

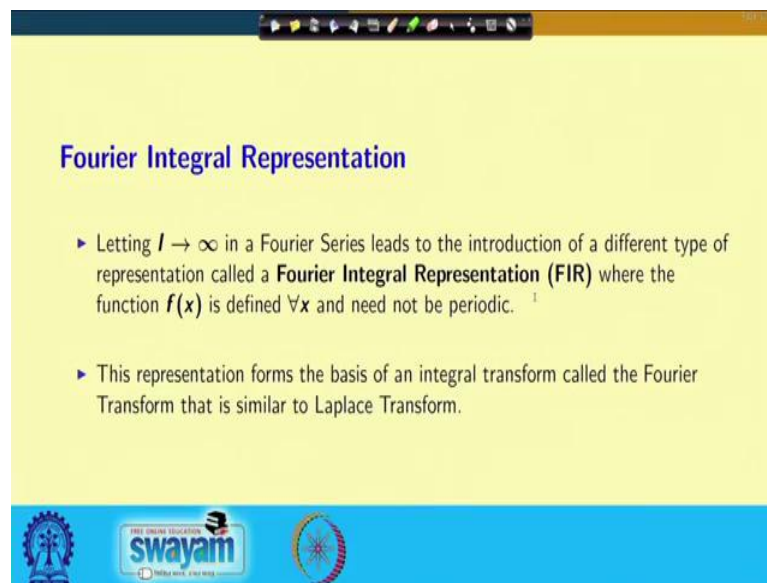
Transform Calculus and its applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 26
Fourier Integral Representation

From this lecture, effectively we are going to start another transform which we call the Fourier transform. In Laplace transform, basically the interval of the function was from $(0, \infty)$, but in physical problems always it may not be possible that the parameter lies in the interval $(0, \infty)$, but it may lie in the interval $(-\infty, \infty)$.




So, we want to study how to handle the situations where one function is defined in $(-\infty, \infty)$ and we want to find out a solution. For that basically, the Fourier transform was developed which we will discuss later, but initially let us start with the Fourier integral representation of a function.

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Fourier Integral Representation

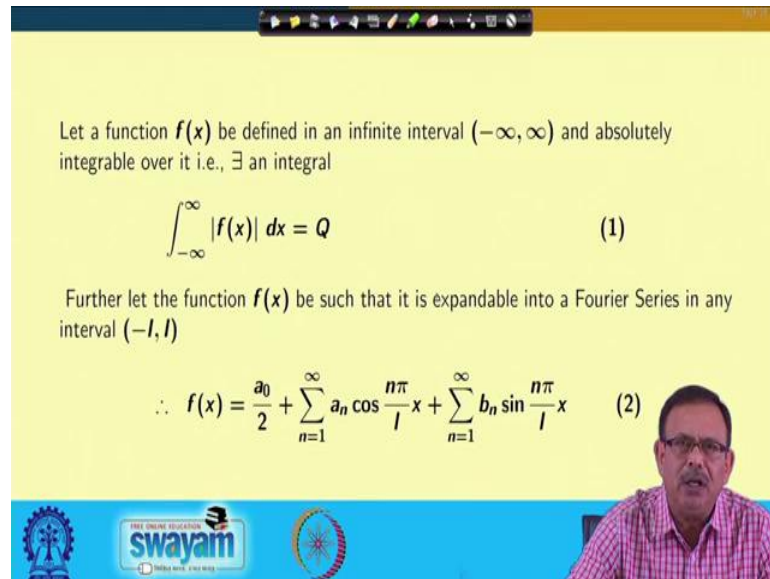
- ▶ Letting $l \rightarrow \infty$ in a Fourier Series leads to the introduction of a different type of representation called a **Fourier Integral Representation (FIR)** where the function $f(x)$ is defined $\forall x$ and need not be periodic. ¹
- ▶ This representation forms the basis of an integral transform called the Fourier Transform that is similar to Laplace Transform.

