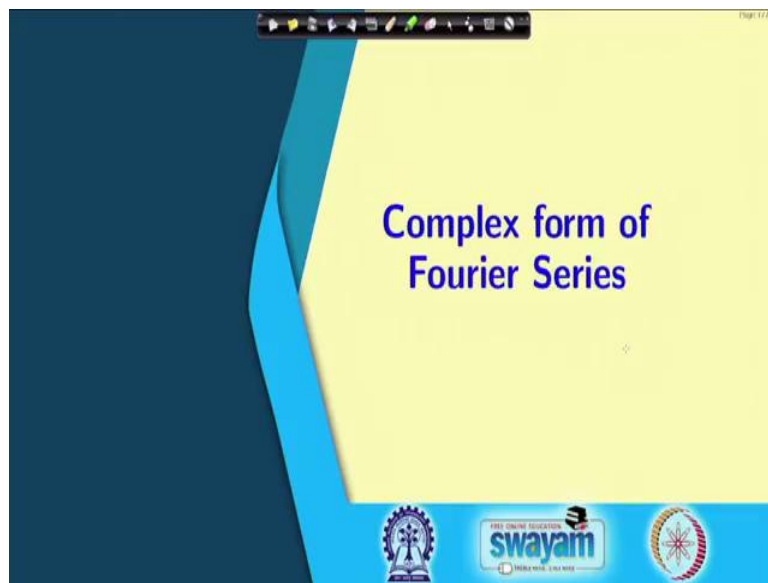


**Transform Calculus and its Applications in Differential Equations**  
**Prof. Adrijit Goswami**  
**Department of Mathematics**  
**Indian Institute of Technology, Kharagpur**

**Lecture – 25**  
**Complex form of Fourier Series**

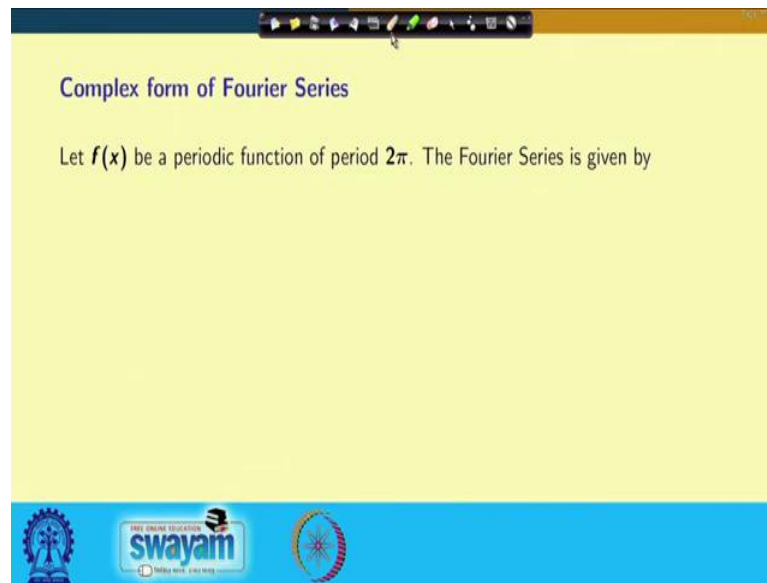
In this particular lecture, we will study the Complex form of Fourier series. Please note that, earlier whatever we have studied, that is a function of  $\cos nx$  and  $\sin nx$ .

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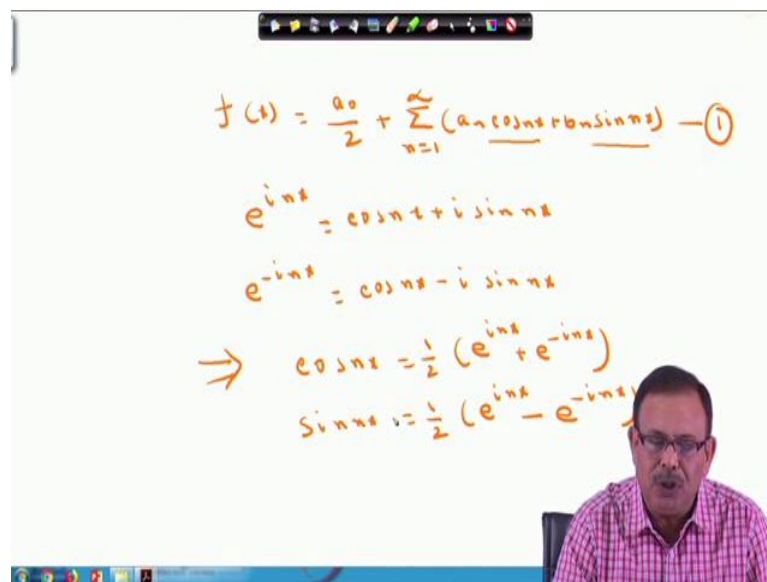
Now, we want to see what is the complex form of a Fourier series that is in the exponential form.

