Transform Calculus and its Applications in Differential Equations
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Lecture – 25
Complex form of Fourier Series

In this particular lecture, we will study the Complex form of Fourier series. Please note that, earlier whatever we have studied, that is a function of $cos\ nx$ and $\sin nx$.

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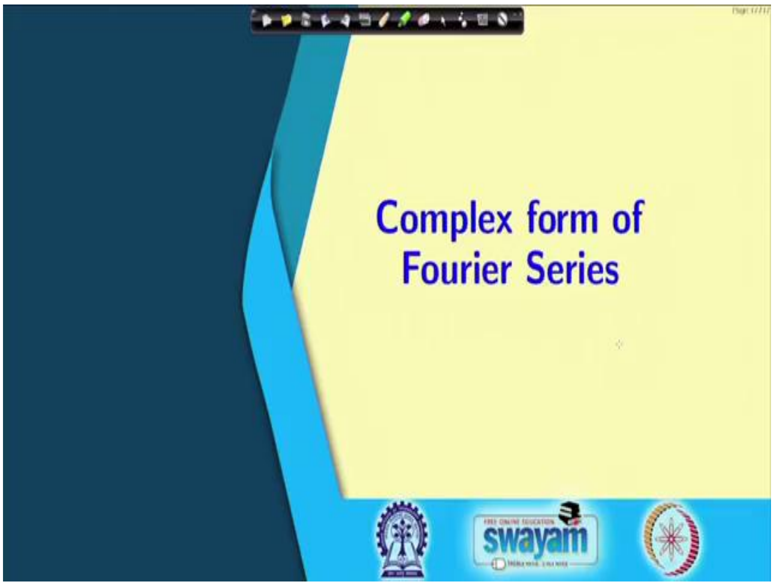

Now, we want to see what is the complex form of a Fourier series that is in the exponential form.

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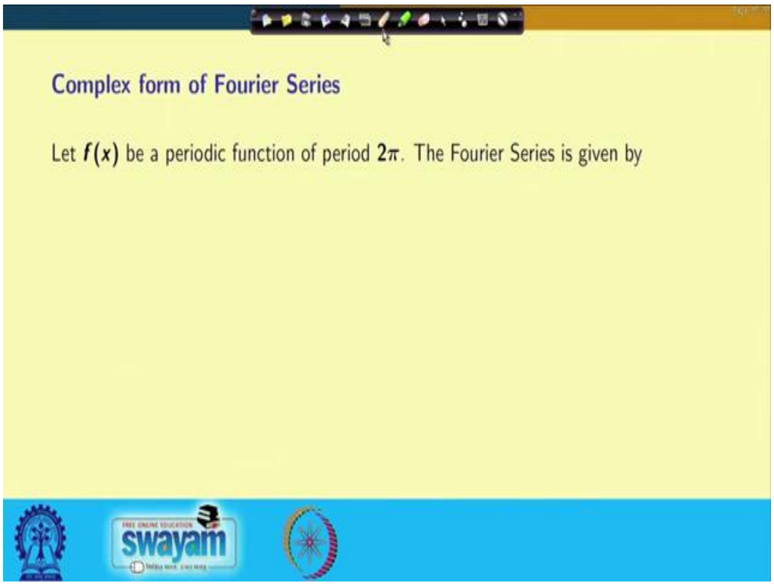

So, for the complex form of the Fourier series, it is given that $f(x)$ is a periodic function of period 2π. From here, we will start and check the results.

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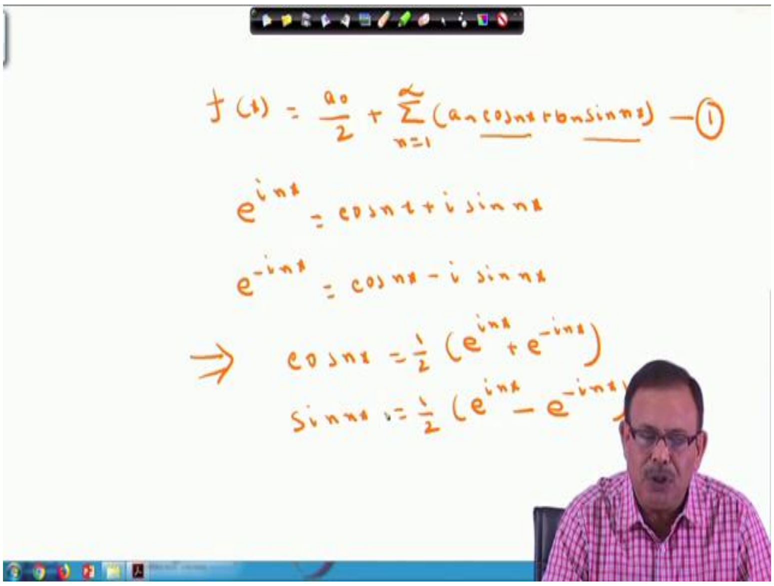

If $f(x)$ be a periodic function of period 2π, then the Fourier series of the function $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1)$$

Now, we also know these two things that

$$e^{inx} = \cos nx + i \sin nx$$
$$e^{-inx} = \cos nx - i \sin nx$$

From these two equations, we can express $\cos nx$ and $\sin nx$ in terms of e^{inx} and e^{-inx} as

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$$
$$\sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

We will now substitute these values of $cos\ nx$ and $sin\ nx$ in equation (1).

