

Transform Calculus and its Applications in Differential Equations

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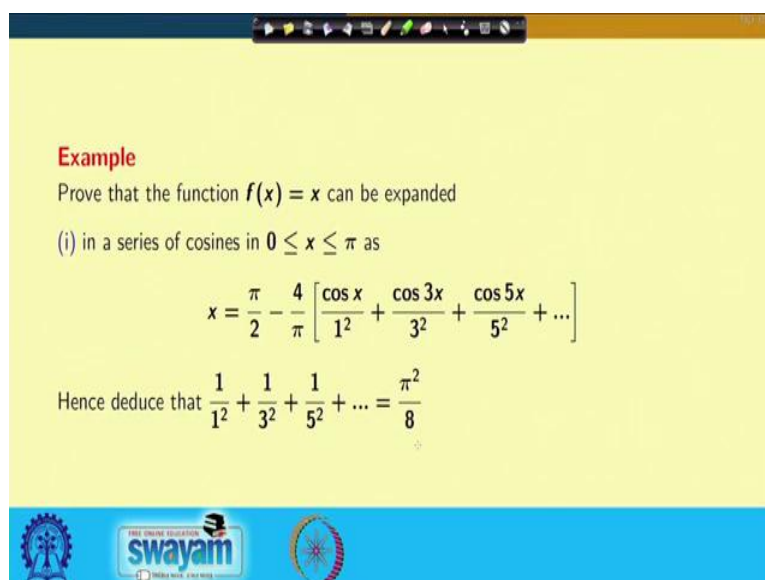
Indian Institutes of Technology Kharagpur

Lecture – 23

Half Range Fourier Series

Welcome back. In the last lecture, we had discussed the Half Range Fourier Sine and Cosine Series and also the Fourier series expansion for Half Range Series. We did one example also.

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Example
Prove that the function $f(x) = x$ can be expanded
(i) in a series of cosines in $0 \leq x \leq \pi$ as

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

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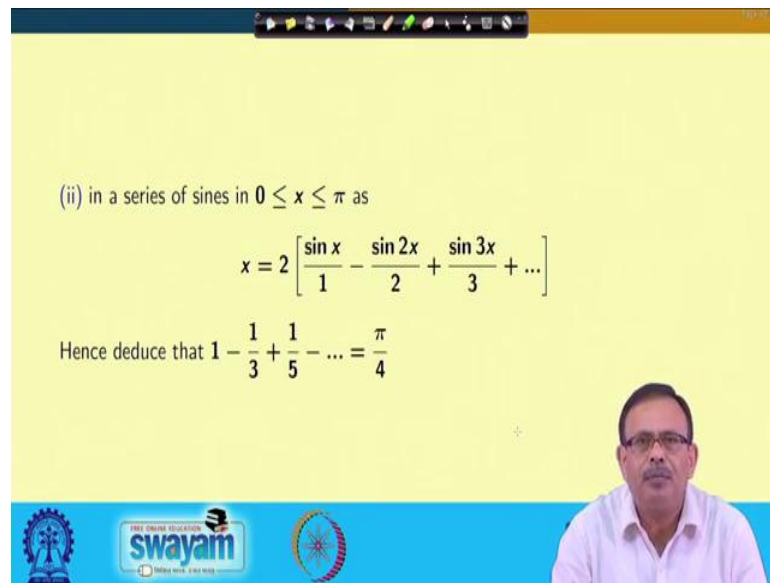
Now, let us take another example. Suppose we have to prove that $f(x) = x$ can be expanded in a series of cosines in $[0, \pi]$. So, we have to expand $f(x) = x$ as a cosine series in $[0, \pi]$ and we have to show that

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right].$$

From here we have to prove that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

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(ii) in a series of sines in $0 \leq x \leq \pi$ as

$$x = 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right]$$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$

Similarly, in the second part of the problem, we have to expand $f(x) = x$ as a sine series in $[0, \pi]$ and we have to show that

$$x = 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right].$$

From here we have to deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}.$$

So, we want to express the same function as a sine series, as a cosine series and from there, we will be able to evaluate different series. Please note that whether we are defining it as a sine series or as a cosine series, the nature of the function $f(x) = x$ in $[0, \pi]$ will remain unaltered.

