Transform Calculus and it is Applications in Differential Equations
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Lecture – 23
Half Range Fourier Series

Welcome back. In the last lecture, we had discussed the Half Range Fourier Sine and Cosine Series and also the Fourier series expansion for Half Range Series. We did one example also.

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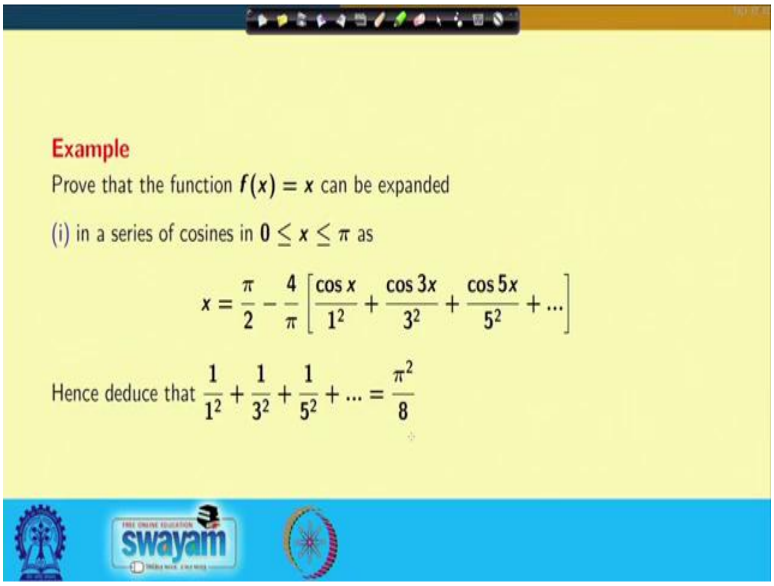

Now, let us take another example. Suppose we have to prove that $f(x) = x$ can be expanded in a series of cosines in $[0, \pi]$. So, we have to expand $f(x) = x$ as a cosine series in $[0, \pi]$ and we have to show that

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right].$$

From here we have to prove that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

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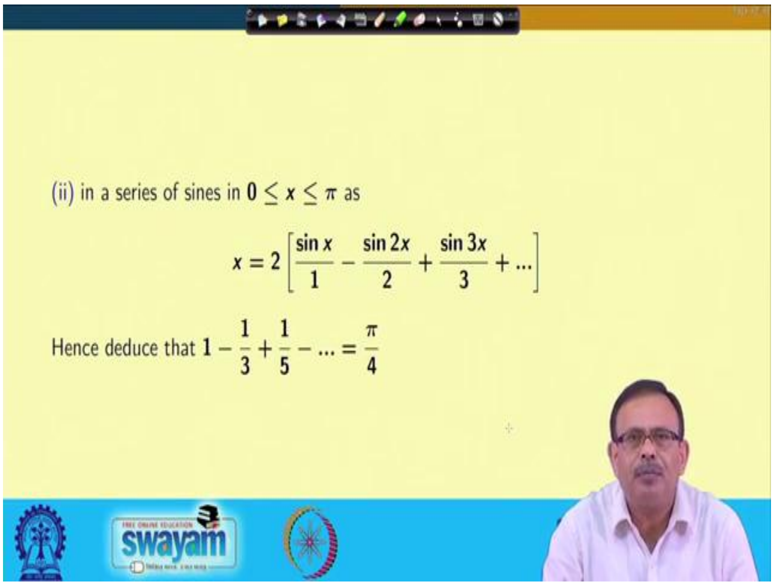

Similarly, in the second part of the problem, we have to expand $f(x) = x$ as a sine series in $[0, \pi]$ and we have to show that

$$x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right].$$

From here we have to deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

So, we want to express the same function as a sine series, as a cosine series and from there, we will be able to evaluate different series. Please note that whether we are defining it as a sine series or as a cosine series, the nature of the function $f(x) = x$ in $[0, \pi]$ will remain unaltered.

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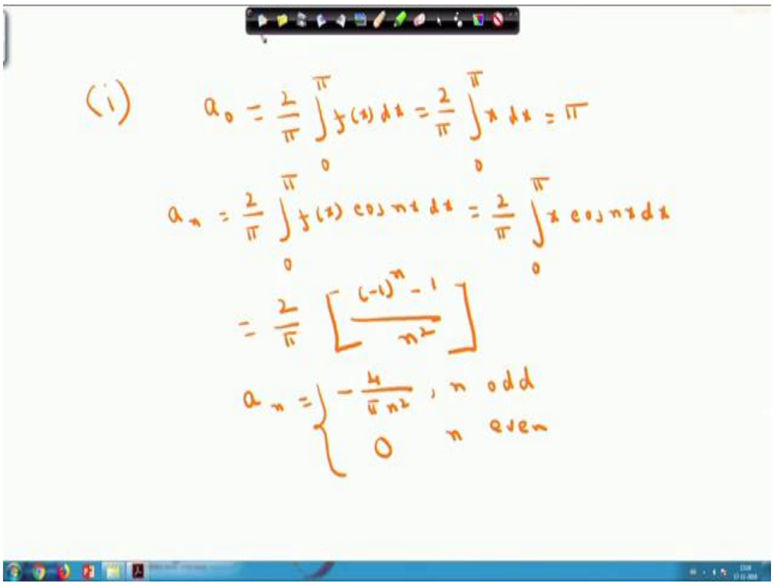

