**Transform Calculus and its Applications in Differential Equations**
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# Lecture - 28
## Derivation of Fourier Cosine Transform and Fourier Sine Transform of Functions

Welcome back. In the last lecture, we have seen how to find out the Fourier transform, Fourier cosine transform and Fourier sine transform of a particular function $f(x)$, when $f(x)$ is defined in either $(-\infty, \infty)$ or $(0, \infty)$.

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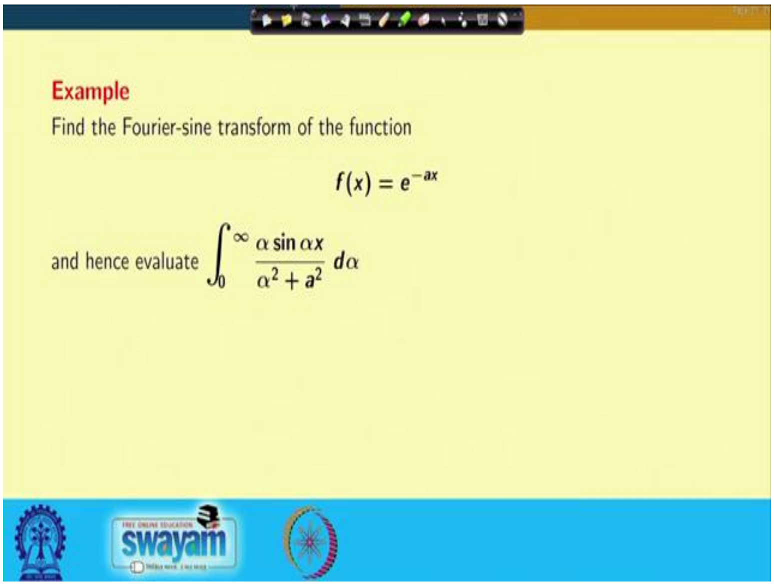


Fourier cosine transform (F.C.T.) of $f(x)$ is denoted by $F_c(\alpha)$ and expressed as

$$\mathcal{F}_c[f(x)] = F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx$$

$$\mathcal{F}_c^{-1}[F_c(\alpha)] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha$$

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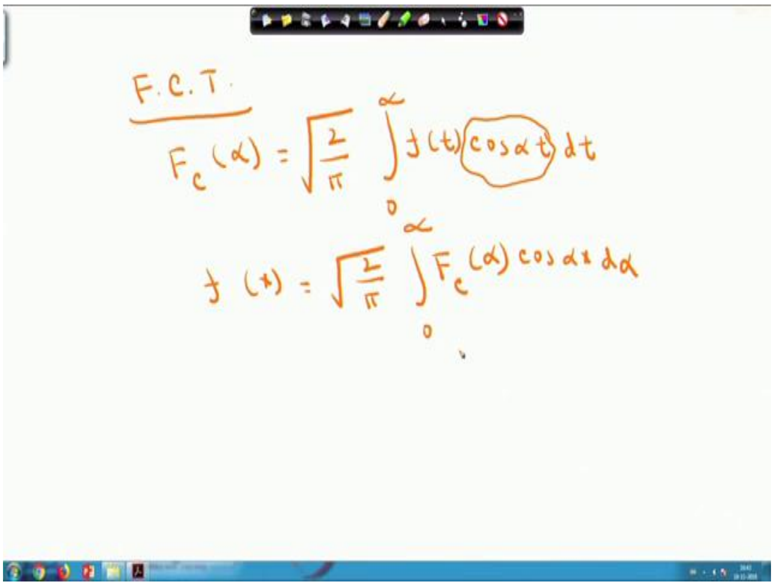


