Transform Calculus and its applications in Differential Equations
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Lecture – 24
Parseval’s Theorem and its Applications

In the last lecture, we have done the half range Fourier sine and cosine series whenever a function is defined in $[0, l]$. The function $f(x)$ which is defined in $[0, l]$ can be expressed as a half range Fourier sine series or can be expressed as a half range Fourier cosine series; accordingly as we are expressing it in only sine terms or in cosine terms respectively.

Now in this particular lecture, we will study a theorem which we call the Parseval’s Theorem and after that, we will study the applications of Parseval’s Theorem as well.

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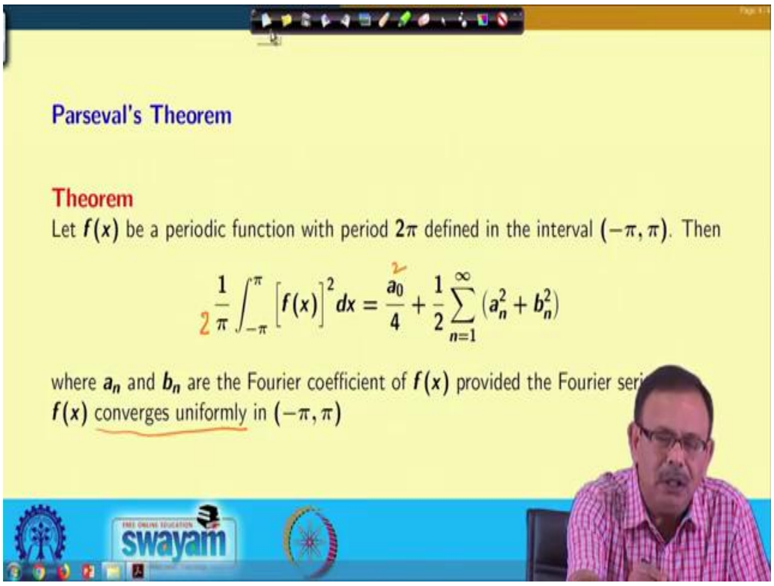

Parseval’s Theorem states: let $f(x)$ be a periodic function with period 2π and defined in the interval $(-\pi, \pi)$, then the following relation holds

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where, a_n and b_n are the Fourier coefficients of $f(x)$ provided the Fourier series of $f(x)$ converges uniformly in $(-\pi, \pi)$.

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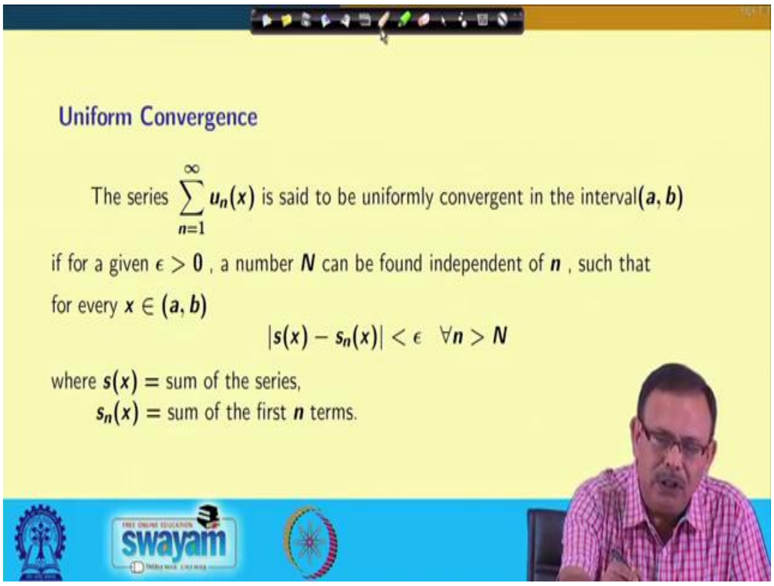

Now we come to a new concept called uniform convergence. The series $\sum_{n=1}^{\infty} u_n(x)$ is said to be uniformly convergent in the interval (a, b) if for a given ϵ, a very small number greater than 0, there exists a number N which is independent of n such that for any x we take in the interval (a, b), we have

$$|s(x) - s_n(x)| < \epsilon \quad \forall\, n > N$$

where, $s(x)$ is the sum of the series and $s_n(x)$ is the sum of the first n terms of the series.