For the solution of the first part that is expressing $f(x) = x$ as a cosine series in $[0, \pi]$, we proceed as follows:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\, dx$$
$$= \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$
$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx\right]_0^{\pi} + \frac{2}{n^2\pi}[\cos nx]_0^{\pi}$$
$$= \frac{2}{n^2\pi}(\cos n\pi - 1)$$
$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$
$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases}$$

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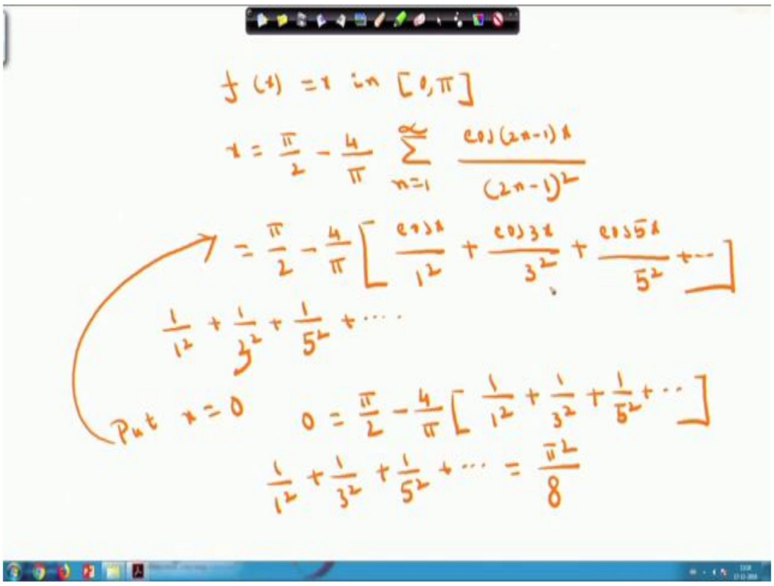

So that now we can express $f(x) = x$ as a cosine series as follows

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

So, this is the half range cosine series of x in $[0, \pi]$ and next, we have to find out the value of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Clearly, from the obtained result, we can see that substituting $x = 0$ will serve our purpose. So, if we put $x = 0$ in this obtained series expansion of x, we will obtain

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}.$$

This completes the first part of the problem. Next, we will express the same function $f(x)=x$ in $[0,\pi]$ in terms of sine series. So, to express it in terms of sine series, there will be only sine terms, so we will calculate only b_n. Therefore,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \\ &= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{2}{n^2\pi}[\sin nx]_0^{\pi} \\ &= -\frac{2}{n}\cos n\pi \\ &= -\frac{2}{n}(-1)^n. \end{aligned}$$

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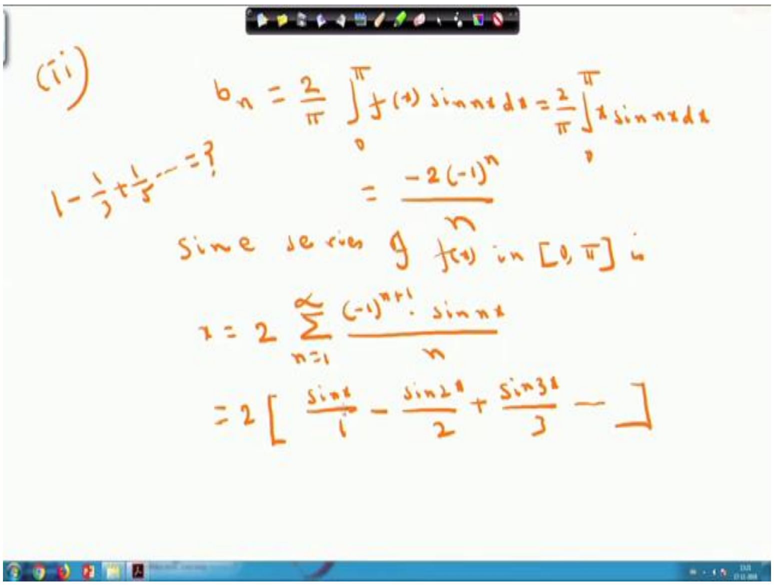

Therefore, the half range sine series of $f(x)=x$ in $[0,\pi]$ is

$$x=\sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

Please note that although we are expressing $f(x) = x$ once as a half range cosine series and once as a half range sine series, they are two different series, but their nature and their graphs will remain same in $[0, \pi]$. Now we have to find out the value of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