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So, for the complex form of the Fourier series, it is given that  $f(x)$  is a periodic function of period  $2\pi$ . From here, we will start and check the results.

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If  $f(x)$  be a periodic function of period  $2\pi$ , then the Fourier series of the function  $f(x)$  can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

Now, we also know these two things that

$$e^{inx} = \cos nx + i \sin nx$$

$$e^{-inx} = \cos nx - i \sin nx$$

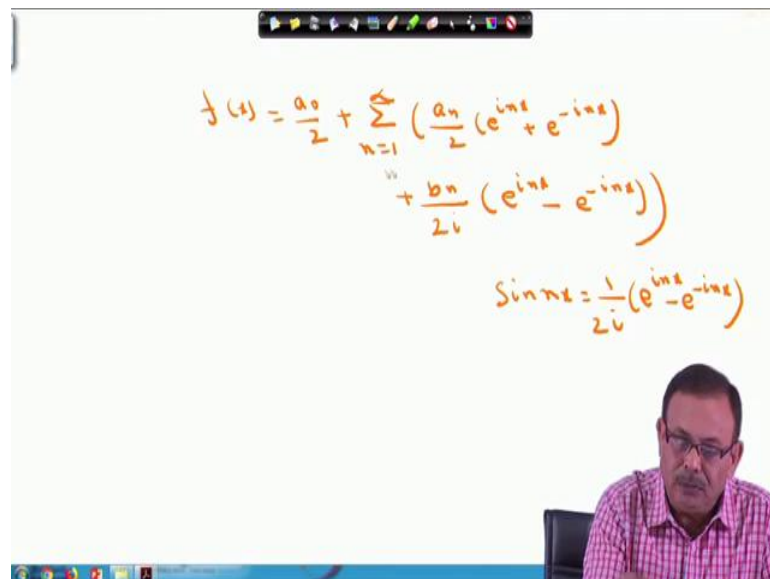
From these two equations, we can express  $\cos nx$  and  $\sin nx$  in terms of  $e^{inx}$  and  $e^{-inx}$  as

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$

$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$$

We will now substitute these values of  $\cos nx$  and  $\sin nx$  in equation (1).

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Then  $f(x)$  can be written as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left[ a_n (e^{inx} + e^{-inx}) + \frac{b_n}{i} (e^{inx} - e^{-inx}) \right] \\ &= \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left[ (a_n - ib_n) e^{inx} + (a_n + ib_n) e^{-inx} \right] \\ &= C_0 + \sum_{n=1}^{\infty} (C_n e^{inx} + C_{-n} e^{-inx}) \end{aligned}$$

where

$$C_0 = \frac{a_0}{2}$$

$$C_n = \frac{1}{2}(a_n - ib_n)$$

$$C_{-n} = \frac{1}{2}(a_n + ib_n)$$

(Refer Slide Time: 04:36)

The image shows a whiteboard with handwritten mathematical derivations. At the top, the function is given as  $f(x) = C_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$ . Below this, the coefficients are defined as  $c_0 = \frac{a_0}{2}$ ,  $c_n = \frac{1}{2}(a_n - ib_n)$ , and  $c_{-n} = \frac{1}{2}(a_n + ib_n)$ . The final line shows the calculation of  $c_0$  as  $c_0 = \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{0 \cdot ix} dx$ .