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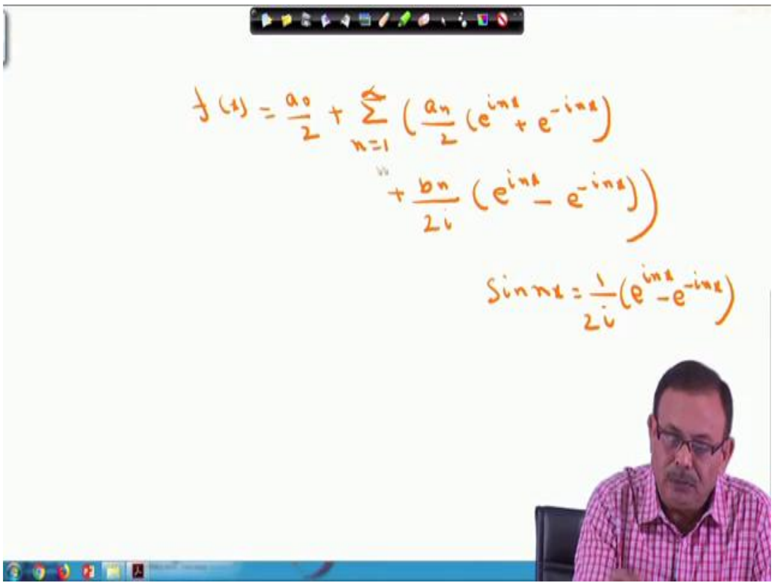

Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n\left(e^{inx} + e^{-inx}\right) + \frac{b_n}{i}\left(e^{inx} - e^{-inx}\right)\right]$$
$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left[(a_n - ib_n)e^{inx} + (a_n + ib_n)e^{-inx}\right]$$
$$= C_0 + \sum_{n=1}^{\infty}\left(C_n e^{inx} + C_{-n} e^{-inx}\right)$$

where

$$C_0 = \frac{a_0}{2}$$

$$C_n = \frac{1}{2}(a_n - ib_n)$$

$$C_{-n} = \frac{1}{2}(a_n + ib_n)$$

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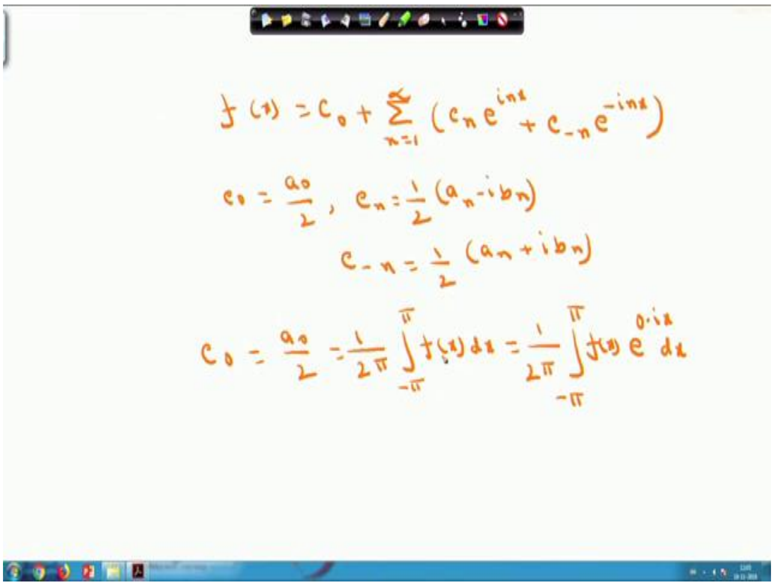

Now, if we substitute the value of a_0, then C_0 is nothing but

$$\begin{aligned} C_0 &= \frac{a_0}{2} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{0.ix}dx \end{aligned}$$

Similarly, substituting the values of a_n and b_n, we have,

$$\begin{aligned} C_n &= \frac{1}{2}(a_n - ib_n) \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\cos nx\, dx - i\int_{-\pi}^{\pi} f(x)\sin nx\, dx\right) \end{aligned}$$

$$\Rightarrow C_n = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(x)\,(\cos nx - i\sin nx)\ dx\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\ dx$$

