Transform Calculus and its applications in Differential Equations

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Lecture – 26
Fourier Integral Representation

From this lecture, effectively we are going to start another transform which we call the Fourier transform. In Laplace transform, basically the interval of the function was from $(0, \infty)$, but in physical problems always it may not be possible that the parameter lies in the interval $(0, \infty)$, but it may lie in the interval $(-\infty, \infty)$.

So, we want to study how to handle the situations where one function is defined in $(-\infty, \infty)$ and we want to find out a solution. For that basically, the Fourier transform was developed which we will discuss later, but initially let us start with the Fourier integral representation of a function.

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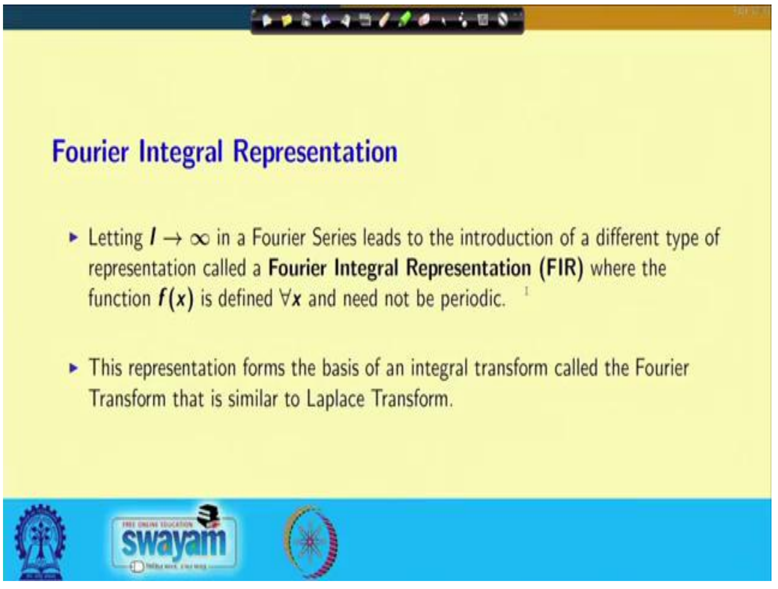

In Fourier Series representation, we can define a function in the interval $(-l, l)$. Now, when l approaches ∞ in a Fourier series, then this leads to introduction of a different type of representation which we call as Fourier Integral Representation or FIR where the function $f(x)$ is defined for all x and need not be periodic.

This representation forms the basis of an integral transform which we call as Fourier transform that is similar to Laplace transform except that in Laplace transform, the function $f(x)$ is defined in $(0, \infty)$, but here the function is defined in $(-\infty, \infty)$. This is because in real life, in many situations a function will be defined in $(-\infty, \infty)$. So, let us see what happens in this case.

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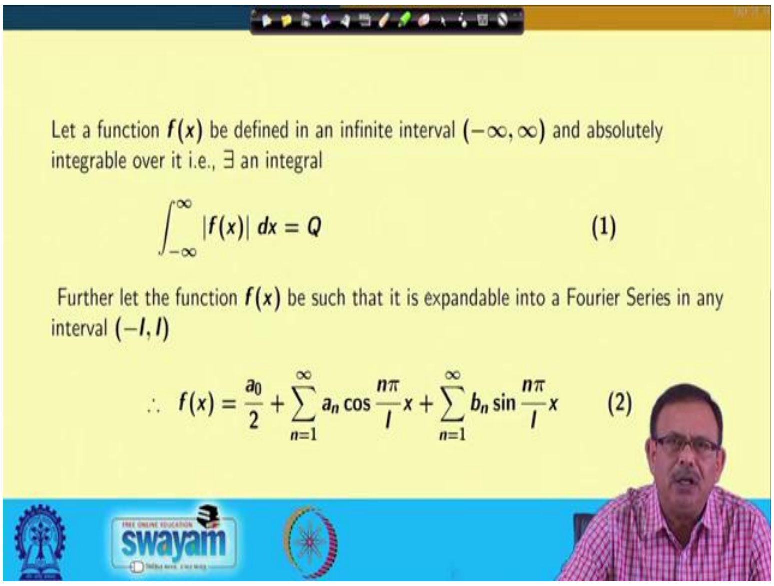

We have a function $f(x)$ which is defined in the interval $(-\infty, \infty)$ and is term by term integrable, that is there exists an integral such that we can write down $\int_{-\infty}^{\infty} |f(x)| dx$ is equal to a finite value Q.