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The image shows a handwritten derivation for the Fourier coefficients of the function $f(x) = x$ on the interval $[0, \pi]$. The derivation is as follows:

$$(i) \quad a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \pi$$
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$
$$= \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right]$$
$$a_n = \begin{cases} -\frac{4}{n^2\pi}, & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

For the solution of the first part that is expressing $f(x) = x$ as a cosine series in $[0, \pi]$, we proceed as follows:

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x dx \\ &= \pi \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx \\ &= \frac{2}{\pi} \left[\frac{x}{n} \sin nx \right]_0^{\pi} + \frac{2}{n^2\pi} [\cos nx]_0^{\pi} \\ &= \frac{2}{n^2\pi} (\cos n\pi - 1) \\ &= \frac{2}{n^2\pi} [(-1)^n - 1] \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

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Handwritten derivation on a whiteboard:

$$f(x) = x \text{ in } [0, \pi]$$

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

Put $x=0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

So that now we can express $f(x) = x$ as a cosine series as follows

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

So, this is the half range cosine series of x in $[0, \pi]$ and next, we have to find out the value of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Clearly, from the obtained result, we can see that substituting $x = 0$ will serve our purpose. So, if we put $x = 0$ in this obtained series expansion of x , we will obtain

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

This completes the first part of the problem. Next, we will express the same function $f(x) = x$ in $[0, \pi]$ in terms of sine series. So, to express it in terms of sine series, there will be only sine terms, so we will calculate only b_n . Therefore,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^\pi x \sin nx \, dx \\ &= \frac{2}{\pi} \left[-\frac{x}{n} \cos nx \right]_0^\pi + \frac{2}{n^2 \pi} [\sin nx]_0^\pi \\ &= -\frac{2}{n} \cos n\pi \\ &= -\frac{2}{n} (-1)^n. \end{aligned}$$

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(ii) $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx$
 $= \frac{-2(-1)^n}{n}$
 Sine series of $f(x)$ in $[0, \pi]$ is
 $x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$
 $= 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]$

Therefore, the half range sine series of $f(x) = x$ in $[0, \pi]$ is

$$x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}\Rightarrow x &= -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \\ &= 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]\end{aligned}$$

Please note that although we are expressing $f(x) = x$ once as a half range cosine series and once as a half range sine series, they are two different series, but their nature and their graphs will remain same in $[0, \pi]$. Now we have to find out the value of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \dots$$

So, in order to obtain the value of the above series, we put $x = \frac{\pi}{2}$ in the sine series expansion of $f(x)$. Therefore,


$$\begin{aligned}\frac{\pi}{2} &= 2 \left[\frac{\sin \frac{\pi}{2}}{1} - \frac{\sin \pi}{2} + \frac{\sin \frac{3\pi}{2}}{3} - \dots \right] \\ &= 2 \left[1 - \frac{1}{3} + \frac{1}{5} - \dots \right] \\ \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \dots &= \frac{\pi}{4}.\end{aligned}$$

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This completes the second part of the given problem as well.

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Solution: (i)


$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \pi$$
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$
$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$
$$= \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right]$$
$$\therefore a_n = \begin{cases} -\frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$


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\therefore The cosine series for $f(x) = x$ in $[0, \pi]$ is given by

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$
$$= \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

Putting $x = 0$ in the above result


$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$
$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$


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(ii)

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$
$$= \frac{-2(-1)^n}{n}$$

\therefore The sine series for $f(x)$ in $[0, \pi]$ is


$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$
$$= 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]$$