So, one may ask, why this uniformly convergent concept is required here. The reason for uniform convergence is that we will see, whenever we are taking the Fourier series and try to make the term by term integration, then it will be only possible if the series is uniformly convergent. For that reason, this particular assumption has been taken care of in this particular case. Now, let us see the proof.

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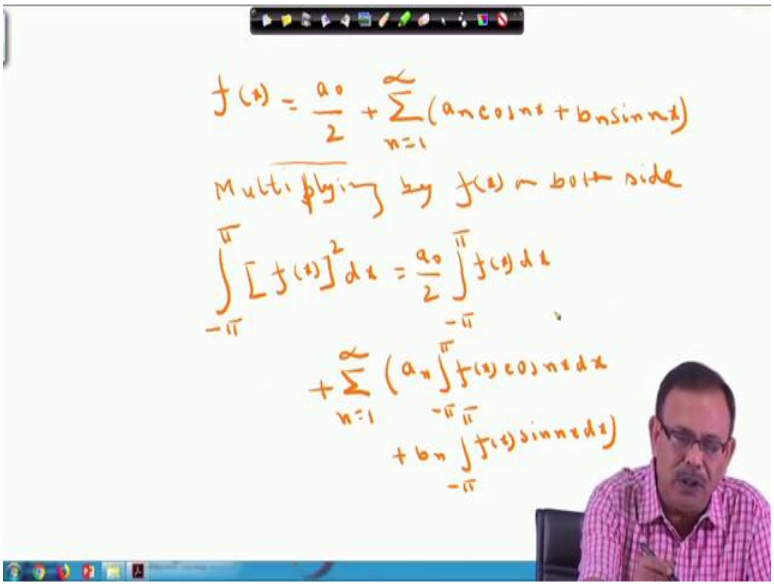

We can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Multiplying $f(x)$ on both sides, we have,

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) f(x)$$

Now integrating the same from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0}{2} \int_{-\pi}^{\pi} f(x) dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} f(x) \cos nx \, dx + b_n \int_{-\pi}^{\pi} f(x) \sin nx \, dx \right) \quad (1)$$

and whenever we are integrating each term from $-\pi$ to π, then the concept of uniform convergence arises. Now this term by term integration of this series on the right hand side is possible only when $f(x)$ is uniformly convergent and for that reason, the assumption has been taken that $f(x)$ is uniformly convergent in the given interval.

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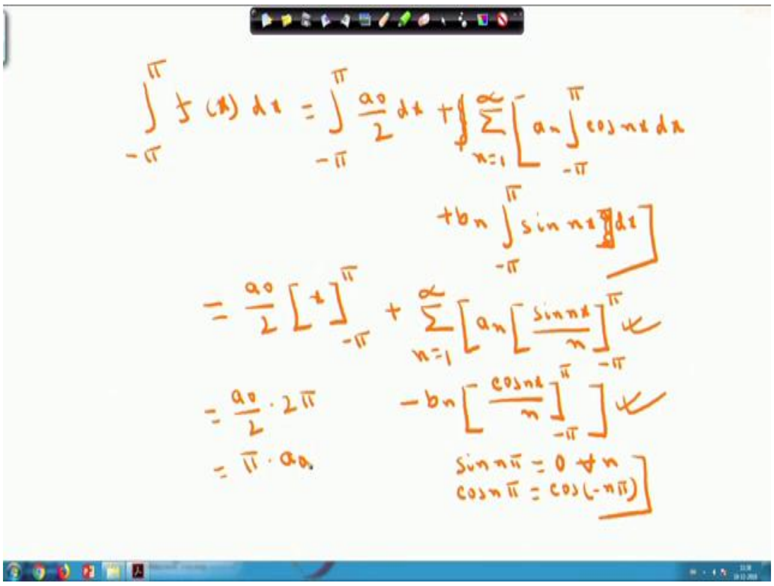