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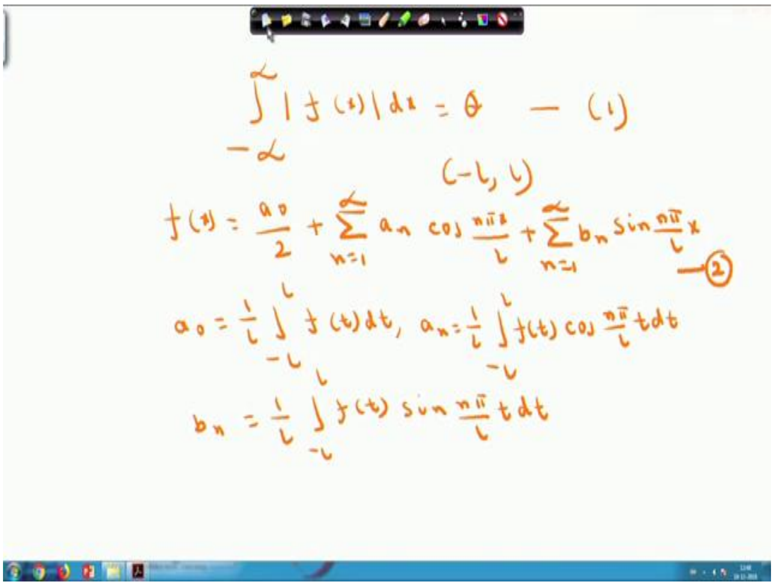

So,

$$\int_{-\infty}^{\infty} |f(x)|dx = Q \tag{1}$$

If we evaluate the integral, it has a definite value. For that, suppose the function $f(x)$ can be expanded in terms of Fourier series in the interval $(-l, l)$ and we can write down

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \tag{2}$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(t)dt$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}t\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi}{l}t\,dt$$

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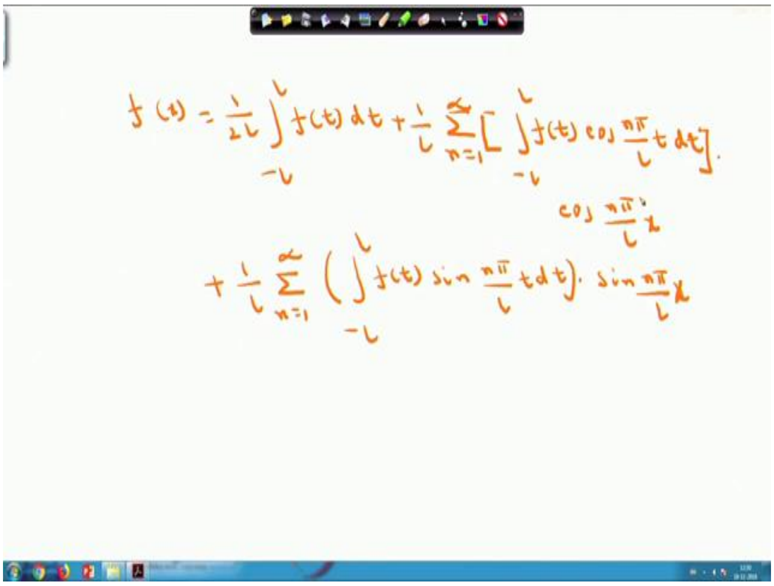

So, if we put the values of these coefficients a_0, a_n and b_n in the equation (2), we will get,

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{l}\sum_{n=1}^{\infty}\left[\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}t\,dt\right]\cos\frac{n\pi}{l}x$$
$$+\frac{1}{l}\sum_{n=1}^{\infty}\left[\int_{-l}^{l} f(t)\sin\frac{n\pi}{l}t\,dt\right]\sin\frac{n\pi}{l}x$$