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Putting $x = \frac{\pi}{2}$ in the above result

$$\frac{\pi}{2} = 2 \left[\frac{\sin \frac{\pi}{2}}{1} - \frac{\sin \pi}{2} + \frac{\sin \frac{3\pi}{2}}{3} - \frac{\sin 2\pi}{4} + \dots \right]$$

i.e., $\frac{\pi}{2} = 2 \left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \dots \right]$

$$\therefore 1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$$


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Example
Find the half-range cosine series for the function $f(x) = x^2$ in $0 \leq x \leq \pi$ and hence find the sum of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Solution:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$
$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

Let us solve one more example. Consider a function $f(x) = x^2$ in $[0, \pi]$ whose half range cosine series is to be computed and then, we want to find the value of the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

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$f(x) = x^2$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$
$$= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi}$$
$$= \frac{2}{\pi} \cdot \frac{2\pi \cos n\pi}{n^2} = \frac{4(-1)^n}{n^2}$$

Therefore,

$$\begin{aligned}a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 dx \\ &= \frac{2}{3} \pi^2\end{aligned}$$

$$\begin{aligned}a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx \\ &= \frac{2}{\pi} \left[\frac{x^2}{n} \sin nx \right]_0^{\pi} - \frac{4}{n\pi} \int_0^{\pi} x \sin nx dx \\ &= -\frac{4}{n\pi} \left[-\frac{x}{n} \cos nx \right]_0^{\pi} + \frac{4}{n^3\pi} [\sin nx]_0^{\pi} \\ &= \frac{4}{n^2} \cos n\pi \\ &= \frac{4}{n^2} (-1)^n\end{aligned}$$

Therefore, the half range cosine series of $f(x) = x^2$ is

$$\begin{aligned}x^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx \\ &= \frac{\pi^2}{3} - 4 \left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right]\end{aligned}$$

Now we need to evaluate the value of the following series

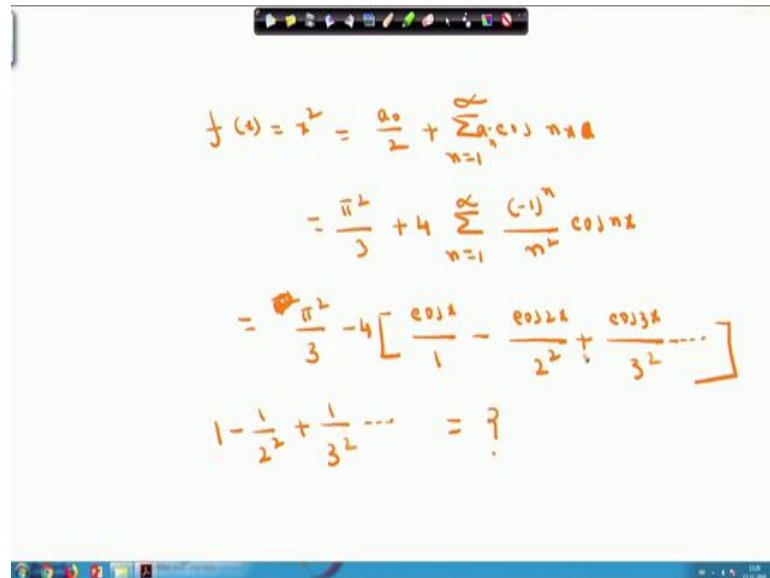
$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

which can be easily achieved by putting $x = 0$ in the obtained half range cosine series of $f(x)$. So we have

$$0 = \frac{\pi^2}{3} - 4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right]$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

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Handwritten mathematical derivation on a whiteboard:

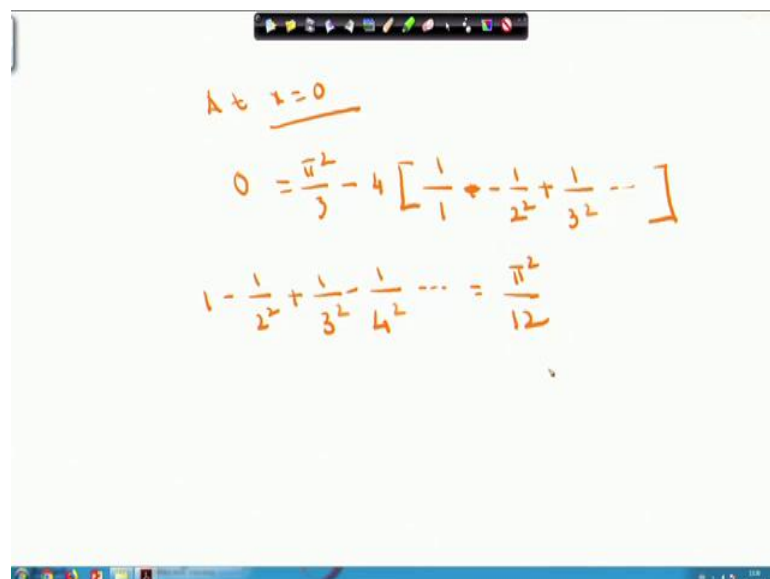
$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} - 4 \left[\frac{\cos x}{1} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right]$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = ?$$

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Handwritten mathematical derivation on a whiteboard:

At $x=0$

$$0 = \frac{\pi^2}{3} - 4 \left[\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right]$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \dots = \frac{\pi^2}{12}$$

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$$\begin{aligned}\Rightarrow a_n &= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^\pi \\ &= \frac{2}{\pi} \cdot \frac{2\pi \cos n\pi}{n^2} \\ &= \frac{4(-1)^n}{n^2}\end{aligned}$$
$$\begin{aligned}\therefore x^2 &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx\end{aligned}$$

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$$\begin{aligned}\Rightarrow x^2 &= \frac{\pi^2}{3} + 4 \left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots \right] \\ &= \frac{\pi^2}{3} - 4 \left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right] \\ \therefore 0 &= \frac{\pi^2}{3} - 4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right] \quad [\text{put } x = 0]\end{aligned}$$
$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

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Example
Expand $f(x) = \cos x$, $0 \leq x \leq \pi$ in the half range sine series.

Solution:

The Fourier sine series of $f(x)$ in $0 \leq x \leq \pi$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \, dx$$

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Let us take one more example. We need to expand $f(x) = \cos x$ as a half range sine series in $[0, \pi]$.

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$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$
$$= \frac{2}{\pi} \int_0^{\pi} \cos x \cdot \sin nx \, dx =$$
$$b_1 = \frac{2}{\pi} \int_0^{\pi} \cos x \sin x \, dx = \frac{2}{\pi} \left[\frac{\sin^2 x}{2} \right]_0^{\pi} = 0$$

The whiteboard shows the step-by-step calculation of the Fourier sine coefficients for the function f(x) = cos x over the interval [0, pi].

So, half range sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx \\
&= \frac{2}{\pi} \int_0^\pi \cos x \sin nx \, dx \\
&= \frac{1}{\pi} \int_0^\pi [\sin(n+1)x + \sin(n-1)x] \, dx \\
&= -\frac{1}{\pi} \left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^\pi \quad \text{for } n \neq 1 \\
&= -\frac{1}{\pi} \left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} - \frac{1}{n-1} \right] \quad \text{for } n \neq 1 \\
&= \frac{1}{\pi} \left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{2n}{n^2-1} \right] \quad \text{for } n \neq 1 \\
&= \frac{1}{\pi} \left[\frac{2n}{n^2-1} (-1)^n + \frac{2n}{n^2-1} \right] \quad \text{for } n \neq 1 \\
&= \frac{2n[(-1)^n + 1]}{\pi(n^2-1)} \quad \text{for } n \neq 1 \\
&= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{4n}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