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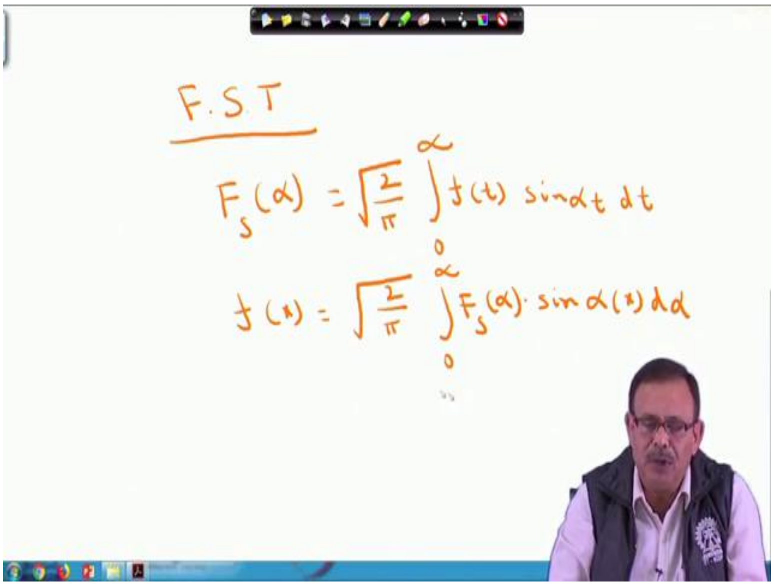


Similarly, Fourier sine transform (F.S.T.) of $f(x)$ is denoted by $F_s(\alpha)$ and expressed as,

$$\mathcal{F}_s[f(x)] = F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \alpha x \, dx$$

$$\mathcal{F}_s^{-1}[F_s(\alpha)] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\alpha) \sin \alpha x \, d\alpha$$

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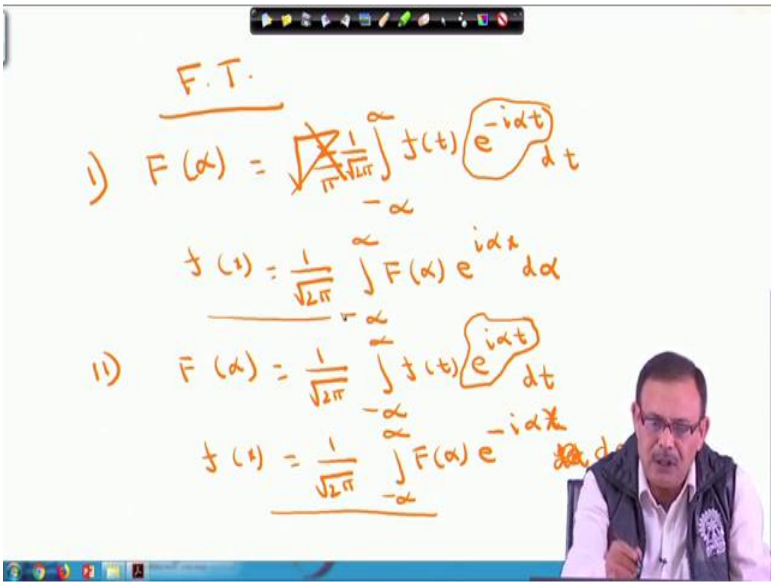


Fourier transform (F.T.) of $f(x)$ is denoted by $F(\alpha)$ and expressed as,

$$\mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx$$

$$\mathcal{F}^{-1}[F(\alpha)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}\,d\alpha$$

or

$$\mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx$$

$$\mathcal{F}^{-1}[F(\alpha)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\,d\alpha$$

Therefore, for the first case, we have taken the kernel as $e^{-i\alpha x}$, whereas for the second case, we have taken the kernel as $e^{i\alpha x}$. Any one of these two forms can be used and we will get the correct result for both cases.