Now, let us perform the integrations term by term. Therefore,

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx$$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx\, dx\right]$$

$$= \pi a_0 + \sum_{n=1}^{\infty}\left[a_n\left[\frac{1}{n}\sin nx\right]_{-\pi}^{\pi} - b_n\left[\frac{1}{n}\cos nx\right]_{-\pi}^{\pi}\right]$$

$$= \pi a_0$$

So, from here we can tell, what is the value of a_0. Again,

$$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)\right]\cos nx\, dx$$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi} \cos nx\, dx + \sum_{k=1}^{\infty}\left[a_k \int_{-\pi}^{\pi} \cos kx \cos nx\, dx + b_k \int_{-\pi}^{\pi} \sin kx \cos nx\, dx\right]$$

$$= 0 \;\text{ for } k \neq n$$

And for $k = n$, we have,

$$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos nx\, dx$$

$$= \frac{a_n}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} - \frac{b_n}{2}\left[\frac{\cos 2nx}{2n}\right]_{-\pi}^{\pi}$$

$$= \pi a_n$$

Therefore, we have,

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \pi a_n$$

Similarly, we can show that

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \pi b_n$$

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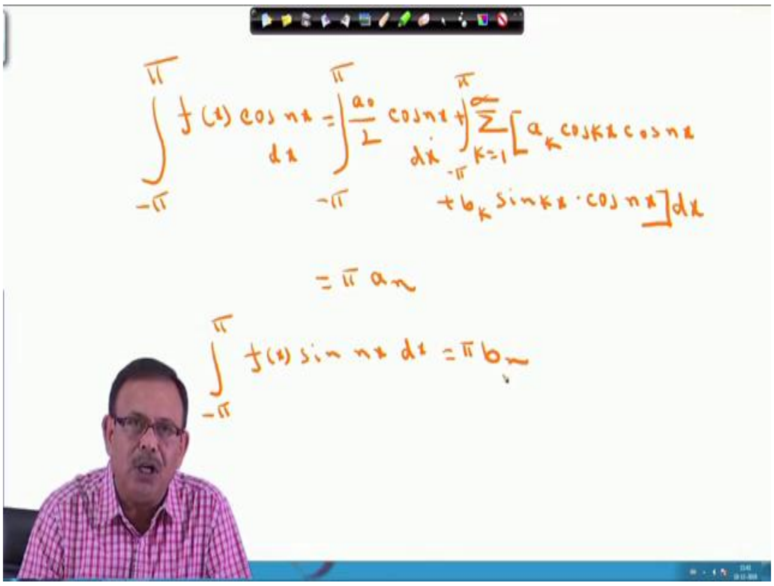

Now substituting the obtained results in equation (1), we have,

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0}{2}\pi a_0 + \sum_{n=1}^{\infty}(a_n \pi a_n + b_n \pi b_n)$$

$$= \frac{\pi a_0^2}{2} + \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

This completes the proof of Parseval’s Theorem.