However, for $n = 1$, we have,

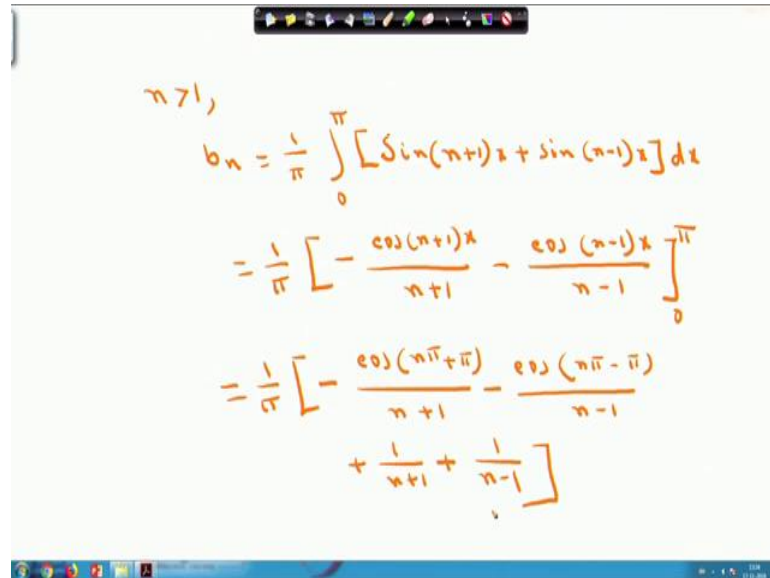
$$\begin{aligned}
b_1 &= \frac{2}{\pi} \int_0^\pi \cos x \sin x \, dx \\
&= \frac{1}{\pi} \int_0^\pi \sin 2x \, dx \\
&= -\frac{1}{2\pi} [\cos 2x]_0^\pi \\
&= 0
\end{aligned}$$

Therefore the half range sine series of $f(x) = \cos x$ in $[0, \pi]$ is given by

$$\begin{aligned}
\cos x &= \sum_{n=2}^{\infty} b_n \sin nx \\
&= \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{n[(-1)^n + 1]}{n^2-1} \sin nx \\
&= \frac{2}{\pi} \left[\frac{4}{3} \sin 2x + \frac{8}{15} \sin 4x + \frac{12}{35} \sin 6x + \dots \right]
\end{aligned}$$

$$\Rightarrow \cos x = \frac{8}{\pi} \left[\frac{\sin 2x}{3} + \frac{2 \sin 4x}{15} + \frac{3 \sin 6x}{35} + \dots \right]$$

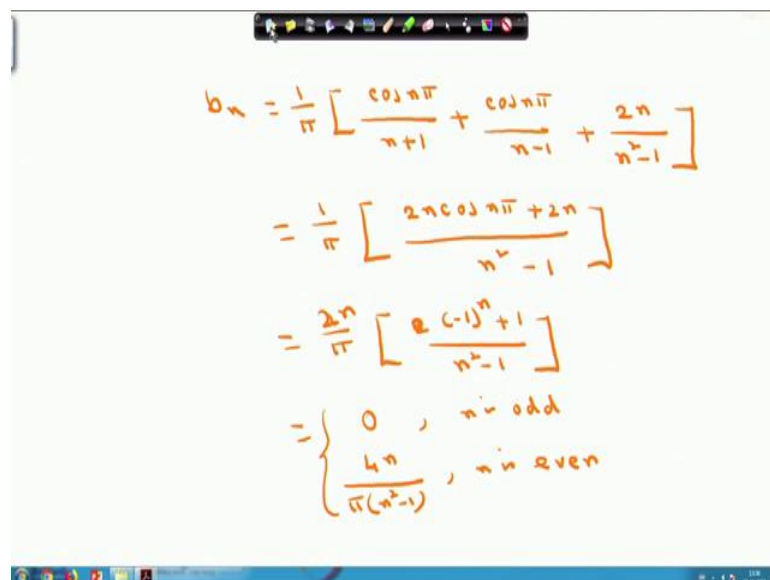
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Handwritten mathematical derivation for b_n on a whiteboard. The derivation starts with $n > 1$ and uses the integral definition of b_n from 0 to π . It involves integration by parts, leading to a final expression for b_n in terms of n .