Now, if we substitute the value of  $a_0$ , then  $C_0$  is nothing but

$$\begin{aligned} C_0 &= \frac{a_0}{2} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{0 \cdot ix} dx \end{aligned}$$

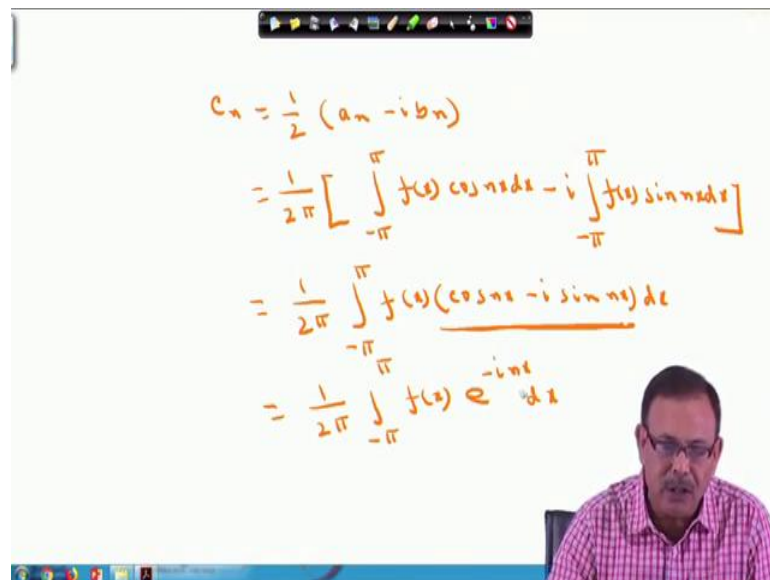
Similarly, substituting the values of  $a_n$  and  $b_n$ , we have,

$$\begin{aligned} C_n &= \frac{1}{2}(a_n - ib_n) \\ &= \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} f(x) \cos nx dx - i \int_{-\pi}^{\pi} f(x) \sin nx dx \right) \end{aligned}$$

$$\Rightarrow C_n = \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

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A video frame showing a whiteboard with handwritten mathematical derivations. The equations are:

$$C_n = \frac{1}{2} (a_n - ib_n)$$

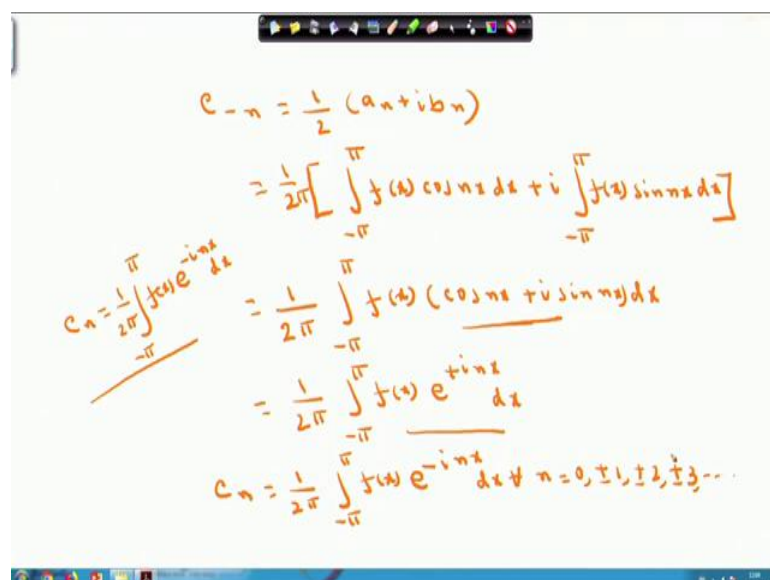
$$= \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} f(x) \cos nx dx - i \int_{-\pi}^{\pi} f(x) \sin nx dx \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

A man in a pink checkered shirt is visible in the bottom right corner of the frame.

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A video frame showing a whiteboard with handwritten mathematical derivations. The equations are:

$$C_{-n} = \frac{1}{2} (a_n + ib_n)$$

$$= \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} f(x) \cos nx dx + i \int_{-\pi}^{\pi} f(x) \sin nx dx \right]$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad \forall n = 0, \pm 1, \pm 2, \pm 3, \dots$$