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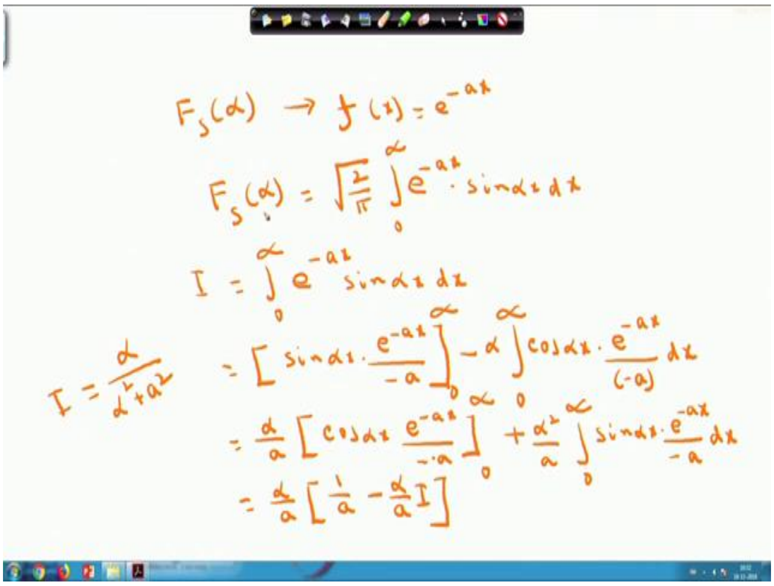

Now, let us solve certain problems using these transforms. The first one is, we want to find out

$$\mathcal{F}_s[e^{-ax}]$$

So, from the definition of Fourier sine transform, we have,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin \alpha x \, dx$$

So, let us assume that

$$I = \int_0^{\infty} e^{-ax} \sin \alpha x \, dx$$

Using integration by parts, we get,

$$I = \left[\frac{e^{-ax}}{-a} \sin \alpha x\right]_{x=0}^{\infty} + \frac{\alpha}{a} \int_0^{\infty} e^{-ax} \cos \alpha x \, dx$$

$$= \frac{\alpha}{a}\left[\frac{e^{-ax}}{-a} \cos \alpha x\right]_{x=0}^{\infty} - \frac{\alpha^2}{a^2} \int_0^{\infty} e^{-ax} \sin \alpha x \, dx$$

$$= \frac{\alpha}{a}\left[\frac{1}{a} - \frac{\alpha}{a} I\right]$$

$$\therefore I = \frac{\alpha}{\alpha^2 + a^2}$$

And therefore, we have,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + a^2}$$

which solves our problem.

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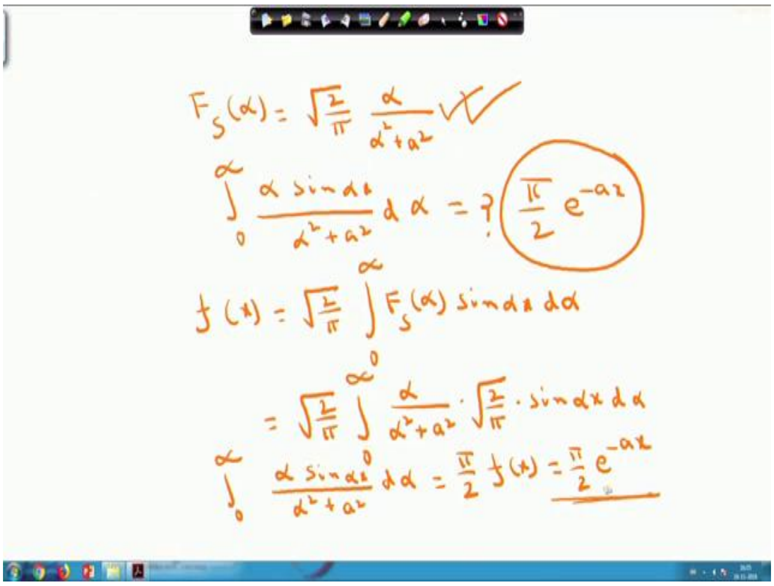

Next we have to evaluate the value of the following integral,

$$\int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + a^2} d\alpha$$

Using the inverse Fourier sine transform, we can write,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) \sin \alpha x \, d\alpha = \sqrt{\frac{2}{\pi}} \int_0^\infty \left( \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + a^2} \right) \sin \alpha x \, d\alpha$$

$$\therefore \int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + a^2} d\alpha = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-ax}$$

This completes the solution.