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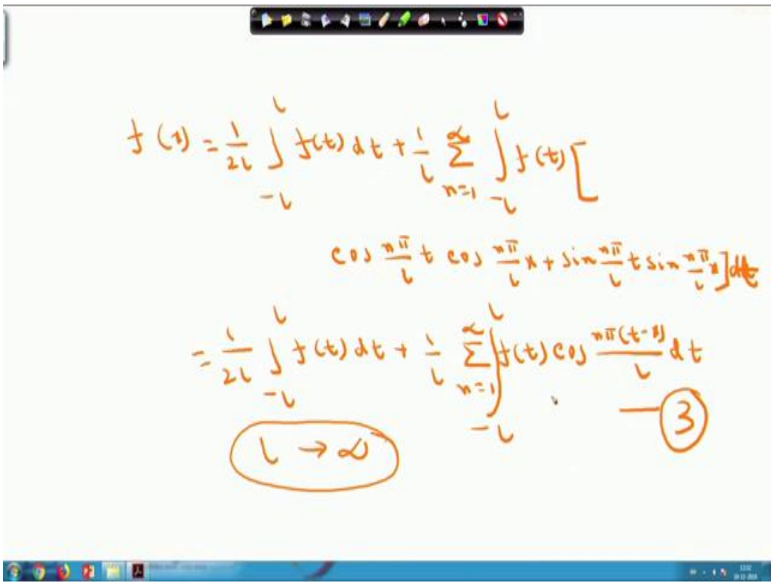

This we can write down as

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\left[\cos\frac{n\pi}{l}t\cos\frac{n\pi}{l}x + \sin\frac{n\pi}{l}t\sin\frac{n\pi}{l}x\right]dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{n\pi(t-x)}{l}dt \tag{3}$$

Now we want to investigate what will be the form of the expression (3) whenever l approaches ∞.

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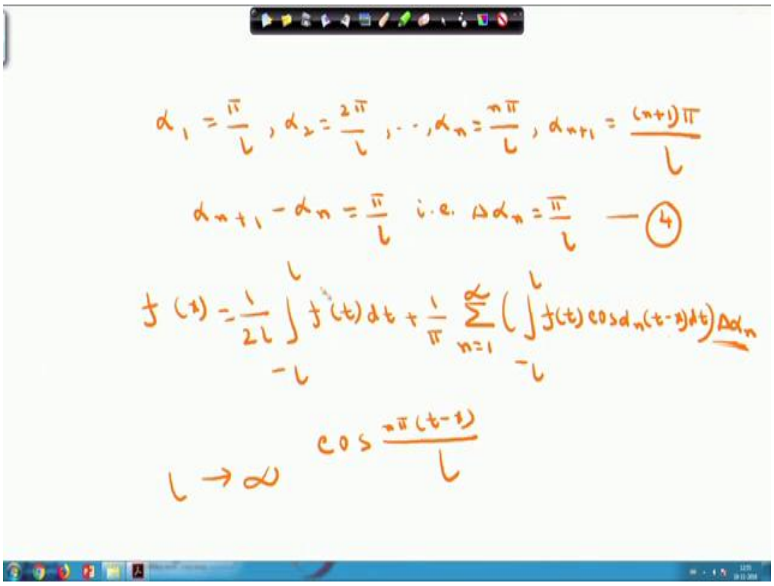

Let us introduce some new notations, say $\alpha_1 = \frac{\pi}{l}, \alpha_2 = \frac{2\pi}{l}, \cdots, \alpha_n = \frac{n\pi}{l}$

$$\therefore \alpha_{n+1} - \alpha_n = \Delta\alpha_n = \frac{\pi}{l} \tag{4}$$

Now, we substitute this value from (4) into equation (3) to obtain

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-l}^{l} f(t)\cos\alpha_n(t-x)\,dt\right)\Delta\alpha_n \tag{5}$$

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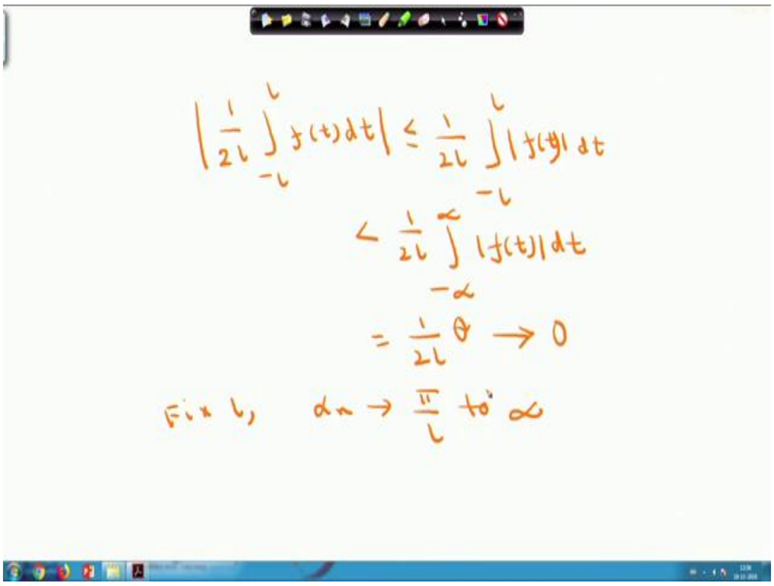

So, whenever $l \to \infty$, the first term $\frac{1}{2l}\int_{-l}^{l} f(t)dt$ approaches 0 since $\int_{-\infty}^{\infty} |f(x)|dx = Q$.