So, in order to obtain the value of the above series, we put $x = \frac{\pi}{2}$ in the sine series expansion of $f(x)$. Therefore,

$$\frac{\pi}{2} = 2\left[\frac{\sin\frac{\pi}{2}}{1} - \frac{\sin\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} - \cdots\right]$$

$$= 2\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right]$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}.$$

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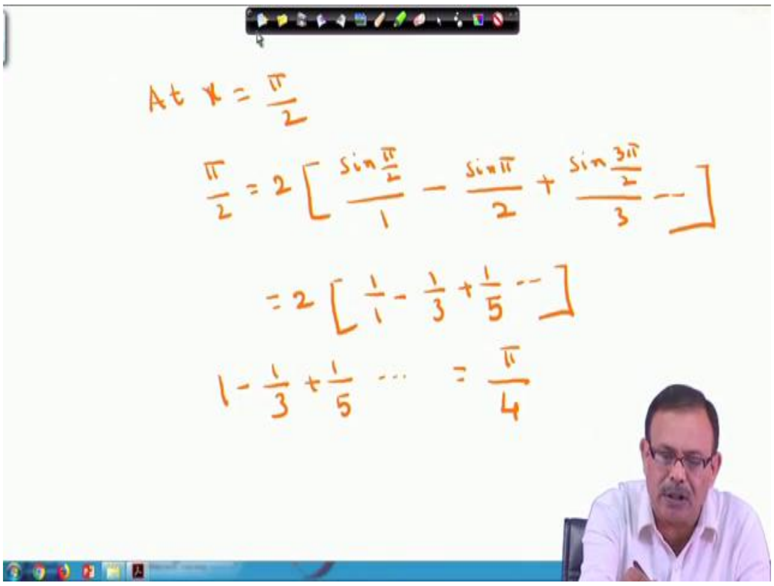

This completes the second part of the given problem as well.

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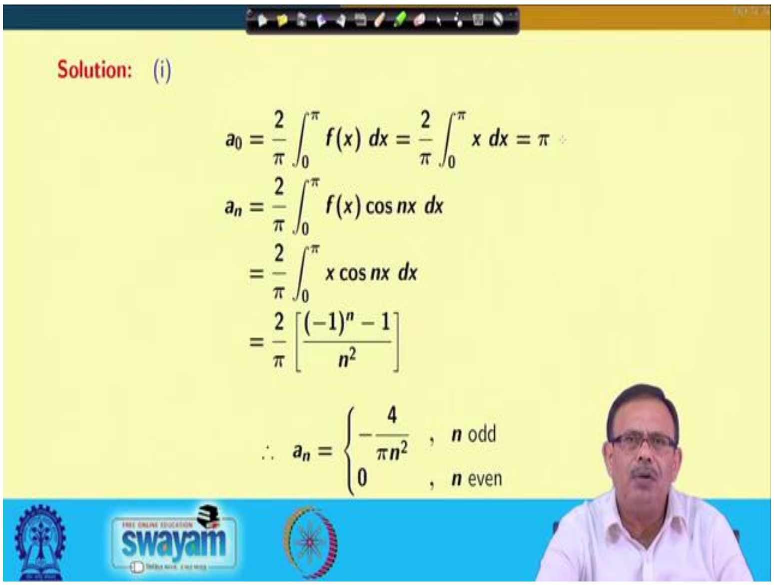

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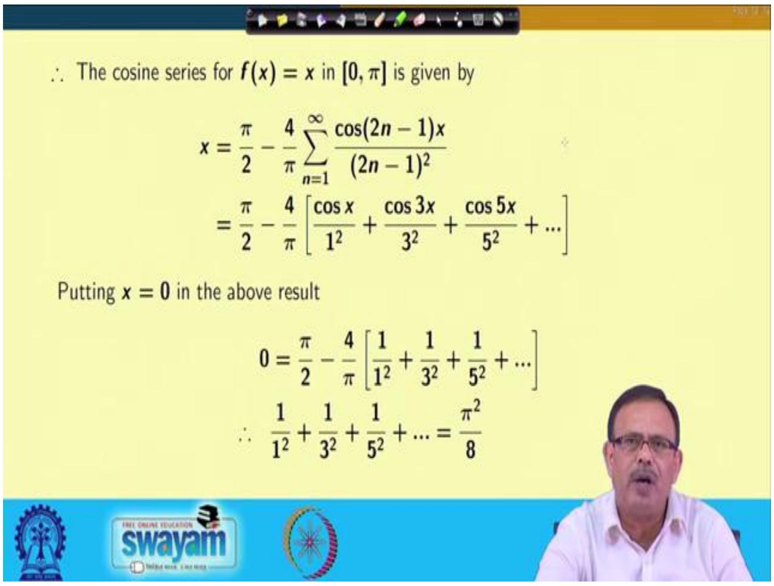