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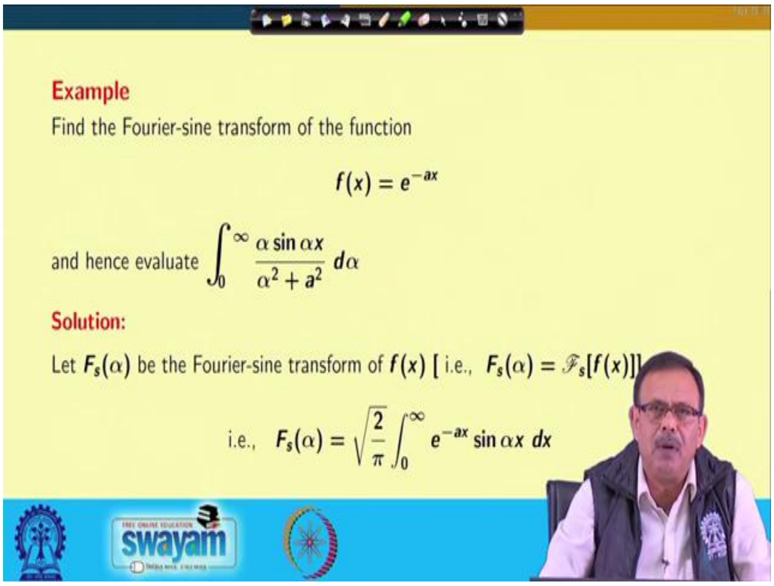


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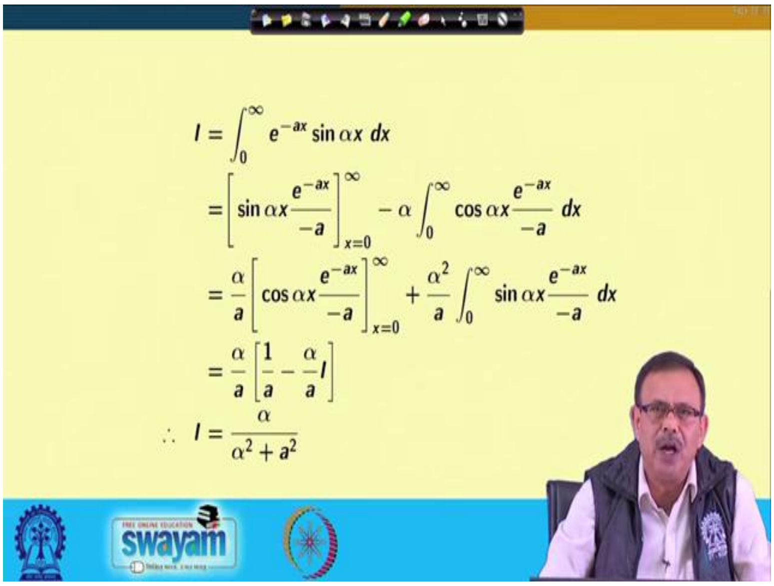