In Fourier Series representation, we can define a function in the interval $(-l, l)$. Now, when l approaches ∞ in a Fourier series, then this leads to introduction of a different type of representation which we call as Fourier Integral Representation or FIR where the function $f(x)$ is defined for all x and need not be periodic.

This representation forms the basis of an integral transform which we call as Fourier transform that is similar to Laplace transform except that in Laplace transform, the function $f(x)$ is defined in $(0, \infty)$, but here the function is defined in $(-\infty, \infty)$. This is because in real life, in many situations a function will be defined in $(-\infty, \infty)$. So, let us see what happens in this case.

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Let a function $f(x)$ be defined in an infinite interval $(-\infty, \infty)$ and absolutely integrable over it i.e., \exists an integral

$$\int_{-\infty}^{\infty} |f(x)| dx = Q \quad (1)$$

Further let the function $f(x)$ be such that it is expandable into a Fourier Series in any interval $(-l, l)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad (2)$$

We have a function $f(x)$ which is defined in the interval $(-\infty, \infty)$ and is term by term integrable, that is there exists an integral such that we can write down $\int_{-\infty}^{\infty} |f(x)| dx$ is equal to a finite value Q .

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Handwritten mathematical notes on a whiteboard:

$$\int_{-\infty}^{\infty} |f(x)| dx = Q \quad (1)$$

(-l, l)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad (2)$$
$$a_0 = \frac{1}{l} \int_{-l}^l f(t) dt, \quad a_n = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{n\pi}{l} t dt$$
$$b_n = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{n\pi}{l} t dt$$

So,

$$\int_{-\infty}^{\infty} |f(x)| dx = Q \quad (1)$$

If we evaluate the integral, it has a definite value. For that, suppose the function $f(x)$ can be expanded in terms of Fourier series in the interval $(-l, l)$ and we can write down

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad (2)$$

where

$$a_0 = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_n = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{n\pi}{l} t dt$$

$$b_n = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{n\pi}{l} t dt$$

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$$f(x) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \left[\int_{-l}^l f(t) \cos \frac{n\pi}{l} t dt \right] \cos \frac{n\pi}{l} x + \frac{1}{l} \sum_{n=1}^{\infty} \left[\int_{-l}^l f(t) \sin \frac{n\pi}{l} t dt \right] \sin \frac{n\pi}{l} x$$

So, if we put the values of these coefficients a_0 , a_n and b_n in the equation (2), we will get,

$$f(x) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \left[\int_{-l}^l f(t) \cos \frac{n\pi}{l} t dt \right] \cos \frac{n\pi}{l} x + \frac{1}{l} \sum_{n=1}^{\infty} \left[\int_{-l}^l f(t) \sin \frac{n\pi}{l} t dt \right] \sin \frac{n\pi}{l} x$$

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$$f(x) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l f(t) \left[\cos \frac{n\pi}{l} t \cos \frac{n\pi}{l} x + \sin \frac{n\pi}{l} t \sin \frac{n\pi}{l} x \right] dt$$

$$= \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l f(t) \cos \frac{n\pi(t-x)}{l} dt$$

$l \rightarrow \infty$
3

This we can write down as

$$\begin{aligned}
f(x) &= \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l f(t) \left[\cos \frac{n\pi}{l} t \cos \frac{n\pi}{l} x + \sin \frac{n\pi}{l} t \sin \frac{n\pi}{l} x \right] dt \\
&= \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l f(t) \cos \frac{n\pi(t-x)}{l} dt
\end{aligned} \tag{3}$$

Now we want to investigate what will be the form of the expression (3) whenever l approaches ∞ .

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The image shows a whiteboard with handwritten mathematical work. The first line defines a sequence of angles: $\alpha_1 = \frac{\pi}{l}, \alpha_2 = \frac{2\pi}{l}, \dots, \alpha_n = \frac{n\pi}{l}, \alpha_{n+1} = \frac{(n+1)\pi}{l}$. The second line shows the difference between consecutive terms: $\alpha_{n+1} - \alpha_n = \frac{\pi}{l}$, which is labeled as $\Delta\alpha_n = \frac{\pi}{l}$ and circled with a 4. The third line shows the function $f(x)$ as a sum of integrals: $f(x) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_{-l}^l f(t) \cos \alpha_n(t-x) dt \right) \Delta\alpha_n$. The final line shows the limit as $l \rightarrow \infty$, where the sum becomes an integral: $\cos \frac{\pi(t-x)}{l}$.