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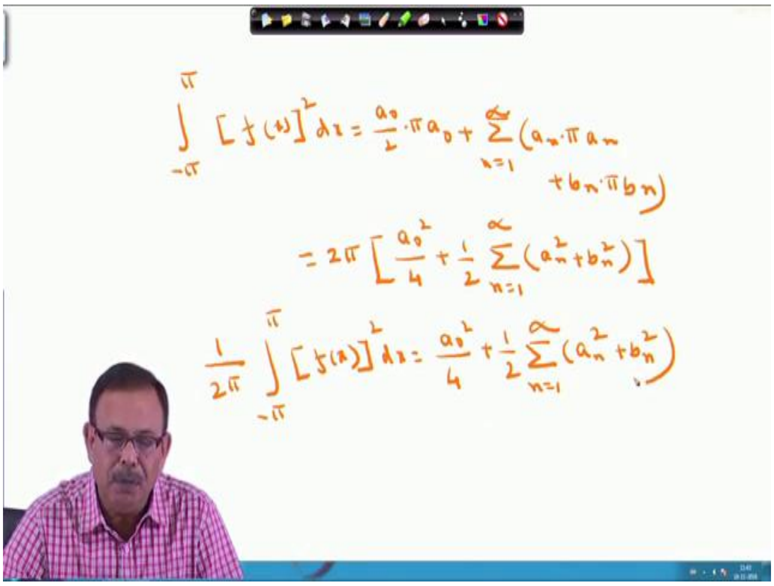

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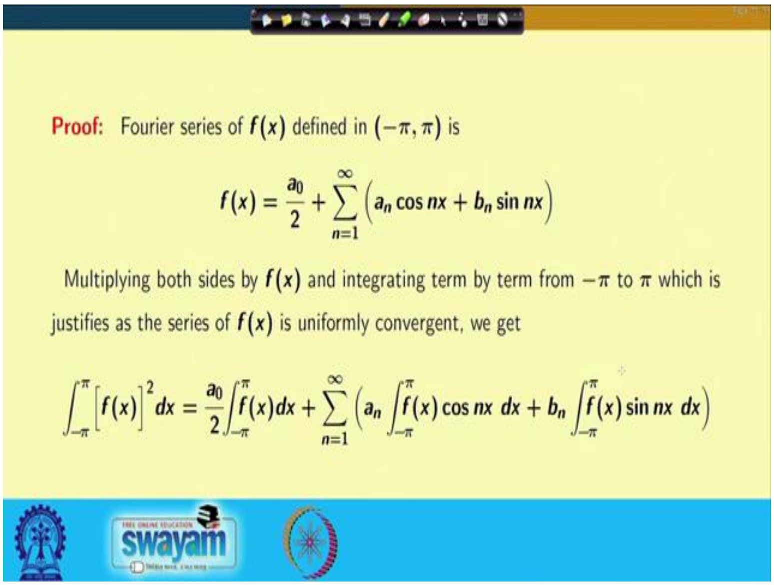

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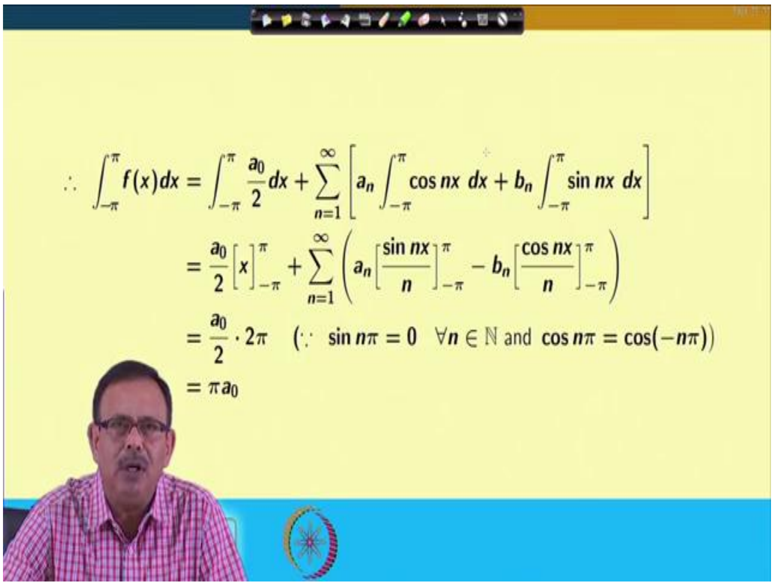