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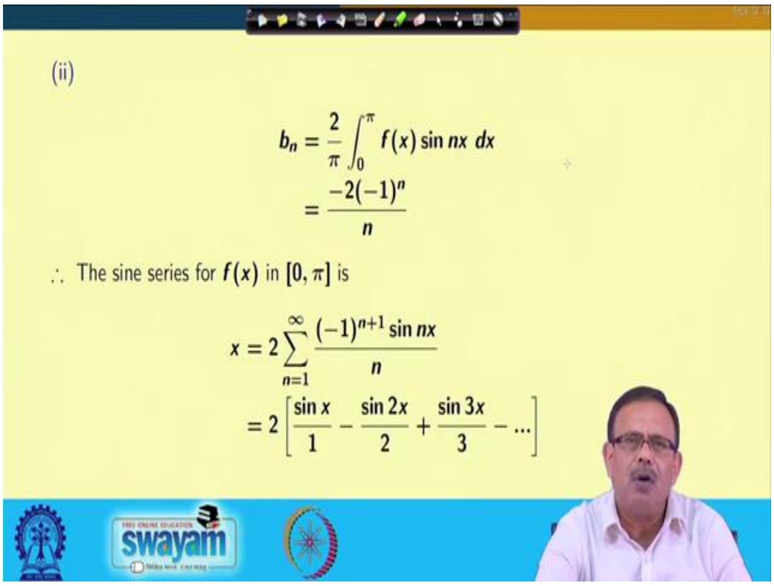

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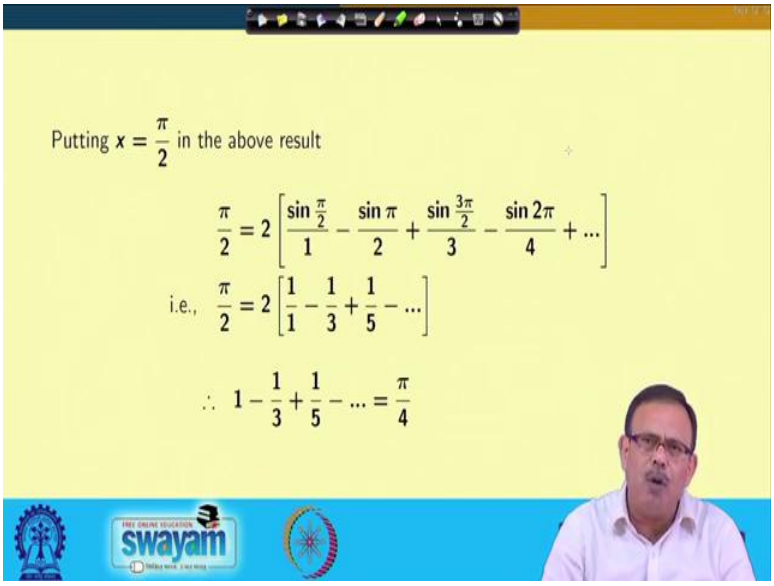

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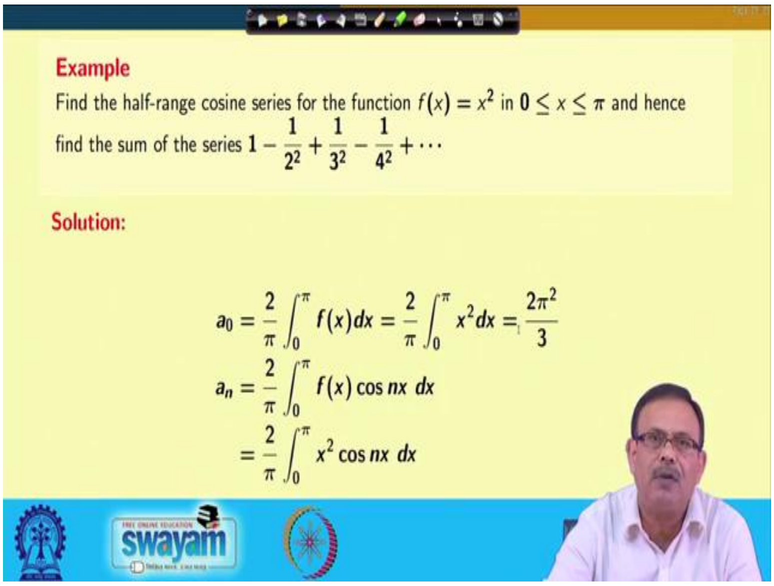

Let us solve one more example. Consider a function $f(x) = x^2$ in $[0, \pi]$ whose half range cosine series is to be computed and then, we want to find the value of the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