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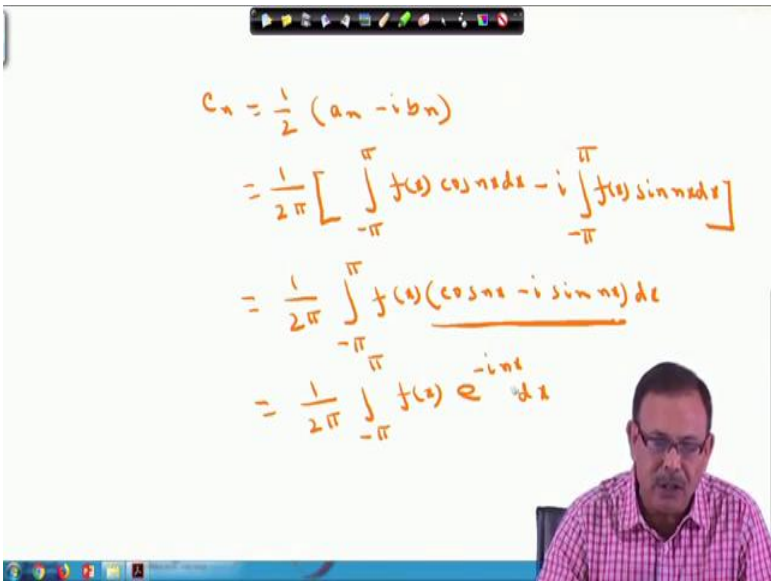

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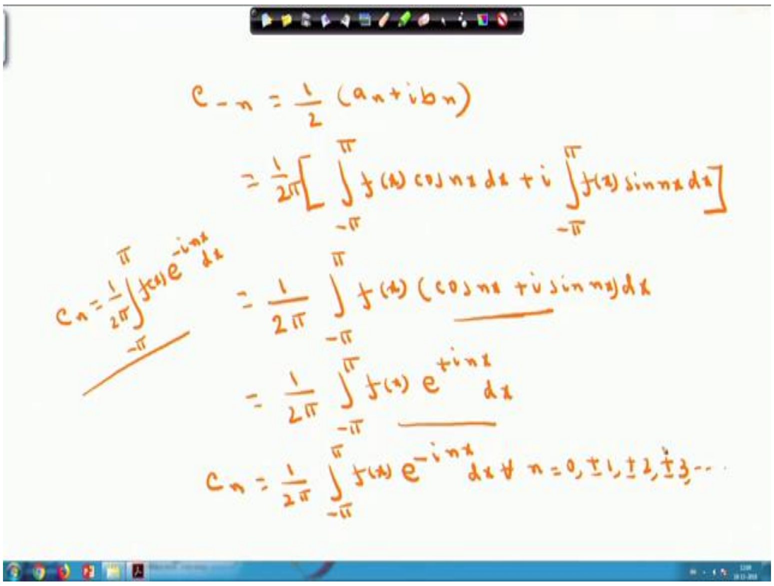

And

$$
\begin{aligned}
C_{-n} &= \frac{1}{2}(a_n + ib_n) \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\cos nx\, dx + i\int_{-\pi}^{\pi} f(x)\sin nx\, dx\right) \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(x)\,(\cos nx + i\sin nx)\ dx\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\ dx
\end{aligned}
$$

Combining all $C_n{}'s$, we can write

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\ dx \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \qquad (2)$$

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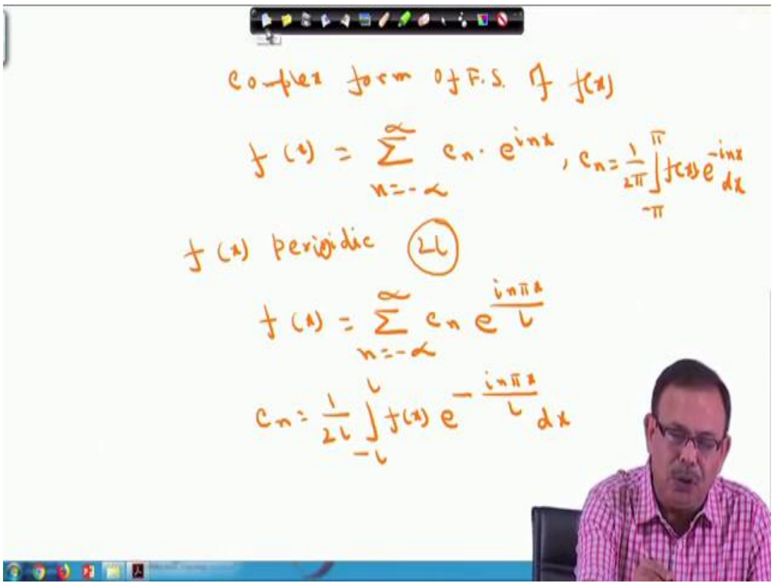

Therefore, the complex form of the Fourier series of $f(x)$ can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

where C_n is given by (2).