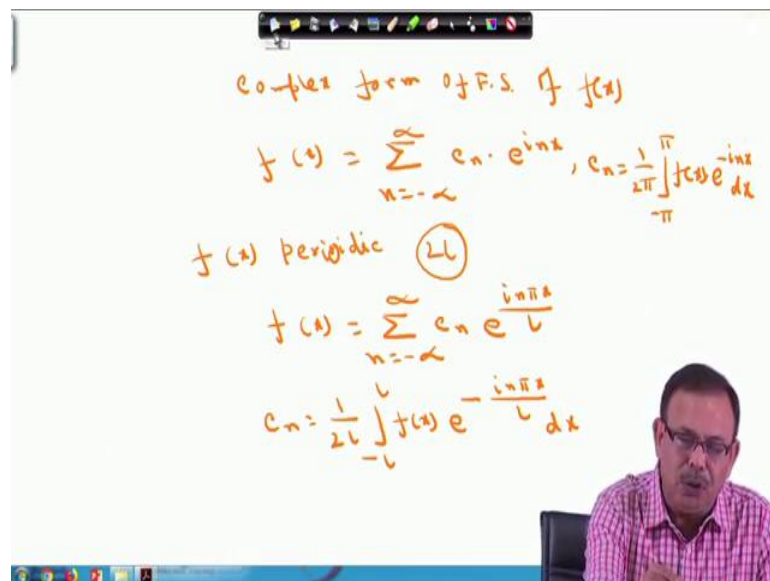
And

$$\begin{aligned}C_{-n} &= \frac{1}{2}(a_n + ib_n) \\&= \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} f(x) \cos nx \, dx + i \int_{-\pi}^{\pi} f(x) \sin nx \, dx \right) \\&= \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx) \, dx \right] \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx\end{aligned}$$

Combining all  $C_n$ 's, we can write

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots \quad (2)$$

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Therefore, the complex form of the Fourier series of  $f(x)$  can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

where  $C_n$  is given by (2).

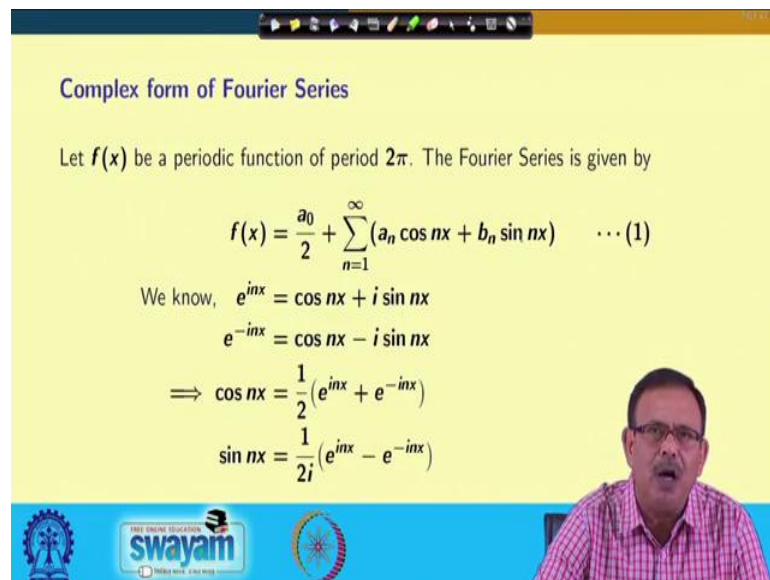
If  $f(x)$  is a periodic function with period  $2l$  say, in that case we have the complex form of the Fourier series of  $f(x)$  as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i n \pi x}{l}}$$

where

$$C_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{i n \pi x}{l}} dx \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

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**Complex form of Fourier Series**

Let  $f(x)$  be a periodic function of period  $2\pi$ . The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \dots (1)$$

We know,  $e^{inx} = \cos nx + i \sin nx$   
 $e^{-inx} = \cos nx - i \sin nx$

$$\Rightarrow \cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$

$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$$


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Substituting the values of  $\cos nx$  and  $\sin nx$  in equation (1) ,



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n}{2} (e^{inx} + e^{-inx}) + \frac{b_n}{2i} (e^{inx} - e^{-inx}) \right)$$
$$= C_0 + \sum_{n=1}^{\infty} (C_n e^{inx} + C_{-n} e^{-inx})$$

where,  $C_0 = \frac{a_0}{2}$ ,  $C_n = \frac{1}{2}(a_n - i b_n)$  and  $C_{-n} = \frac{1}{2}(a_n + i b_n)$



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Now,  $C_0 = \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{0 \cdot ix} dx$

$$C_n = \frac{1}{2}(a_n - i b_n)$$
$$= \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} f(x) \cos nx dx - i \int_{-\pi}^{\pi} f(x) \sin nx dx \right)$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$




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$$\begin{aligned}C_{-n} &= \frac{1}{2}(a_n + i b_n) \\&= \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} f(x) \cos nx \, dx + i \int_{-\pi}^{\pi} f(x) \sin nx \, dx \right) \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx) \, dx \\&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{+inx} \, dx \\ \therefore C_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad \forall n = 0, \pm 1, \pm 2, \pm 3, \dots\end{aligned}$$

The slide includes logos for Swamyam and other educational institutions at the bottom.