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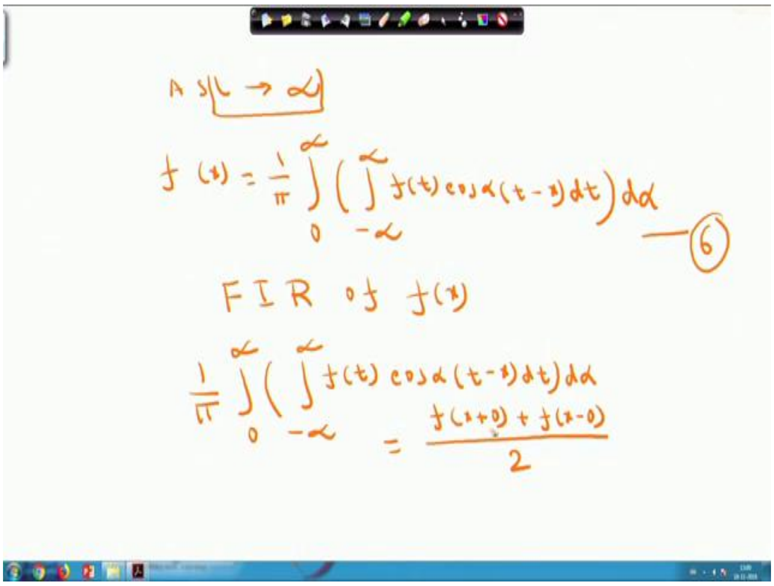

And, as $l \to \infty$, equation (5) reduces to,

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right)d\alpha \qquad (6)$$

And this expression is known as the Fourier integral representation of the function $f(x)$.

Equation (6) occurs at all points where the function is continuous. If the function is discontinuous at a particular point, by the concept what we have told in the Fourier series, we can write down

$$\frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right)d\alpha = \frac{f(x+0)+f(x-0)}{2}$$

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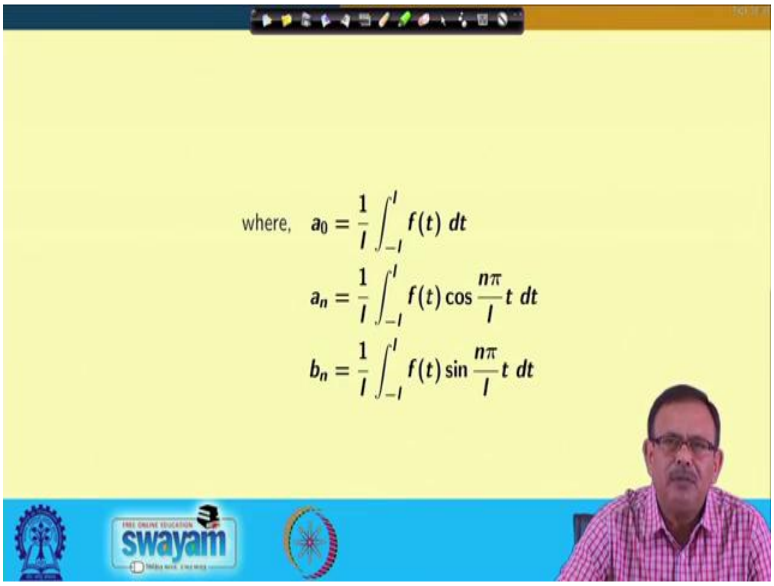