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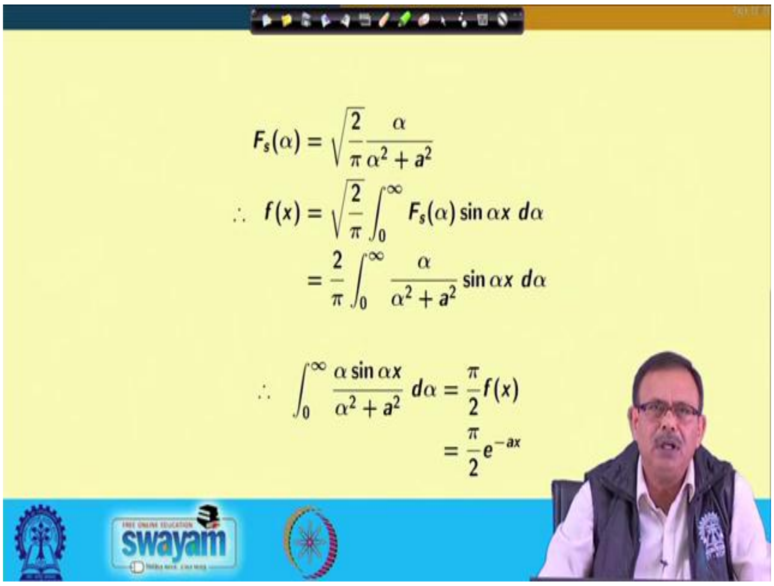


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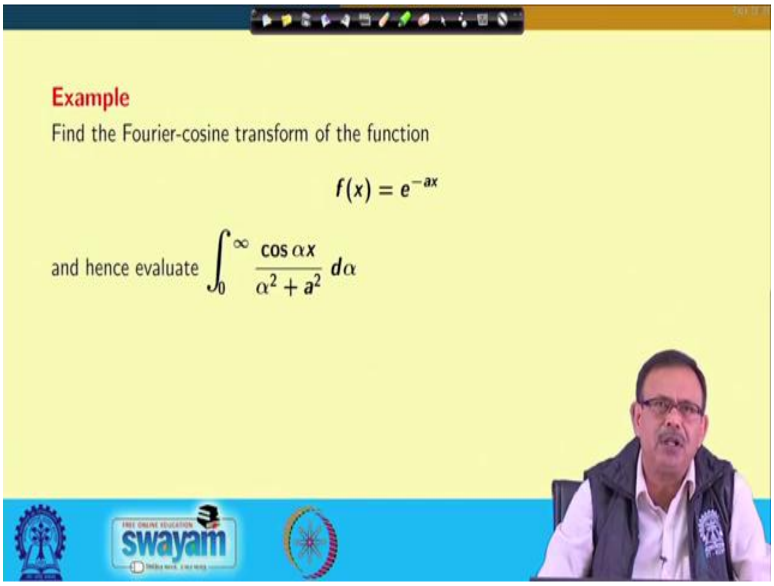


Now, we want to find out the Fourier cosine transform of $e^{-ax}$ and from there, we want to evaluate the value of the following integral

$$\int_0^\infty \frac{\cos \alpha x}{\alpha^2 + a^2} d\alpha$$

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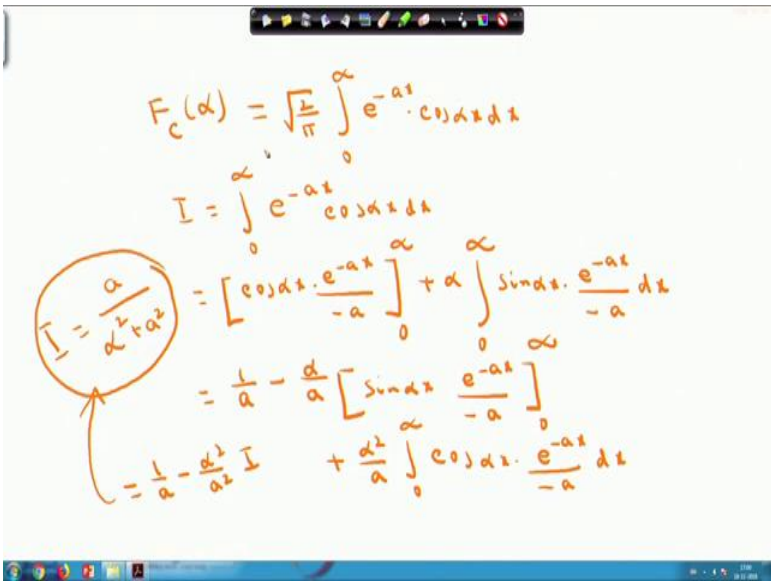