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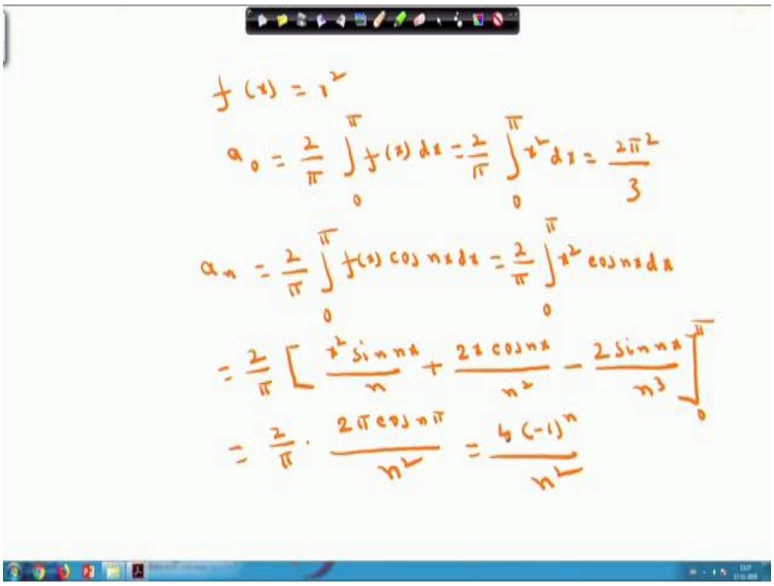

Therefore,

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2\,dx \\ &= \frac{2}{3}\pi^2 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{x^2}{n}\sin nx\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x\sin nx\,dx \\ &= -\frac{4}{n\pi}\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{4}{n^3\pi}[\sin nx]_0^{\pi} \\ &= \frac{4}{n^2}\cos n\pi \\ &= \frac{4}{n^2}(-1)^n \end{aligned}$$

Therefore, the half range cosine series of $f(x) = x^2$ is

$$\begin{aligned} x^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx \\ &= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right] \end{aligned}$$

Now we need to evaluate the value of the following series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

which can be easily achieved by putting $x = 0$ in the obtained half range cosine series of $f(x)$. So we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}.$$

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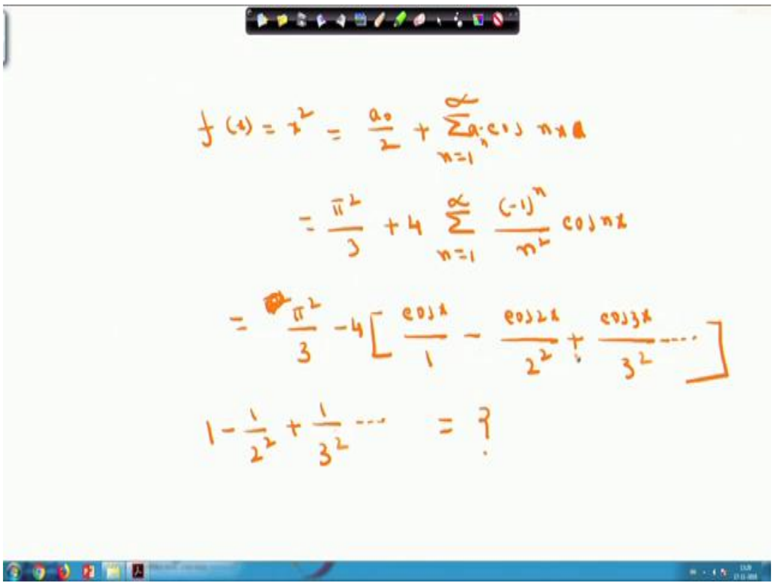

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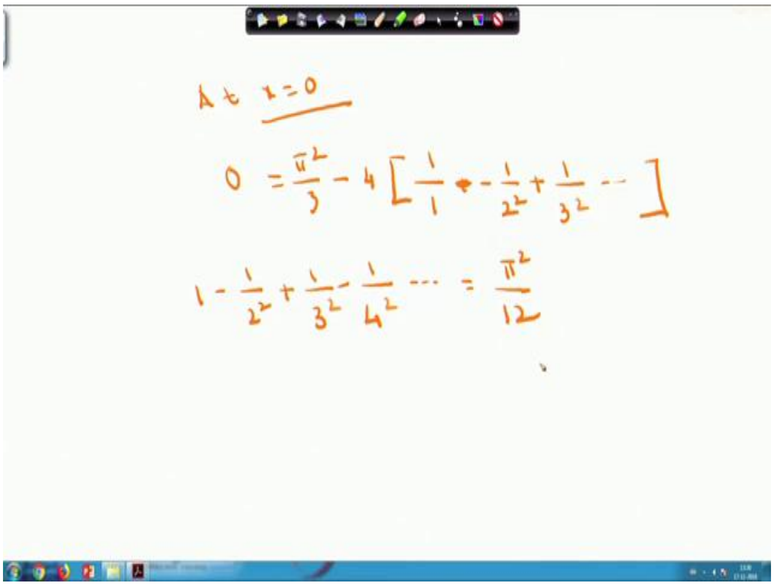