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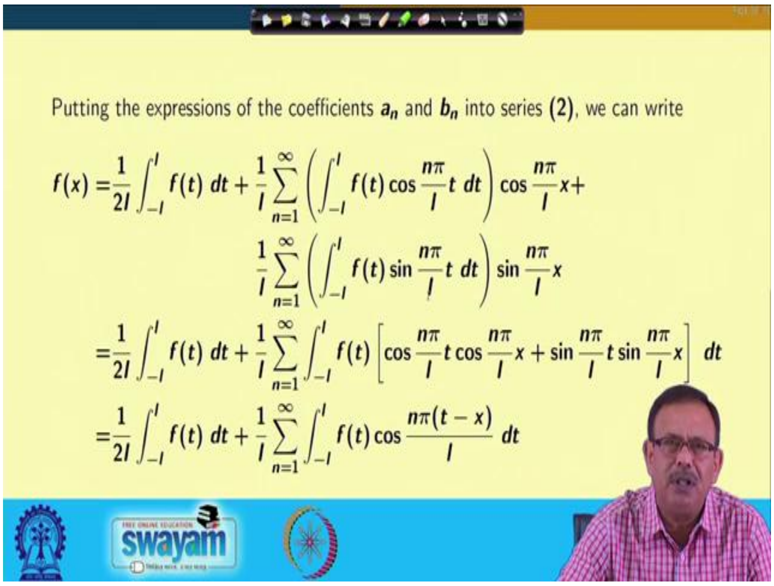

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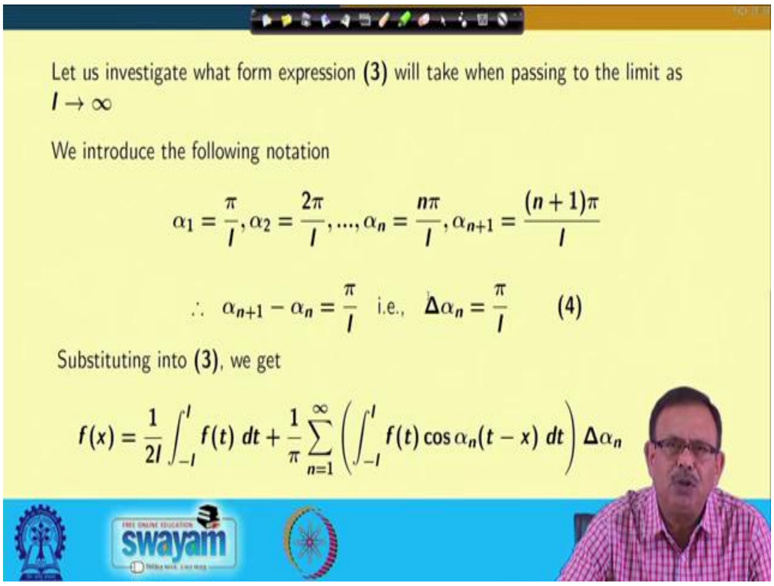

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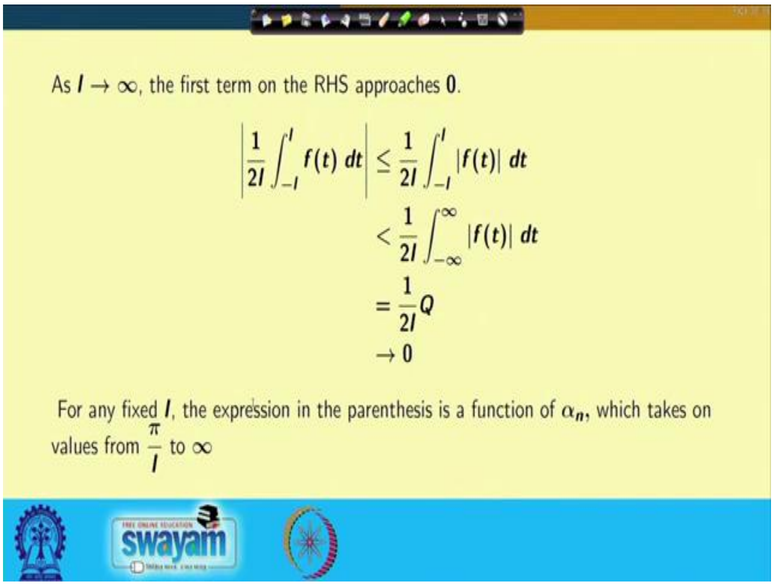