If $f(x)$ is a periodic function with period $2l$ say, in that case we have the complex form of the Fourier series of $f(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

where

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

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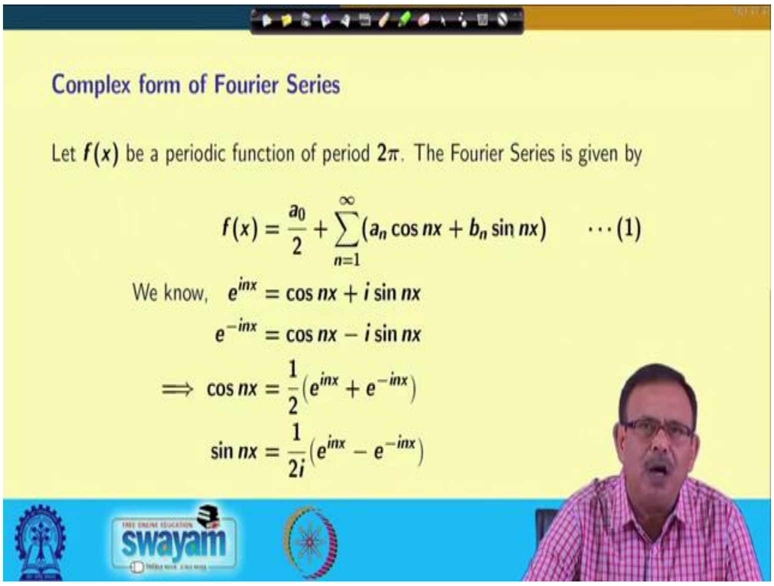

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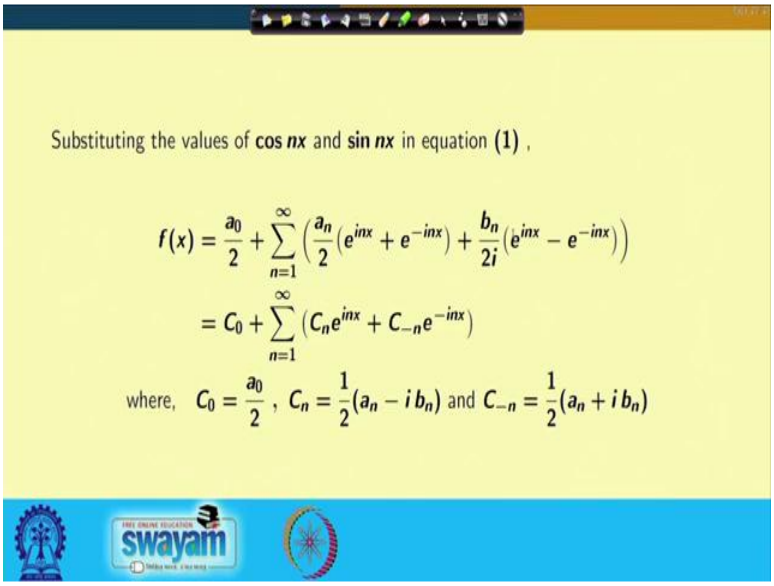

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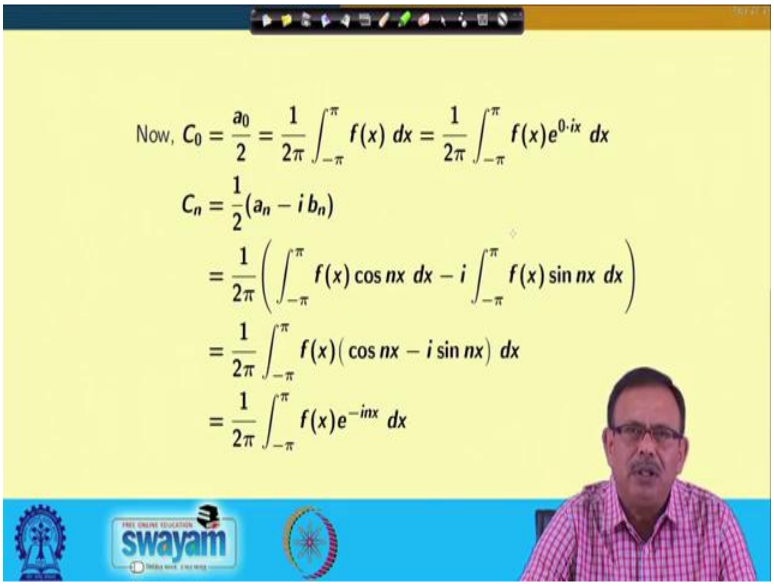