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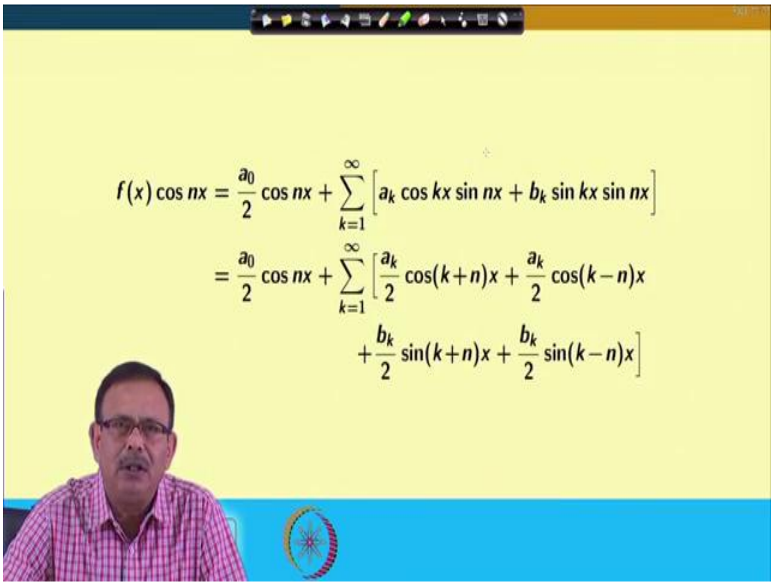

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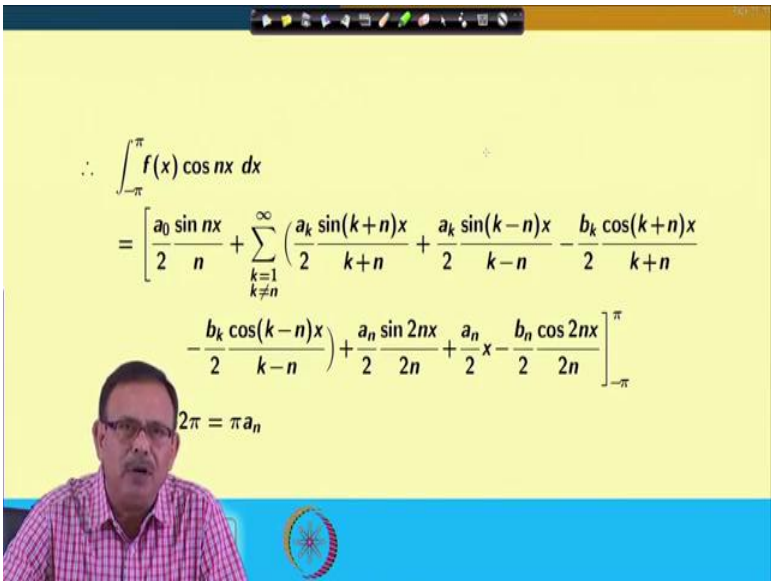

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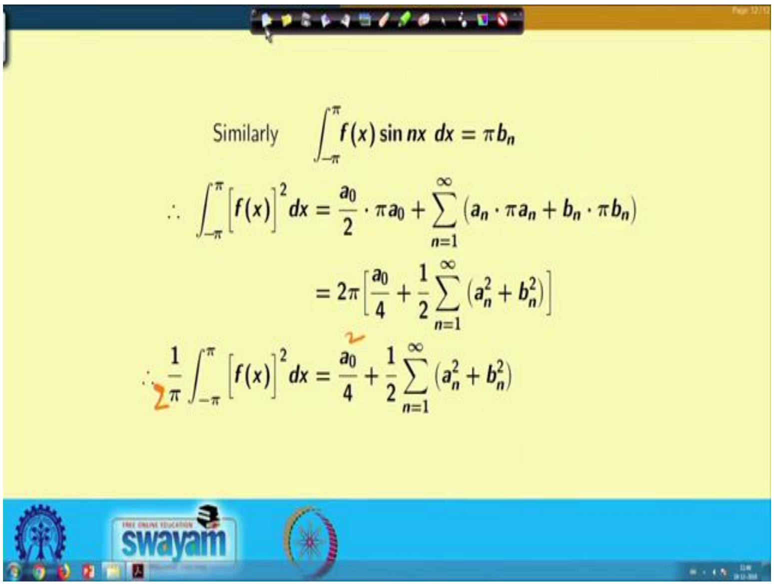