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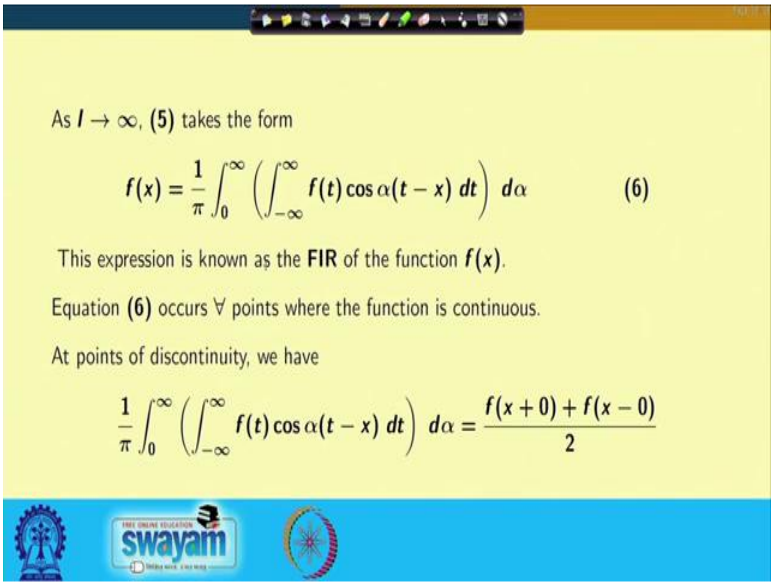

Now, let us expand $\cos\alpha(t-x)$ in the equation (6).

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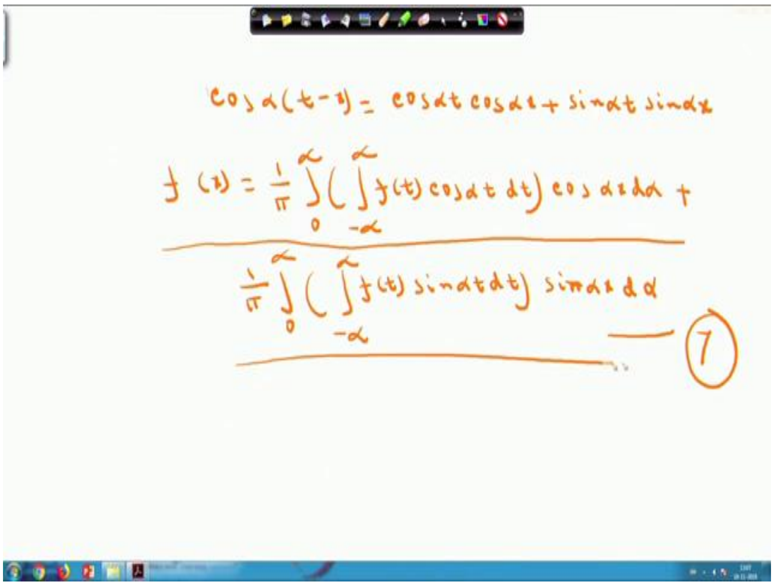

$$\cos\alpha(t-x) = \cos\alpha t\cos\alpha x + \sin\alpha t\sin\alpha x$$

Putting this in equation (6), we have,

$$f(x) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(t)\cos \alpha t\, dt\right)\cos \alpha x\, d\alpha$$
$$+\frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(t)\sin \alpha t\, dt\right)\sin \alpha x\, d\alpha \qquad (7)$$

Let us take some particular cases of this equation (7)

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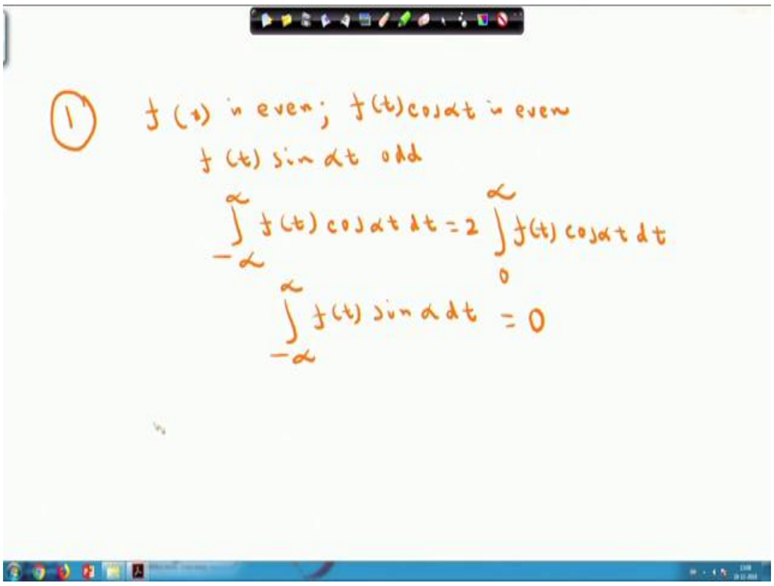