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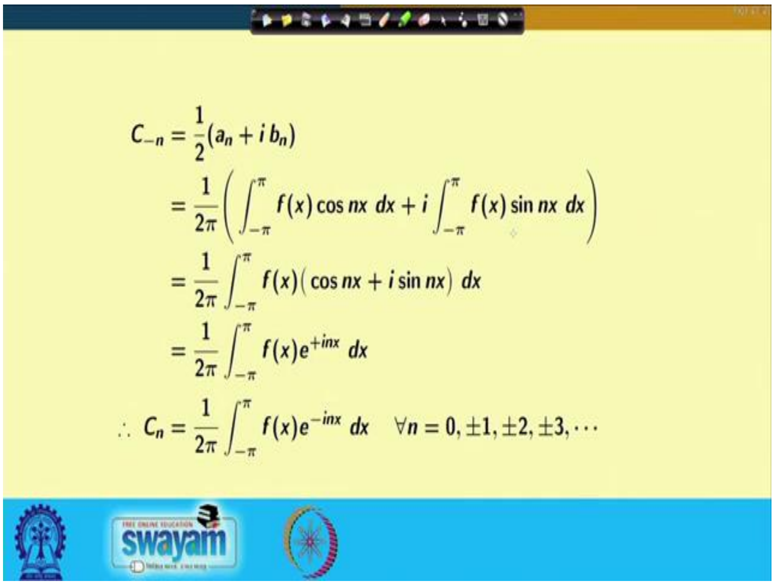

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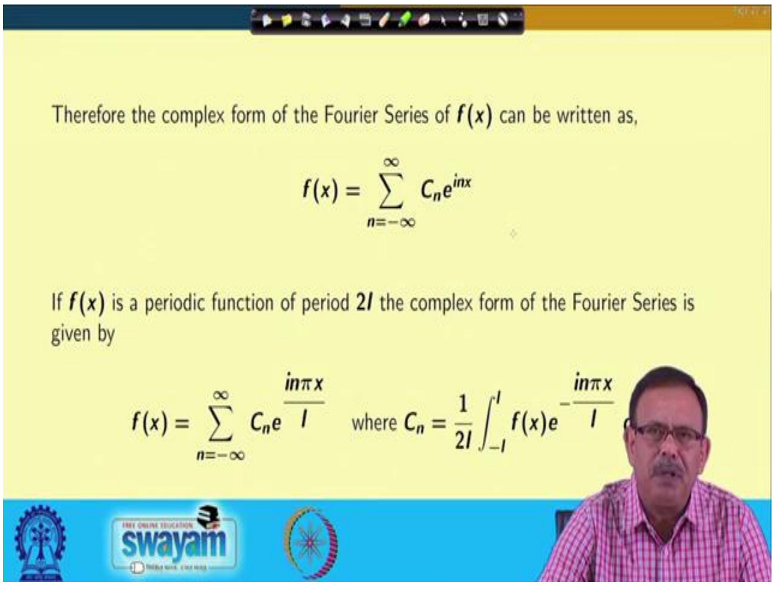

So, this is the complex form of the Fourier representation of a function $f(x)$ which may be defined in $(-\pi, \pi)$ or in $(-l, l)$.

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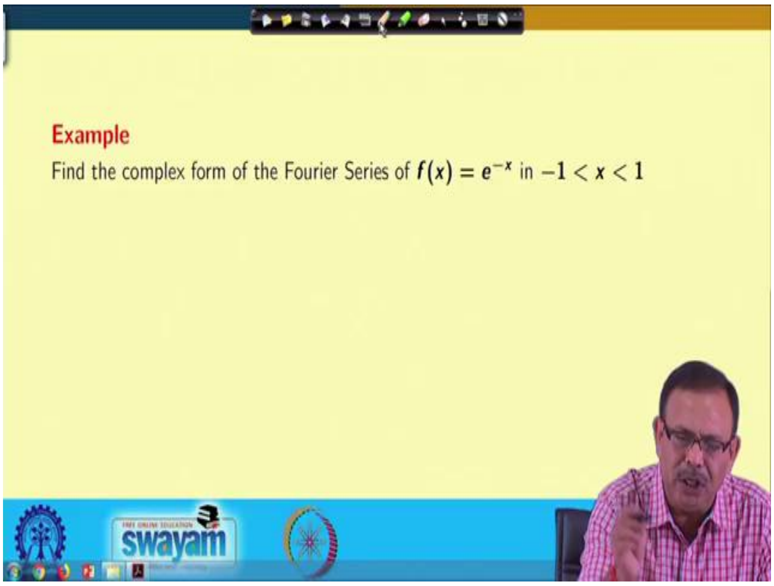

Now, let us take an example. Suppose we want to find out the complex form of the Fourier series of the function $f(x) = e^{-x}$ in $(-1,1)$.