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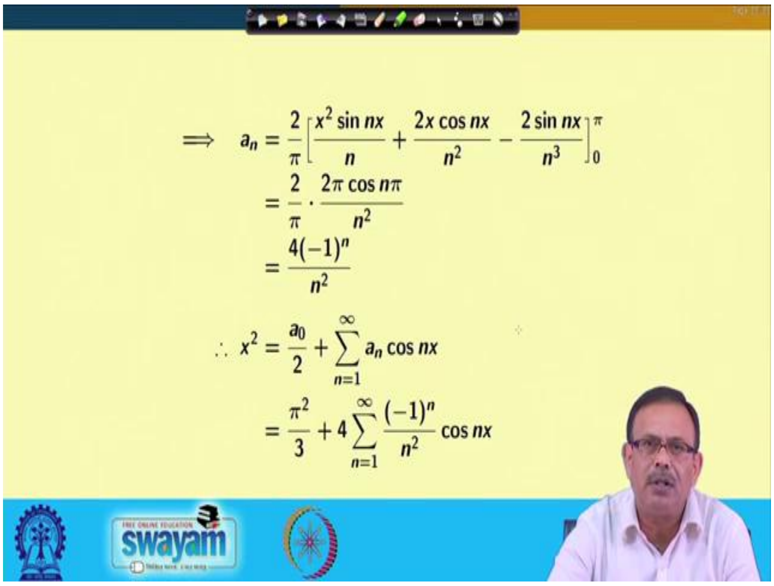

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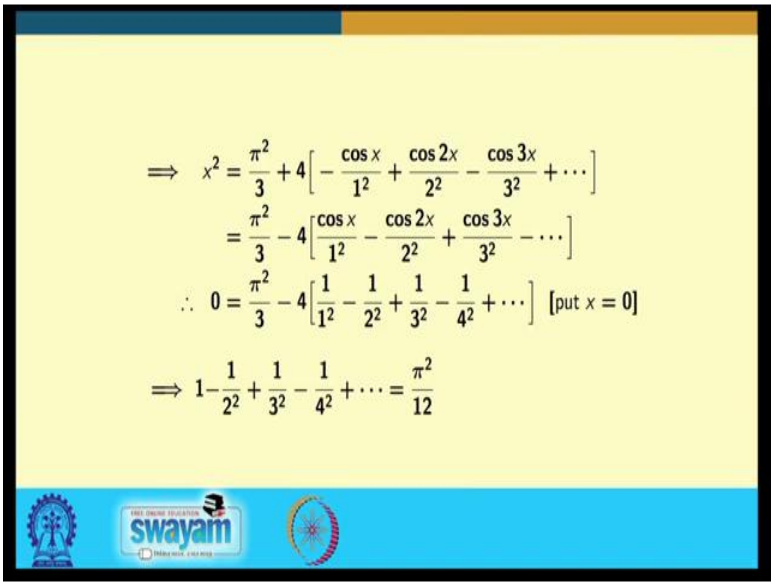

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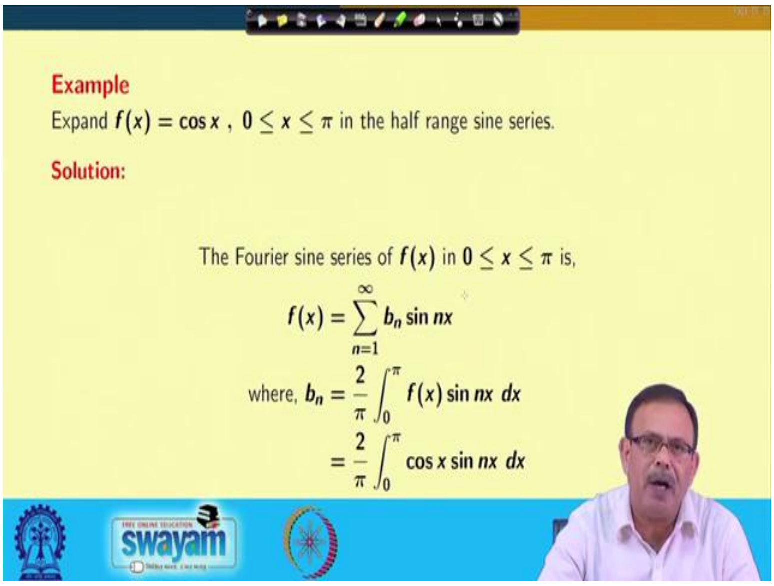

Let us take one more example. We need to expand $f(x) = \cos x$ as a half range sine series in $[0, \pi]$.