So, from the definition of Fourier cosine transform, we have,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos \alpha x \, dx$$

So, let us assume that

$$I = \int_0^\infty e^{-ax} \cos \alpha x \, dx$$

Using integration by parts, we get,

$$I = \left[\frac{e^{-ax}}{-a} \cos \alpha x\right]_{x=0}^{\infty} - \frac{\alpha}{a} \int_0^\infty e^{-ax} \sin \alpha x \, dx$$

$$= \frac{1}{a} - \frac{\alpha}{a} \left[\frac{e^{-ax}}{-a} \sin \alpha x\right]_{x=0}^{\infty} - \frac{\alpha^2}{a^2} \int_0^\infty e^{-ax} \cos \alpha x \, dx$$

$$= \frac{1}{a} - \frac{\alpha^2}{a^2} I$$

$$\therefore I = \frac{a}{\alpha^2 + a^2}$$

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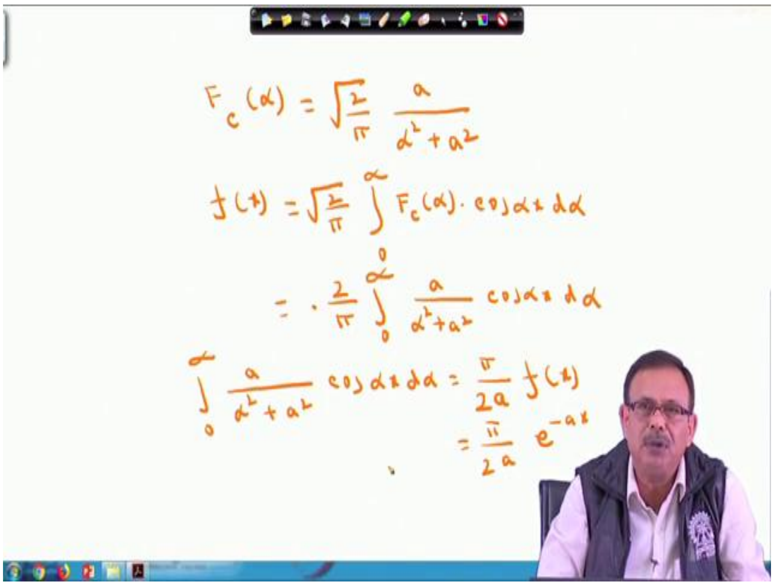

And therefore, we have,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \frac{a}{\alpha^2 + a^2}$$

Using the inverse Fourier cosine transform, we can write,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\alpha) \cos \alpha x \, d\alpha = \frac{2}{\pi} \int_0^{\infty} \frac{a}{\alpha^2 + a^2} \cos \alpha x \, d\alpha$$

$$\therefore \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + a^2} d\alpha = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} e^{-ax}$$

So, like this way we can find out the Fourier sine transform, Fourier cosine transform of different functions.