Let us introduce some new notations, say $\alpha_1 = \frac{\pi}{l}, \alpha_2 = \frac{2\pi}{l}, \dots, \alpha_n = \frac{n\pi}{l}$

$$\therefore \alpha_{n+1} - \alpha_n = \Delta\alpha_n = \frac{\pi}{l} \tag{4}$$

Now, we substitute this value from (4) into equation (3) to obtain

$$f(x) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_{-l}^l f(t) \cos \alpha_n(t-x) dt \right) \Delta\alpha_n \tag{5}$$

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Handwritten derivation on a whiteboard:

$$\left| \frac{1}{2l} \int_{-l}^l f(t) dt \right| \leq \frac{1}{2l} \int_{-l}^l |f(t)| dt$$

$$< \frac{1}{2l} \int_{-\infty}^{\infty} |f(t)| dt$$

$$= \frac{1}{2l} Q \rightarrow 0$$

Fix l , $2l \rightarrow \frac{\pi}{\alpha}$ to ∞

So, whenever $l \rightarrow \infty$, the first term $\frac{1}{2l} \int_{-l}^l f(t) dt$ approaches 0 since $\int_{-\infty}^{\infty} |f(x)| dx = Q$.

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Handwritten derivation on a whiteboard:

As $l \rightarrow \infty$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad \text{--- (6)}$$

FIR of $f(x)$

$$\frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha = \frac{f(x+0) + f(x-0)}{2}$$

And, as $l \rightarrow \infty$, equation (5) reduces to,

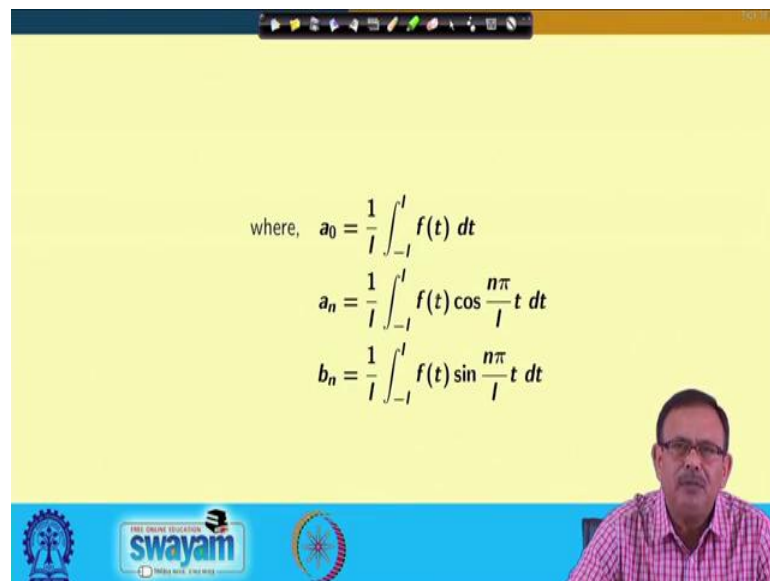
$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad (6)$$

And this expression is known as the Fourier integral representation of the function $f(x)$.