Case 1: Let $f(x)$ be even. Whenever $f(x)$ is even, $f(x)\cos \alpha t$ is even and $f(x)\sin \alpha t$ is odd. So,

$$\int_{-\infty}^\infty f(t)\cos \alpha t\, dt = 2\int_0^\infty f(t)\cos \alpha t\, dt$$
$$\int_{-\infty}^\infty f(t)\sin \alpha t\, dt = 0$$

So, in that case, equation (7) can be written as

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos \alpha t\, dt\right)\cos \alpha x\, d\alpha \qquad (8)$$

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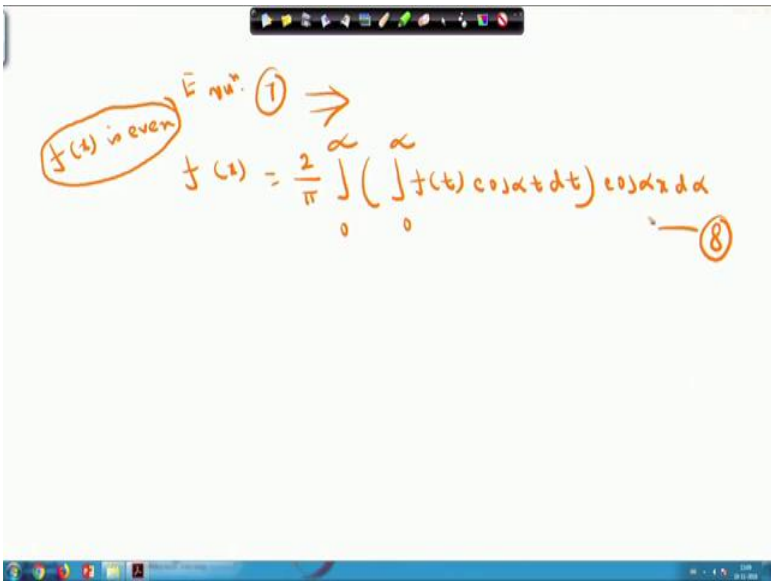

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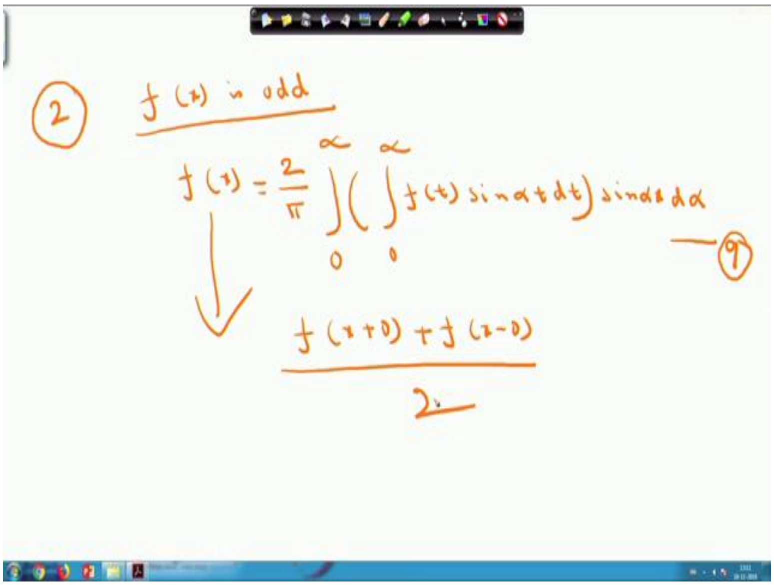

Case 2: Let $f(x)$ be odd. Whenever $f(x)$ is odd, $f(x)\cos\alpha t$ is odd and $f(x)\sin\alpha t$ is even. So,

$$\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt = 0$$

$$\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt = 2\int_{0}^{\infty} f(t)\sin\alpha t\,dt$$

So, if $f(x)$ be odd, equation (7) can be written as

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\int_{0}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\,d\alpha \qquad (9)$$

And again, please note that for both the equation (8) and (9), if at a particular point x, the function $f(x)$ is discontinuous, then the value of $f(x)$ will be equal to

$$\frac{f(x+0)+f(x-0)}{2}$$

Thank you.