Let us now see the applications of this.

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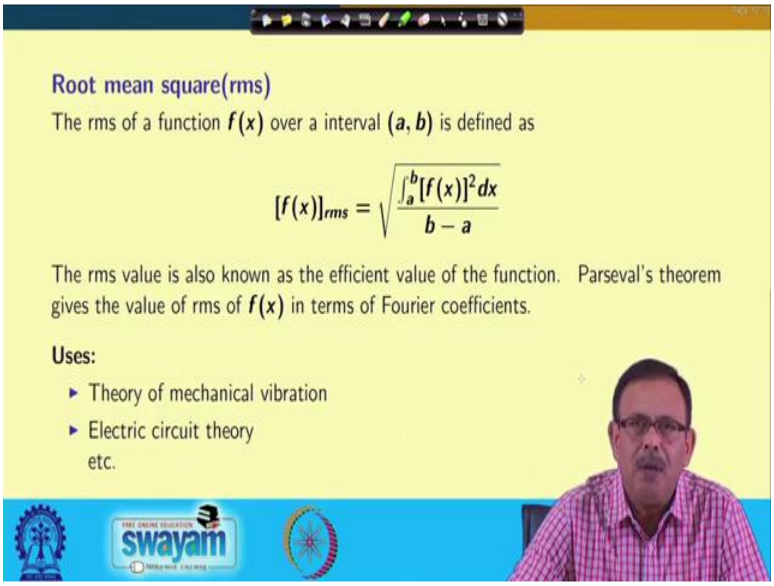

First we come across Root mean square. The root mean square or rms of a function $f(x)$ over an interval (a, b) is defined as

$$[f(x)]_{rms} = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

This value can be obtained using the Parseval's Theorem itself.

The rms value is sometimes also known as the efficient value of the function and the Parseval's Theorem gives the value of rms of $f(x)$ in terms of the Fourier coefficients. One or two applications are theory of mechanical vibration and electric circuit theory. The rms concept is used widely and the involved calculations can be done very easily using Parseval's Theorem.

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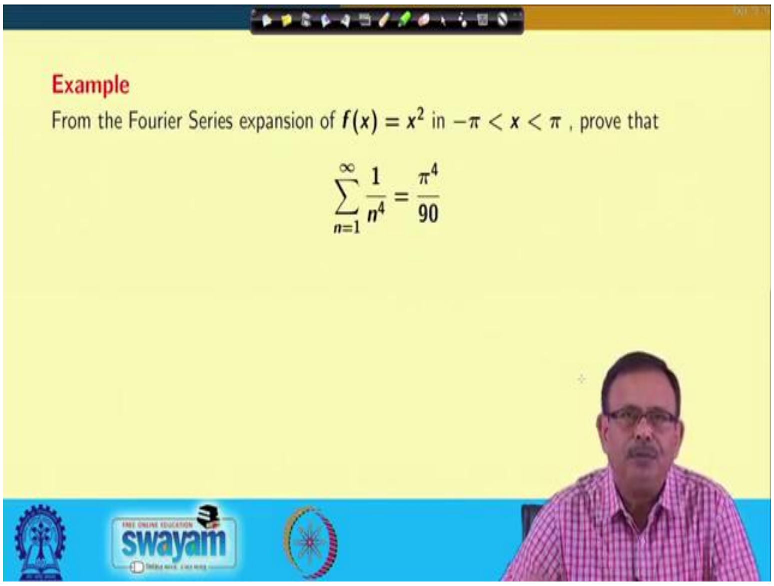

Now, let us see one example. We want to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

from the Fourier series expansion of $f(x) = x^2,\ -\pi < x < \pi$.