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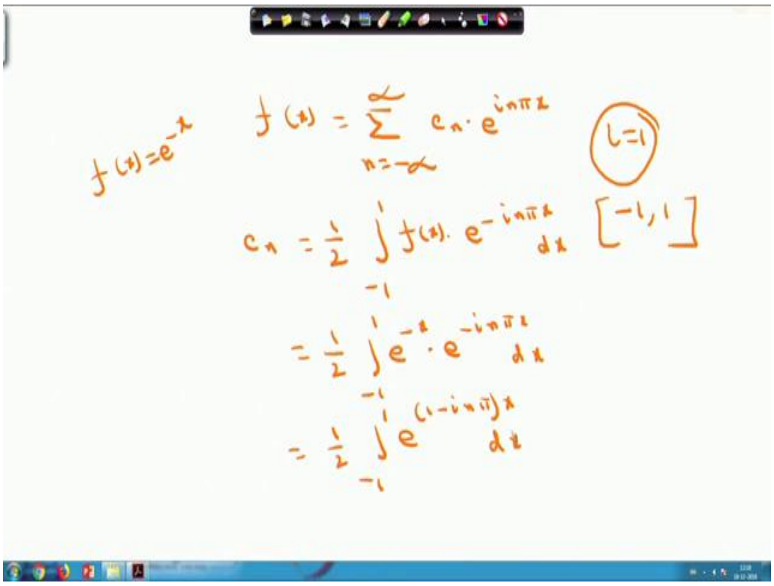

We know that the complex form of the Fourier series of $f(x)$ always can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

since $l = 1$ in this case and

$$C_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-x}\, e^{-in\pi x}\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\ dx$$

$$= \frac{\left[e^{(1+in\pi)} - e^{-(1+in\pi)}\right]}{2(1+in\pi)}$$

Expanding $e^{in\pi}$ and $e^{-in\pi}$ in sines and cosines, we obtain

$$C_n = \frac{(-1)^n(1-in\pi)}{(1+n^2\pi^2)} \sinh 1$$

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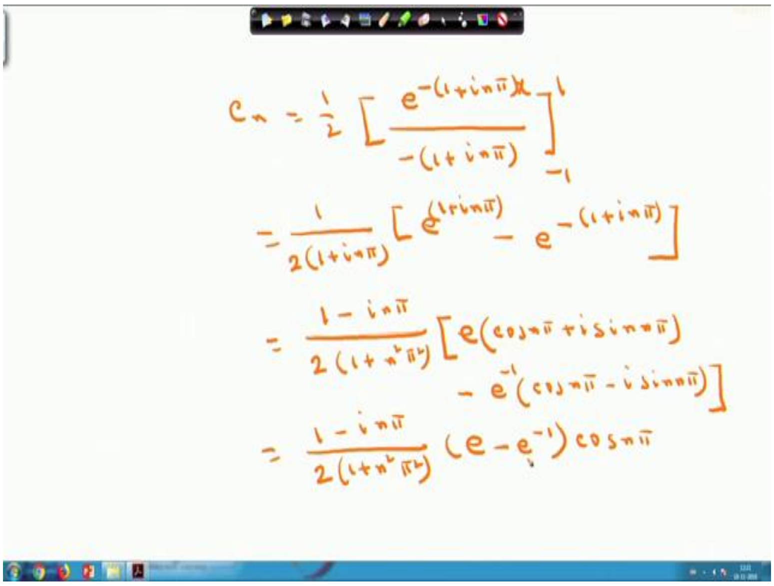

Therefore, the complex form of the Fourier series of $f(x) = e^{-x}$ is expressed as

$$e^{-x} = \sinh 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)}{(1+n^2\pi^2)} e^{in\pi x}$$

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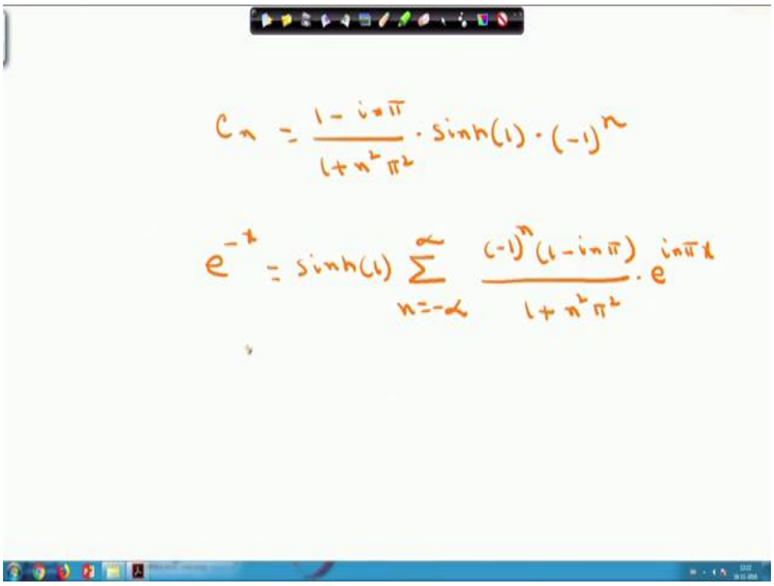

