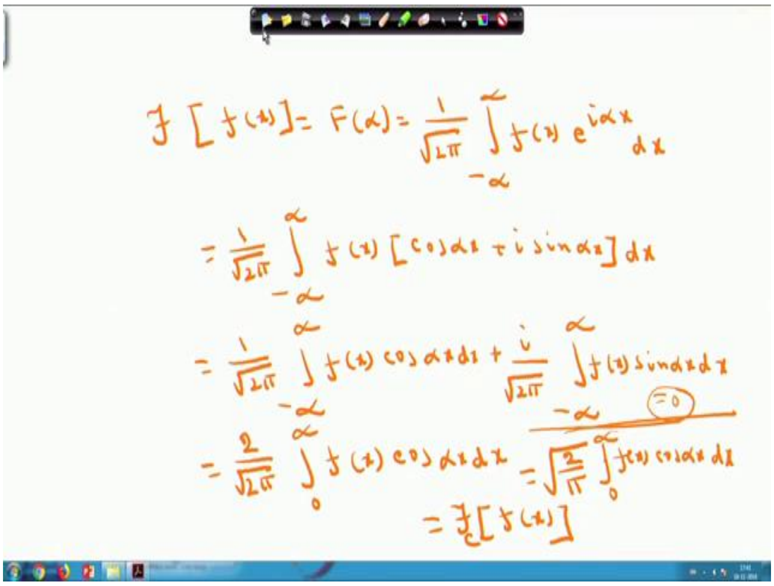

From definition, we get,

$$\mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx$$

Now using $e^{i\alpha x} = \cos \alpha x + i \sin \alpha x$, we can break the above integral as,

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \alpha x\, dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \sin \alpha x\, dx$$

Since $f(x)$ is an even function, so $f(x) \cos \alpha x$ is also an even function and $f(x) \sin \alpha x$ is an odd function. Therefore, using the property of integration, second integral will vanish and $F(\alpha)$ becomes

$$\therefore \mathcal{F}[f(x)] = F(\alpha) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \cos \alpha x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos \alpha x\, dx$$

$$= \mathcal{F}_c[f(x)]$$

Hence proved.

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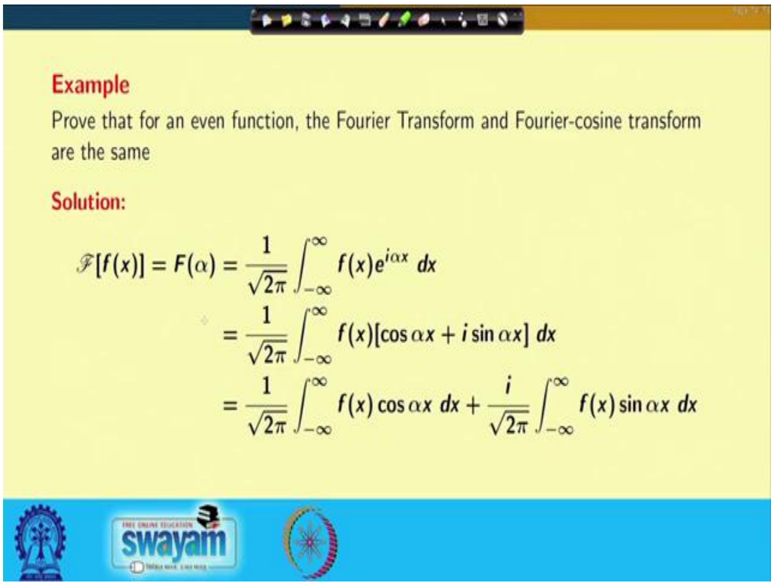

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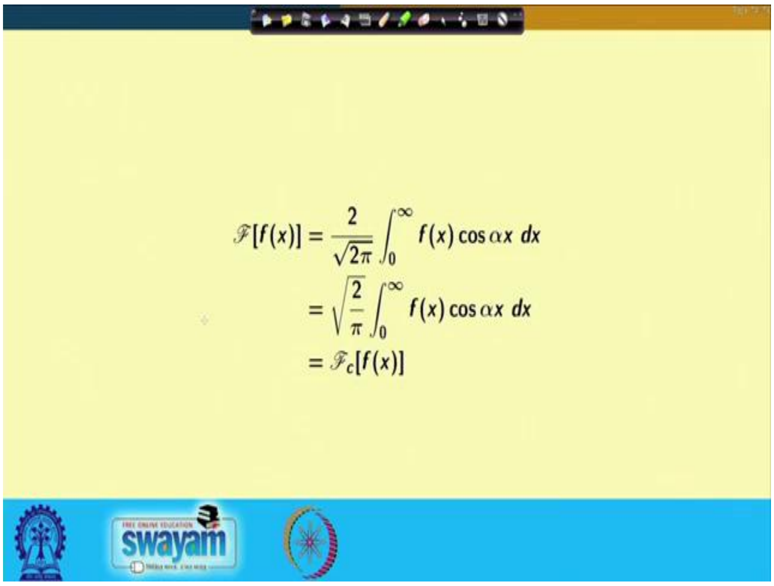

Thank you.