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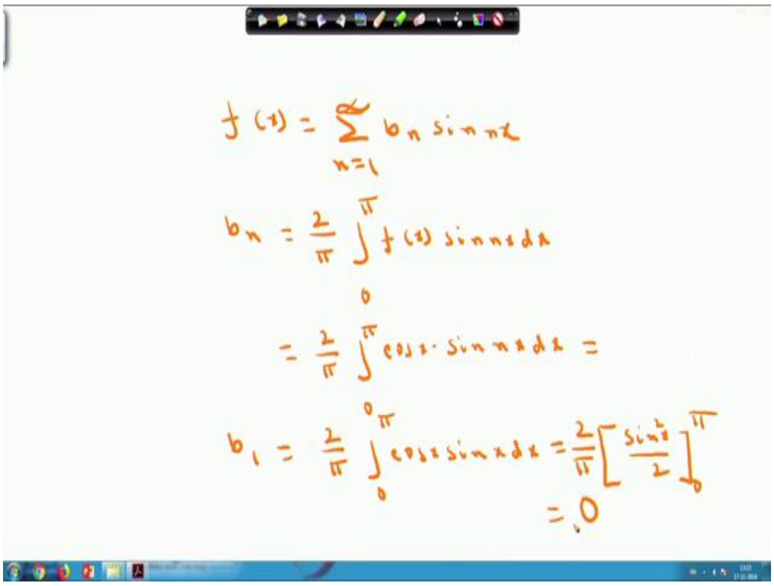

So, half range sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x]dx \\
&= -\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \qquad \text{for } n \neq 1 \\
&= -\frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right] \qquad \text{for } n \neq 1 \\
&= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{2n}{n^2-1}\right] \qquad \text{for } n \neq 1 \\
&= \frac{1}{\pi}\left[\frac{2n}{n^2-1}(-1)^n + \frac{2n}{n^2-1}\right] \qquad \text{for } n \neq 1 \\
&= \frac{2n[(-1)^n+1]}{\pi(n^2-1)} \qquad \text{for } n \neq 1 \\
&= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{4n}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}
\end{aligned}
$$

However, for $n = 1$, we have,

$$
\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} \cos x \sin x\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx \\
&= -\frac{1}{2\pi}[\cos 2x]_0^{\pi} \\
&= 0
\end{aligned}
$$

Therefore the half range sine series of $f(x) = \cos x$ in $[0, \pi]$ is given by

$$
\begin{aligned}
\cos x &= \sum_{n=2}^{\infty} b_n \sin nx \\
&= \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{n[(-1)^n+1]}{n^2-1}\sin nx \\
&= \frac{2}{\pi}\left[\frac{4}{3}\sin 2x + \frac{8}{15}\sin 4x + \frac{12}{35}\sin 6x + \cdots\right]
\end{aligned}
$$

$$\Rightarrow \cos x = \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{2\sin 4x}{15} + \frac{3\sin 6x}{35} + \cdots\right]$$

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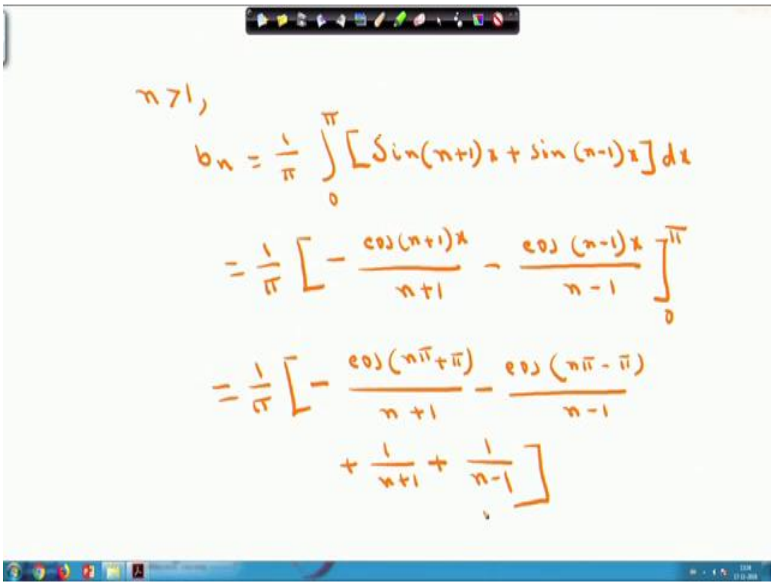

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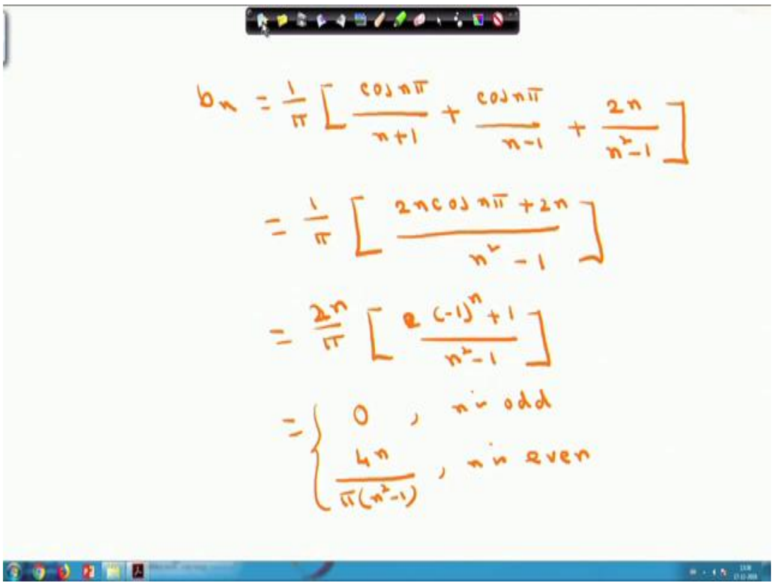