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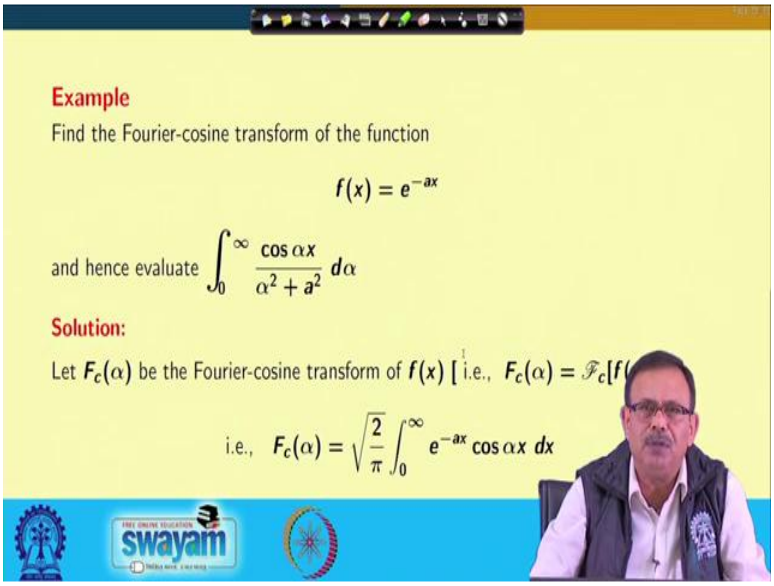


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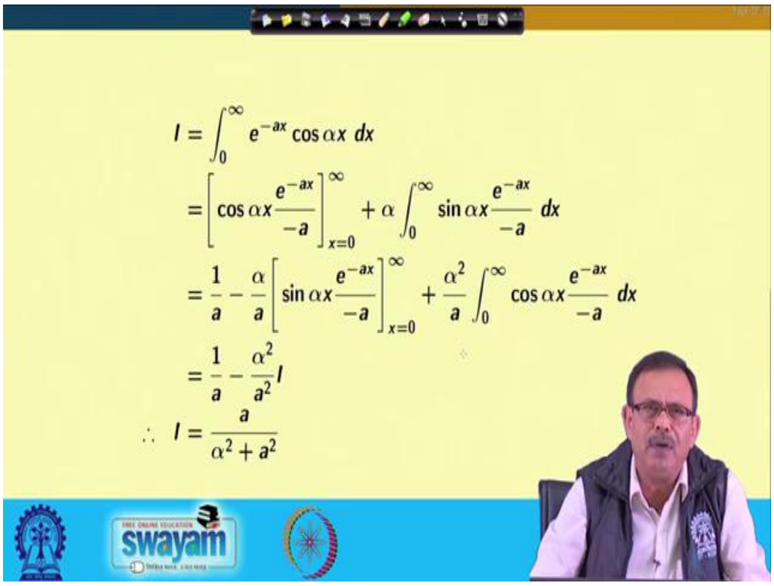

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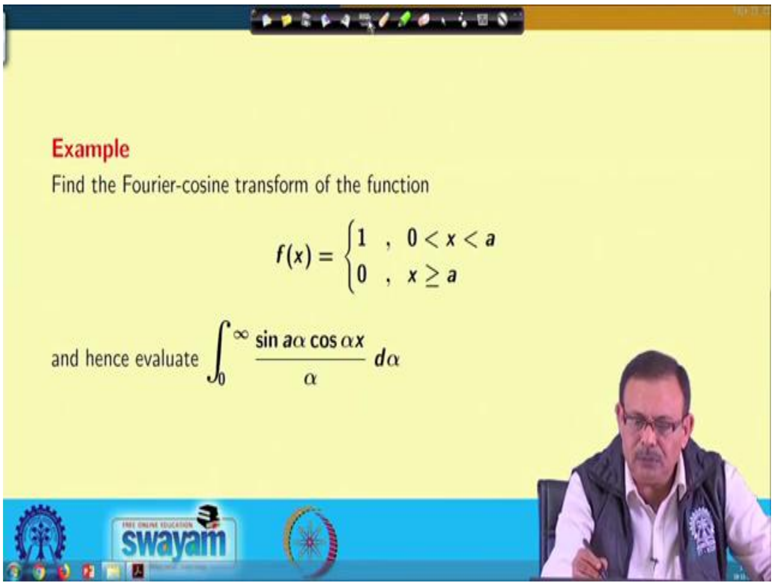