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Therefore the complex form of the Fourier Series of  $f(x)$  can be written as,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

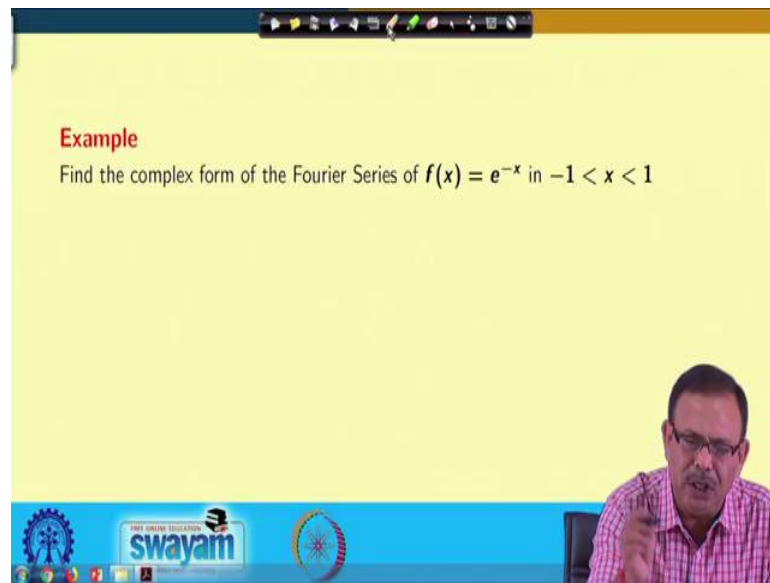
If  $f(x)$  is a periodic function of period  $2l$  the complex form of the Fourier Series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i n \pi x}{l}} \quad \text{where } C_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{i n \pi x}{l}} dx$$

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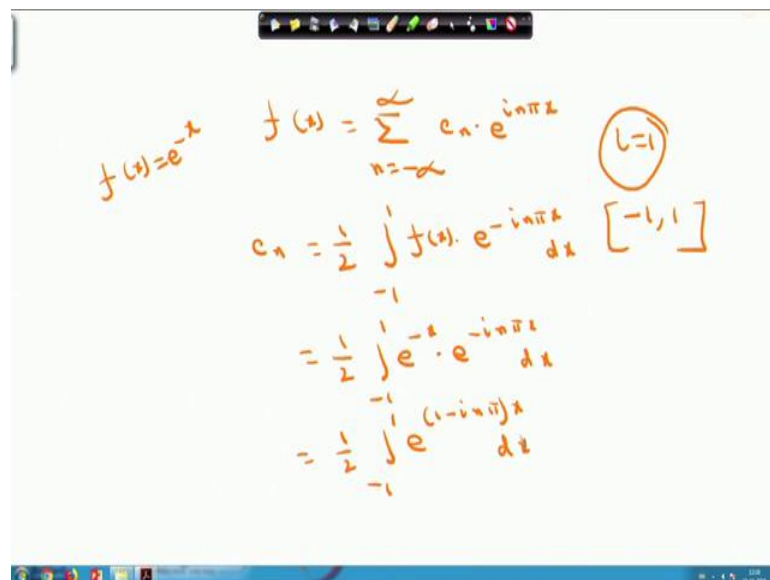
So, this is the complex form of the Fourier representation of a function  $f(x)$  which may be defined in  $(-\pi, \pi)$  or in  $(-l, l)$ .

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Now, let us take an example. Suppose we want to find out the complex form of the Fourier series of the function  $f(x) = e^{-x}$  in  $(-1,1)$ .

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We know that the complex form of the Fourier series of  $f(x)$  always can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

since  $l = 1$  in this case and

$$\begin{aligned}
C_n &= \frac{1}{2} \int_{-1}^1 f(x) e^{-in\pi x} dx \\
&= \frac{1}{2} \int_{-1}^1 e^{-x} e^{-in\pi x} dx \\
&= \frac{1}{2} \int_{-1}^1 e^{-(1+in\pi)x} dx \\
&= \frac{[e^{(1+in\pi)} - e^{-(1+in\pi)}]}{2(1+in\pi)}
\end{aligned}$$