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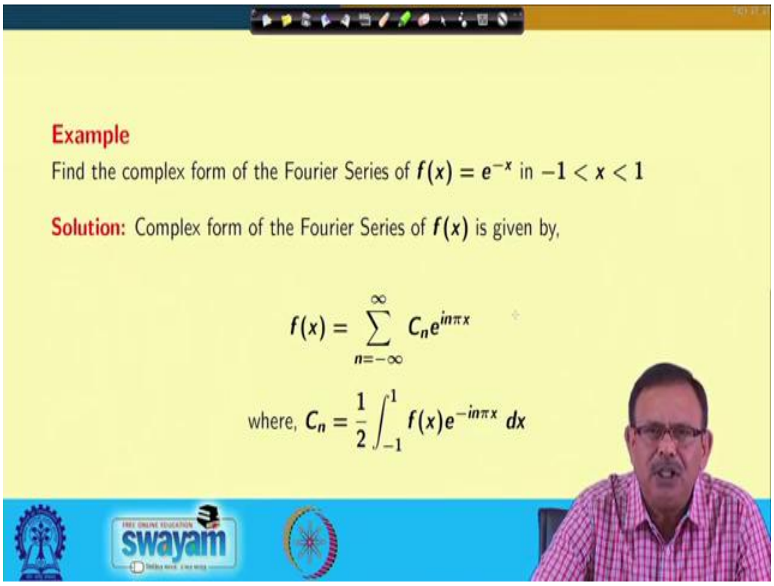

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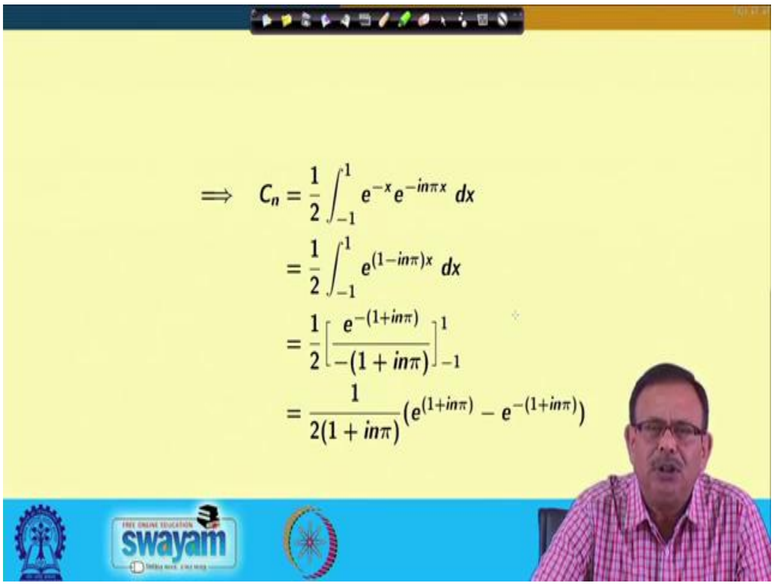

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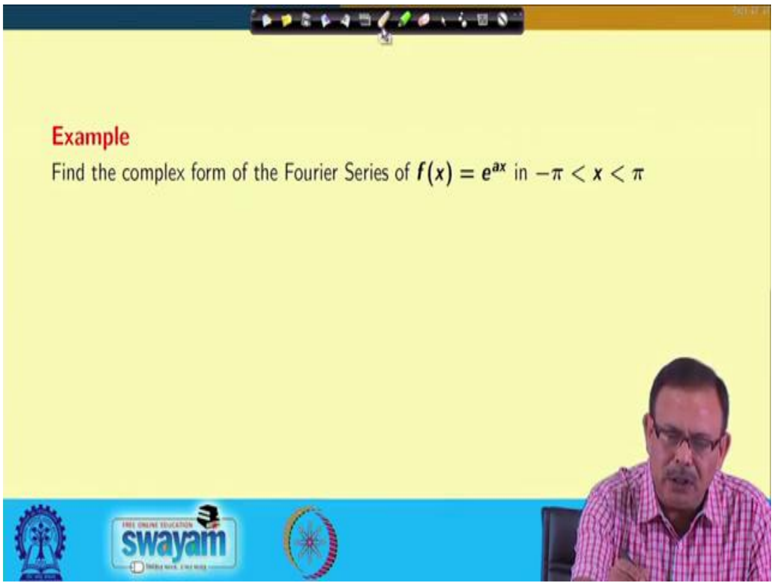

Let us take one more example. Suppose we want to find out the complex form of the Fourier series of the function $f(x) = e^{ax}$ in $(-\pi, \pi)$.