$$\begin{aligned}
 n > 1, \\
 b_n &= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] dx \\
 &= \frac{1}{\pi} \left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi} \\
 &= \frac{1}{\pi} \left[-\frac{\cos(n\pi + \pi)}{n+1} - \frac{\cos(n\pi - \pi)}{n-1} \right. \\
 &\quad \left. + \frac{1}{n+1} + \frac{1}{n-1} \right]
 \end{aligned}$$

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Handwritten mathematical derivation for b_n on a whiteboard. The derivation shows the simplification of the expression for b_n and its final value based on whether n is odd or even.

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{2n}{n^2-1} \right] \\
 &= \frac{1}{\pi} \left[\frac{2n \cos n\pi + 2n}{n^2-1} \right] \\
 &= \frac{2n}{\pi} \left[\frac{e^{(-1)^n} + 1}{n^2-1} \right] \\
 &= \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases}
 \end{aligned}$$

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The image shows a handwritten derivation for the Fourier series of $f(x) = x^2$. The function is periodic with period 2π . The derivation starts with the formula for the Fourier series of a periodic function $f(x)$ with period 2π :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n \cos nx + b_n \sin nx}{2} \right]$$

Since $f(x) = x^2$ is an even function, $b_n = 0$. The coefficients a_n are given by:

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

The derivation shows the first few terms of the series:

$$f(x) = \frac{4}{\pi} \left[\frac{2 \sin 2x}{2^2 - 1} + \frac{4 \sin 4x}{4^2 - 1} + \frac{6 \sin 6x}{6^2 - 1} + \dots \right]$$

And then simplifies it to:

$$= \frac{8}{\pi} \left[\frac{\sin 2x}{3} + \frac{2 \sin 4x}{15} + \frac{3 \sin 6x}{35} + \dots \right]$$

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The image shows a handwritten derivation for the Fourier coefficients b_1 and b_n . The function is $f(x) = \sin^2 x$.

For b_1 :

$$\therefore b_1 = \frac{2}{\pi} \int_0^{\pi} \cos x \sin x \, dx = \frac{2}{\pi} \left[\frac{\sin^2 x}{2} \right]_0^{\pi} = 0$$

if $n > 1$

$$b_n = \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] \, dx$$
$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$
$$= \frac{1}{\pi} \left[-\frac{\cos(n\pi + \pi)}{n+1} - \frac{\cos(n\pi - \pi)}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right]$$

The slide also features logos for Swamyam and other educational institutions at the bottom.

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$$\begin{aligned} \Rightarrow b_n &= \frac{1}{\pi} \left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{2n}{n^2-1} \right] \\ &= \frac{1}{\pi} \left[\frac{2n \cos n\pi + 2n}{n^2-1} \right] \\ &= \frac{2n}{\pi} \left[\frac{(-1)^n + 1}{n^2-1} \right] \\ &= \begin{cases} 0 & , \text{ if } n \text{ is odd} \\ \frac{4n}{\pi(n^2-1)} & , \text{ if } n \text{ is even} \end{cases} \end{aligned}$$

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$$\begin{aligned} \therefore f(x) &= \frac{4}{\pi} \left[\frac{2 \sin 2x}{2^2-1} + \frac{4 \sin 4x}{4^2-1} + \frac{6 \sin 6x}{6^2-1} + \dots \right] \\ &= \frac{8}{\pi} \left[\frac{\sin 2x}{3} + \frac{2 \sin 4x}{15} + \frac{3 \sin 6x}{35} + \dots \right] \end{aligned}$$

So therefore, if we are given a function, then we have seen how to express the function in terms of a half range sine or cosine series.

We can now find out the Fourier series of an odd function, Fourier series of an even function and half range Fourier cosine series or half range Fourier sine series of a function as well.

Thank you.