Equation (6) occurs at all points where the function is continuous. If the function is discontinuous at a particular point, by the concept what we have told in the Fourier series, we can write down

$$\frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha = \frac{f(x+0) + f(x-0)}{2}$$

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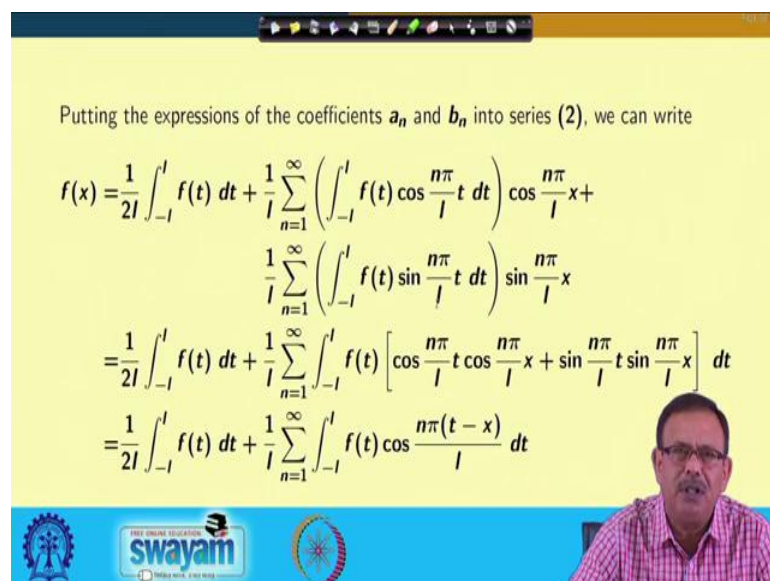
where, $a_0 = \frac{1}{l} \int_{-l}^l f(t) dt$

$$a_n = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{n\pi}{l} t dt$$

$$b_n = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{n\pi}{l} t dt$$

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Putting the expressions of the coefficients a_n and b_n into series (2), we can write

$$f(x) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \left(\int_{-l}^l f(t) \cos \frac{n\pi}{l} t dt \right) \cos \frac{n\pi}{l} x +$$

$$\frac{1}{l} \sum_{n=1}^{\infty} \left(\int_{-l}^l f(t) \sin \frac{n\pi}{l} t dt \right) \sin \frac{n\pi}{l} x$$

$$= \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l f(t) \left[\cos \frac{n\pi}{l} t \cos \frac{n\pi}{l} x + \sin \frac{n\pi}{l} t \sin \frac{n\pi}{l} x \right] dt$$

$$= \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_{n=1}^{\infty} \int_{-l}^l f(t) \cos \frac{n\pi(t-x)}{l} dt$$

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Let us investigate what form expression (3) will take when passing to the limit as $l \rightarrow \infty$

We introduce the following notation

$$\alpha_1 = \frac{\pi}{l}, \alpha_2 = \frac{2\pi}{l}, \dots, \alpha_n = \frac{n\pi}{l}, \alpha_{n+1} = \frac{(n+1)\pi}{l}$$
$$\therefore \alpha_{n+1} - \alpha_n = \frac{\pi}{l} \quad \text{i.e.,} \quad \Delta\alpha_n = \frac{\pi}{l} \quad (4)$$

Substituting into (3), we get


$$f(x) = \frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_{-l}^l f(t) \cos \alpha_n(t-x) dt \right) \Delta\alpha_n$$


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As $l \rightarrow \infty$, the first term on the RHS approaches 0.

$$\begin{aligned} \left| \frac{1}{2l} \int_{-l}^l f(t) dt \right| &\leq \frac{1}{2l} \int_{-l}^l |f(t)| dt \\ &< \frac{1}{2l} \int_{-\infty}^{\infty} |f(t)| dt \\ &= \frac{1}{2l} Q \\ &\rightarrow 0 \end{aligned}$$

For any fixed l , the expression in the parenthesis is a function of α_n , which takes on values from $\frac{\pi}{l}$ to ∞



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As $l \rightarrow \infty$, (5) takes the form

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha \quad (6)$$

This expression is known as the **FIR** of the function $f(x)$.

Equation (6) occurs \forall points where the function is continuous.

At points of discontinuity, we have

$$\frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x) dt \right) d\alpha = \frac{f(x+0) + f(x-0)}{2}$$

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Now, let us expand $\cos \alpha(t-x)$ in the equation (6).