We know that the complex form of the Fourier series of $f(x)$ always can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

since $l = \pi$ in this case and

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\ dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\, e^{-inx}\ dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\ dx$$

$$= \frac{\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]}{2\pi(a-in)}$$

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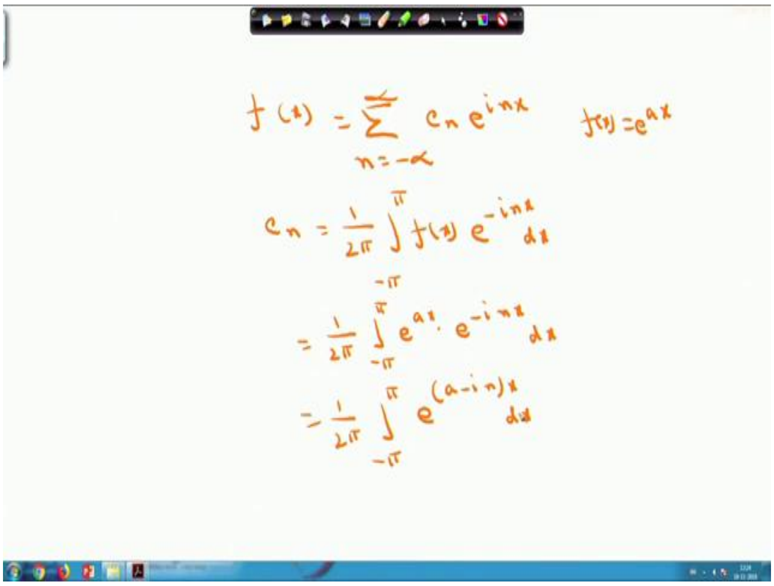

Expanding $e^{in\pi}$ and $e^{-in\pi}$ in sines and cosines, we obtain

$$C_n = \frac{(-1)^n(a+in)}{\pi(a^2+n^2)}\sinh(a\pi)$$

Therefore, the complex form of the Fourier series of $f(x) = e^{ax}$ is expressed as

$$e^{ax} = \frac{\sinh(a\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{(a^2+n^2)}e^{inx}$$

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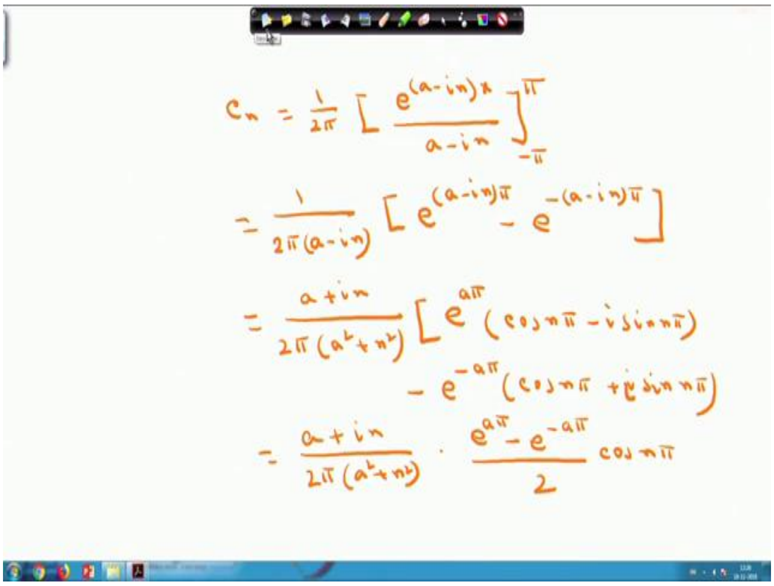

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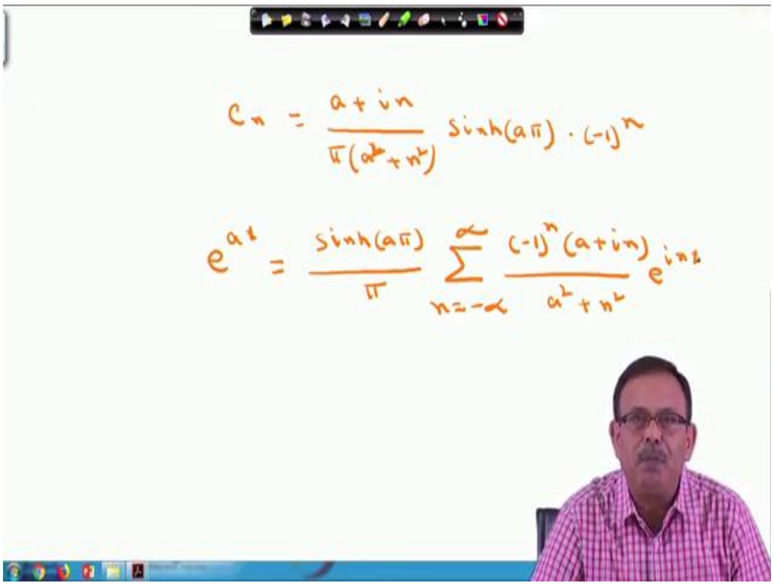

Thank you.