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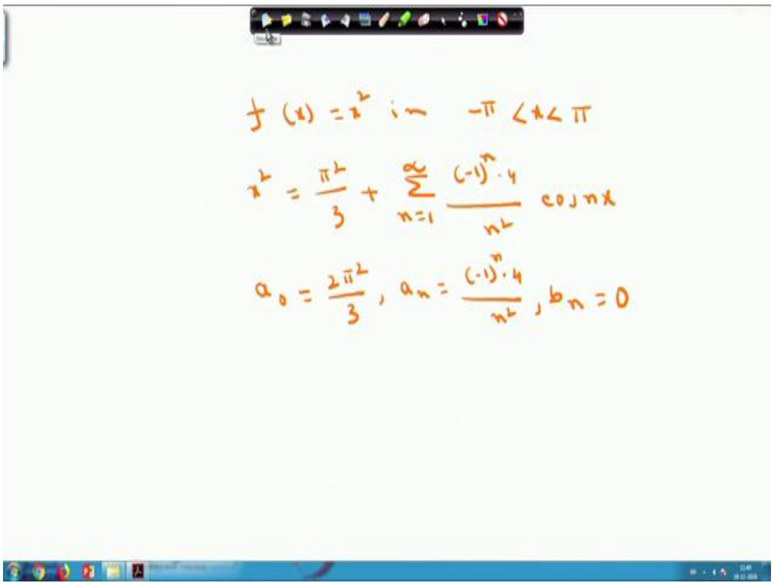

The Fourier series expansion of $f(x) = x^2,\ -\pi < x < \pi$ has already been discussed in the previous lectures where we had obtained

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4}{n^2}(-1)^n, \quad b_n = 0$$

So we can directly write the Fourier series expansion of $f(x) = x^2, \ -\pi < x < \pi$ as

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

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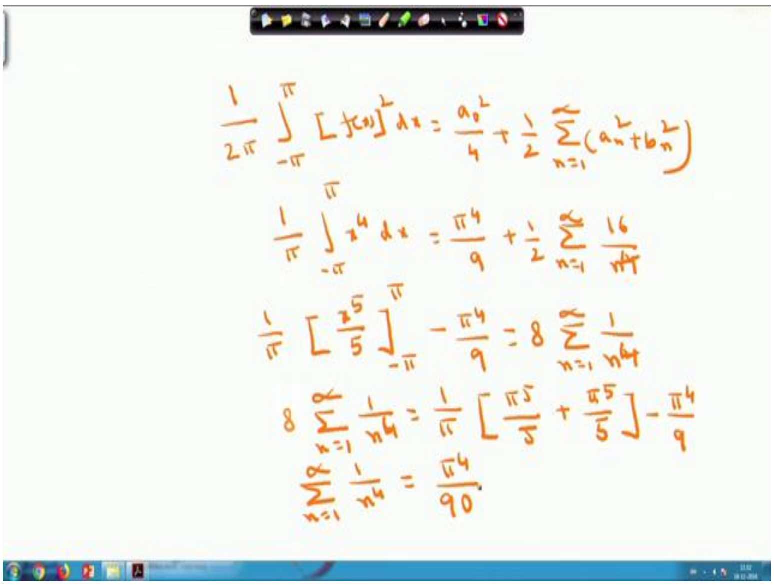

Now, from the Parseval's Theorem, we know that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Substituting here, $f(x) = x^2$, and the values of a_0, a_n and b_n, we obtain,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{1}{2}\frac{4\pi^4}{9} + \sum_{n=1}^{\infty}\left(\frac{16}{n^4} + 0\right)$$

$$\Rightarrow \frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

This solves our problem.