Expanding  $e^{in\pi}$  and  $e^{-in\pi}$  in sines and cosines, we obtain

$$C_n = \frac{(-1)^n(1 - in\pi)}{(1 + n^2\pi^2)} \sinh 1$$

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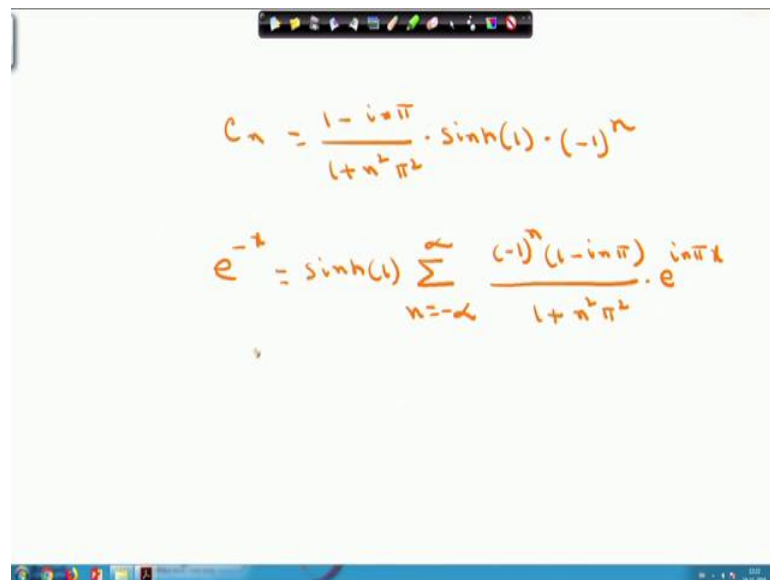
The image shows a handwritten derivation of the complex Fourier coefficient  $C_n$  for the function  $f(x) = e^{-x}$  on the interval  $[-1, 1]$ . The steps are as follows:

$$\begin{aligned}
C_n &= \frac{1}{2} \left[ \frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^1 \\
&= \frac{1}{2(1+in\pi)} [e^{-(1+in\pi)} - e^{(1+in\pi)}] \\
&= \frac{1 - in\pi}{2(1+n^2\pi^2)} [e^{(\cos\pi - i\sin\pi)} - e^{-(\cos\pi + i\sin\pi)}] \\
&= \frac{1 - in\pi}{2(1+n^2\pi^2)} (e - e^{-1}) \cos\pi
\end{aligned}$$

Therefore, the complex form of the Fourier series of  $f(x) = e^{-x}$  is expressed as

$$e^{-x} = \sinh 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1 - in\pi)}{(1 + n^2\pi^2)} e^{in\pi x}$$

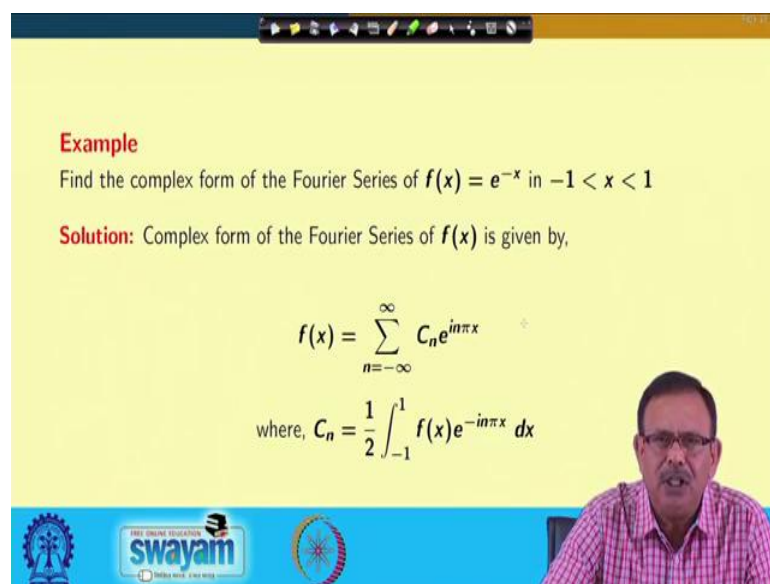
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The image shows a whiteboard with handwritten mathematical expressions. The first expression is  $C_n = \frac{1 - i^n \pi}{1 + n^2 \pi^2} \cdot \sinh(1) \cdot (-1)^n$ . The second expression is  $e^{-x} = \sinh(1) \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - i^n \pi)}{1 + n^2 \pi^2} \cdot e^{in\pi x}$ . The whiteboard has a toolbar at the top and a Windows taskbar at the bottom.