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$$\cos \alpha(t-x) = \cos at \cos ax + \sin at \sin ax$$
$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos at dt \right) \cos ax d\alpha + \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \sin at dt \right) \sin ax d\alpha \quad (7)$$

$$\cos \alpha(t-x) = \cos at \cos ax + \sin at \sin ax$$

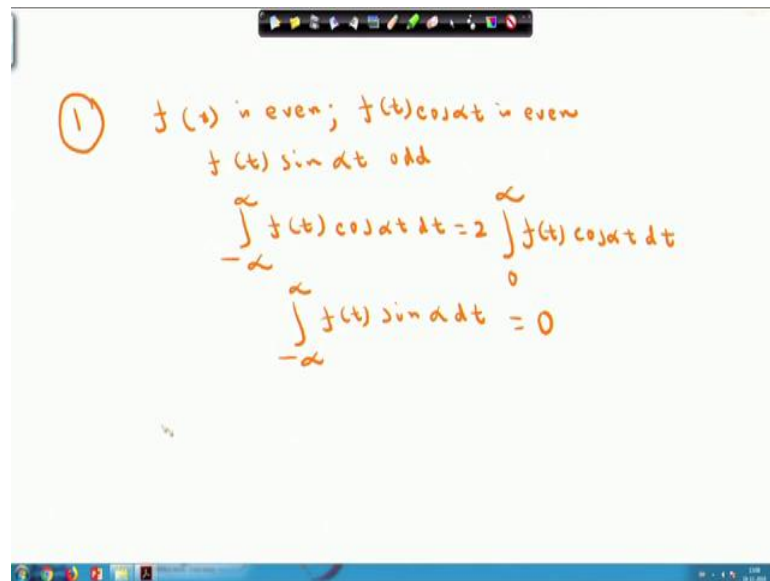
Putting this in equation (6), we have,

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos at \, dt \right) \cos ax \, d\alpha$$

$$+ \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \sin at \, dt \right) \sin ax \, d\alpha \quad (7)$$

Let us take some particular cases of this equation (7)

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Case 1: Let $f(x)$ be even. Whenever $f(x)$ is even, $f(x) \cos at$ is even and $f(x) \sin at$ is odd. So,

$$\int_{-\infty}^{\infty} f(t) \cos at \, dt = 2 \int_0^{\infty} f(t) \cos at \, dt$$

$$\int_{-\infty}^{\infty} f(t) \sin at \, dt = 0$$

So, in that case, equation (7) can be written as

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cos at \, dt \right) \cos ax \, d\alpha \quad (8)$$

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Handwritten mathematical derivation for the Fourier cosine series of an even function. The text is written in orange ink on a white background. It starts with "Eqn. (1) \Rightarrow " and a circled note "f(t) is even". The main equation is
$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cos \alpha t dt \right) \cos \alpha x d\alpha$$
 with a circled "8" at the end. The slide also shows a standard Windows taskbar at the bottom.

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Handwritten mathematical derivation for the Fourier sine series of an odd function. The text is written in orange ink on a white background. It starts with a circled "2" and "f(x) is odd". The main equation is
$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \sin \alpha t dt \right) \sin \alpha x d\alpha$$
 with a circled "9" at the end. Below this, an arrow points to the expression
$$\frac{f(x+0) + f(x-0)}{2}$$
 with a circled "2" underneath. The slide also shows a standard Windows taskbar at the bottom.

Case 2: Let $f(x)$ be odd. Whenever $f(x)$ is odd, $f(x) \cos at$ is odd and $f(x) \sin at$ is even. So,

$$\int_{-\infty}^{\infty} f(t) \cos at dt = 0$$

$$\int_{-\infty}^{\infty} f(t) \sin \alpha t dt = 2 \int_0^{\infty} f(t) \sin \alpha t dt$$

So, if $f(x)$ be odd, equation (7) can be written as

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \sin \alpha t dt \right) \sin \alpha x d\alpha \quad (9)$$

And again, please note that for both the equation (8) and (9), if at a particular point x , the function $f(x)$ is discontinuous, then the value of $f(x)$ will be equal to

$$\frac{f(x + 0) + f(x - 0)}{2}$$

Thank you.