Now, let us take another example.

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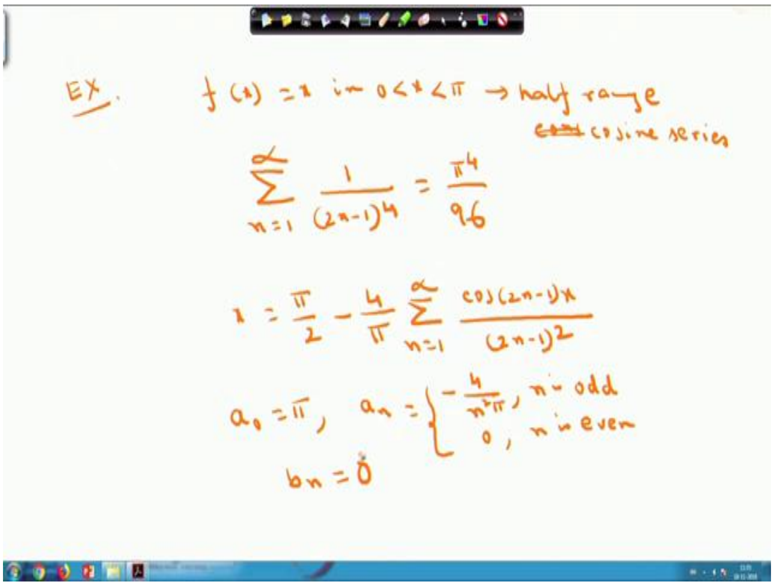

From the half range cosine series expansion of the function $f(x) = x$ which is defined in $(0, \pi)$, we need to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

and the half range cosine series of the function $f(x) = x$ is already known to us. So as already obtained in the previous lectures, we can write the half range cosine series of $f(x) = x$ as

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

where

$$a_0 = \pi, \quad a_n = \begin{cases} -\dfrac{4}{n^2\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Now, from Parseval's Theorem directly we can write,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\Rightarrow \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{2\pi^2}{3} - \frac{\pi^2}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

This completes the proof.

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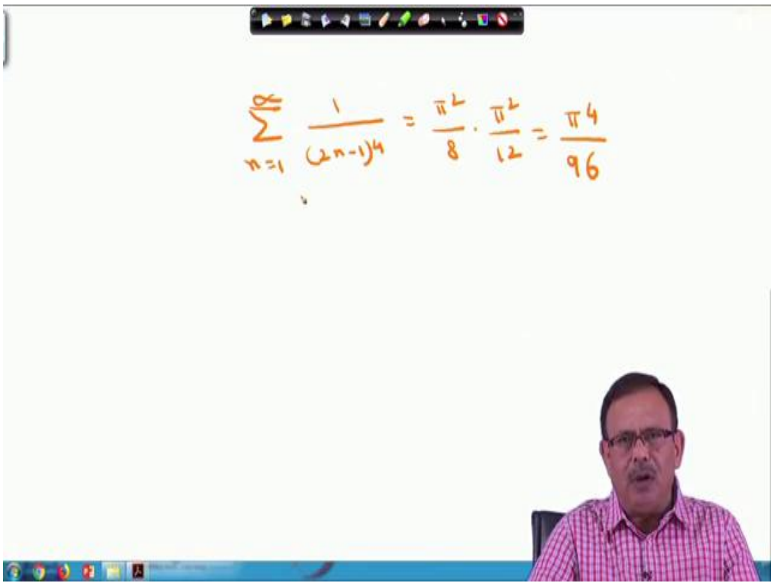

So, effectively the Parseval's Theorem is very useful in electrical engineering cases, where they have to calculate the rms value or the efficient value of the function. Thank you.