$$C_n = \frac{1 - i^n \pi}{1 + n^2 \pi^2} \cdot \sinh(1) \cdot (-1)^n$$
$$e^{-x} = \sinh(1) \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - i^n \pi)}{1 + n^2 \pi^2} \cdot e^{in\pi x}$$

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The slide has a yellow background. It contains the following text: **Example** Find the complex form of the Fourier Series of  $f(x) = e^{-x}$  in  $-1 < x < 1$ . **Solution:** Complex form of the Fourier Series of  $f(x)$  is given by, 
$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$
 where, 
$$C_n = \frac{1}{2} \int_{-1}^1 f(x) e^{-in\pi x} dx$$
 A presenter is visible in the bottom right corner. The slide footer includes logos for IIT Bombay, Swayam, and IIT Madras.

**Example**  
Find the complex form of the Fourier Series of  $f(x) = e^{-x}$  in  $-1 < x < 1$

**Solution:** Complex form of the Fourier Series of  $f(x)$  is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

where, 
$$C_n = \frac{1}{2} \int_{-1}^1 f(x) e^{-in\pi x} dx$$

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The slide displays the following derivation for the complex Fourier coefficient  $C_n$ :

$$\begin{aligned} \Rightarrow C_n &= \frac{1}{2} \int_{-1}^1 e^{-x} e^{-in\pi x} dx \\ &= \frac{1}{2} \int_{-1}^1 e^{(1-in\pi)x} dx \\ &= \frac{1}{2} \left[ \frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^1 \\ &= \frac{1}{2(1+in\pi)} (e^{-(1+in\pi)} - e^{-(1-in\pi)}) \end{aligned}$$

The slide also features a video feed of the instructor in the bottom right corner and logos for Swamyam and other institutions at the bottom.

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The slide is titled "Example" and contains the following text:

Find the complex form of the Fourier Series of  $f(x) = e^{ax}$  in  $-\pi < x < \pi$

The slide also features a video feed of the instructor in the bottom right corner and logos for Swamyam and other institutions at the bottom.

Let us take one more example. Suppose we want to find out the complex form of the Fourier series of the function  $f(x) = e^{ax}$  in  $(-\pi, \pi)$ .

We know that the complex form of the Fourier series of  $f(x)$  always can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

since  $l = \pi$  in this case and

$$\begin{aligned}
C_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx \\
&= \frac{[e^{(a-in)\pi} - e^{-(a-in)\pi}]}{2\pi(a-in)}
\end{aligned}$$

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The image shows a whiteboard with handwritten mathematical work. At the top, it states  $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$  and  $f(x) = e^{ax}$ . Below this, the coefficient  $c_n$  is calculated as  $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$ . This is then simplified to  $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} \cdot e^{-inx} dx$ , and finally to  $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx$ .

Expanding  $e^{in\pi}$  and  $e^{-in\pi}$  in sines and cosines, we obtain

$$C_n = \frac{(-1)^n(a+in)}{\pi(a^2+n^2)} \sinh(a\pi)$$


Therefore, the complex form of the Fourier series of  $f(x) = e^{ax}$  is expressed as

$$e^{ax} = \frac{\sinh(a\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)}{(a^2+n^2)} e^{inx}$$

(Refer Slide Time: 26:10)

$$\begin{aligned}c_n &= \frac{1}{2\pi} \left[ \frac{e^{(a-in)x}}{a-in} \right]_{-\pi}^{\pi} \\&= \frac{1}{2\pi(a-in)} \left[ e^{(a-in)\pi} - e^{-(a-in)\pi} \right] \\&= \frac{a+in}{2\pi(a^2+n^2)} \left[ e^{a\pi} (\cos n\pi - i \sin n\pi) \right. \\&\quad \left. - e^{-a\pi} (\cos n\pi + i \sin n\pi) \right] \\&= \frac{a+in}{2\pi(a^2+n^2)} \cdot \frac{e^{a\pi} - e^{-a\pi}}{2} \cos n\pi\end{aligned}$$

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$$\begin{aligned}c_n &= \frac{a+in}{\pi(a^2+n^2)} \sinh(a\pi) \cdot (-1)^n \\e^{at} &= \frac{\sinh(a\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{a^2+n^2} e^{int}\end{aligned}$$


Thank you.