Next, let us take another problem. We need to find the Fourier cosine transform of the function $f(x)$ defined as

$$f(x) = \begin{cases} 1 & , \ 0 < x < a \\ 0 & , \ x \geq a \end{cases}$$

and using this result, we need to evaluate

$$\int_0^\infty \frac{\sin a\alpha \cos \alpha x}{\alpha} d\alpha$$

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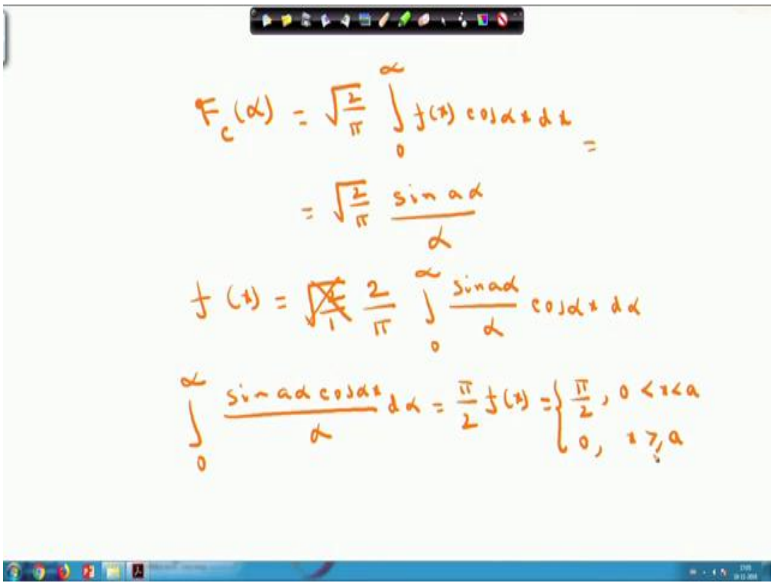

So, from the definition of Fourier cosine transform of $f(x)$, we have,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx$$

And if we evaluate the integral for given $f(x)$, we will have to evaluate it from 0 to $a$ only, then the value is obtained as

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^a \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$$

This completes the first part of the problem.

Using the inverse Fourier cosine transform, we can write,

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin a\alpha}{\alpha} \cos \alpha x \, d\alpha$$

$$\Rightarrow \int_0^\infty \frac{\sin a\alpha \ \cos \alpha x}{\alpha} d\alpha = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & , \quad 0 < x < a \\ 0 & , \quad x \geq a \end{cases}$$

Now let us take another problem, where we want to find out the Fourier sine transform of the earlier function and we want to evaluate the integral

$$\int_0^\infty \frac{(1 - \cos a\alpha) \sin \alpha x}{\alpha} d\alpha$$

So, please note that, not only we are evaluating the Fourier sine transform, Fourier cosine transform of a function, but also simultaneously using this Fourier cosine or Fourier sine transform, we can evaluate the values of some integrals, which may become difficult for us to evaluate using the normal integration processes.

So, from the definition of Fourier sine transform, we have,

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x \, dx$$

And if we evaluate the integral for given $f(x)$, we will have to evaluate it from 0 to $a$ only, then the value is obtained as

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^a \sin \alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{(1 - \cos a\alpha)}{\alpha}$$

Using the inverse Fourier sine transform, we can write,

$$\int_0^\infty \frac{(1 - \cos a\alpha) \sin \alpha x}{\alpha} d\alpha = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & , \quad 0 < x < a \\ 0 & , \quad x \geq a \end{cases}$$

So, we have studied two things in this lecture. One is how to find out the Fourier cosine transform or Fourier sine transform of a function, whenever the function is given to us. And also whenever we are finding out the Fourier transform, Fourier sine or Fourier cosine transform of the function, then using the corresponding inverse transform, we can also find out the value of certain other integrals also. This is the advantage of using Fourier sine and Fourier cosine transforms. Thank you.