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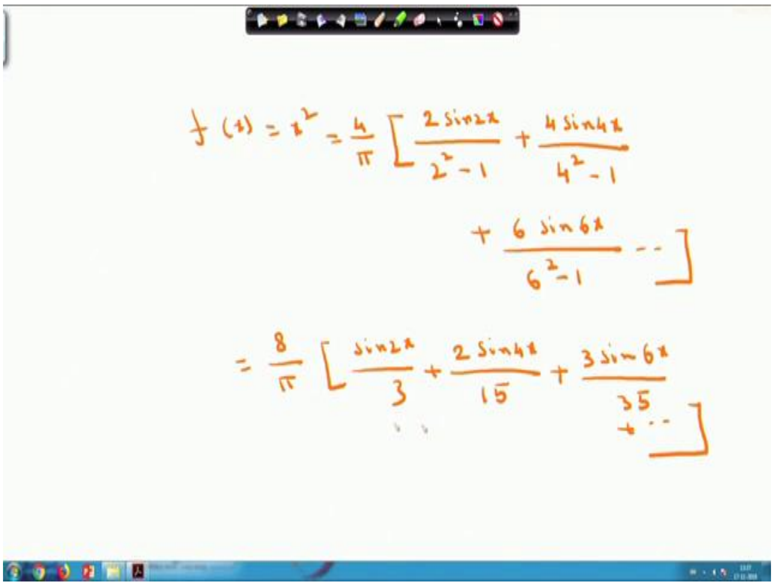

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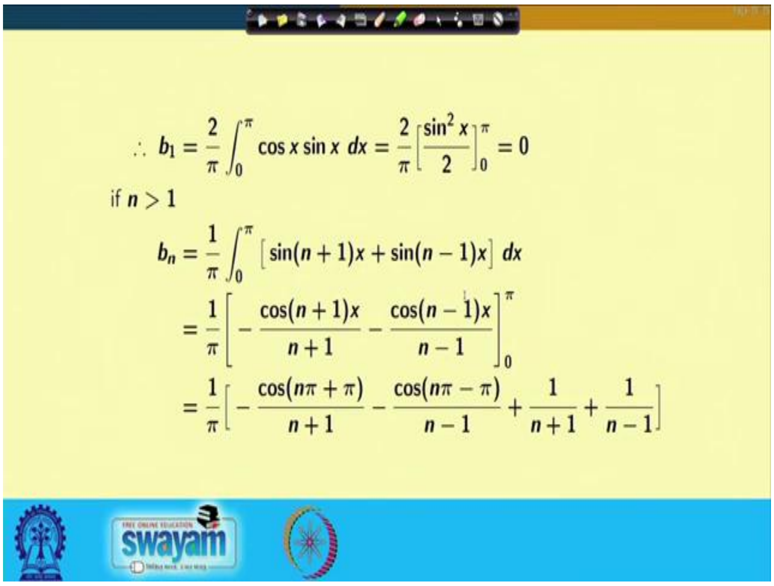

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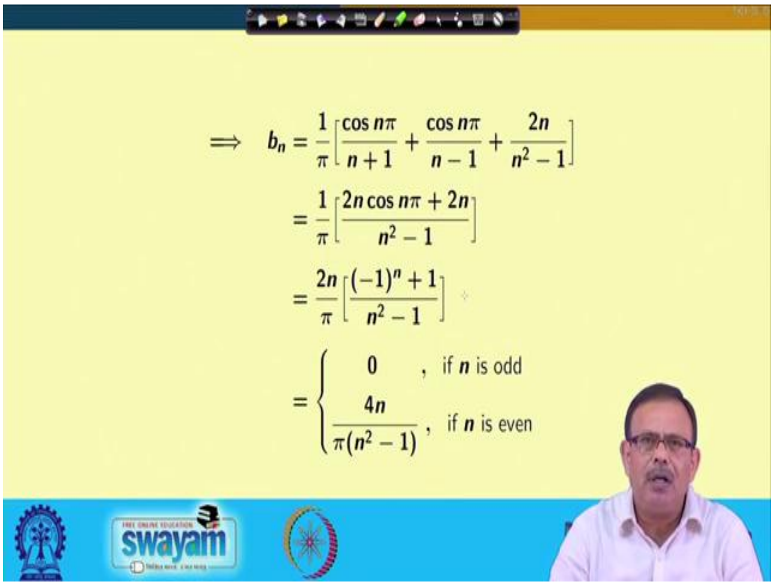

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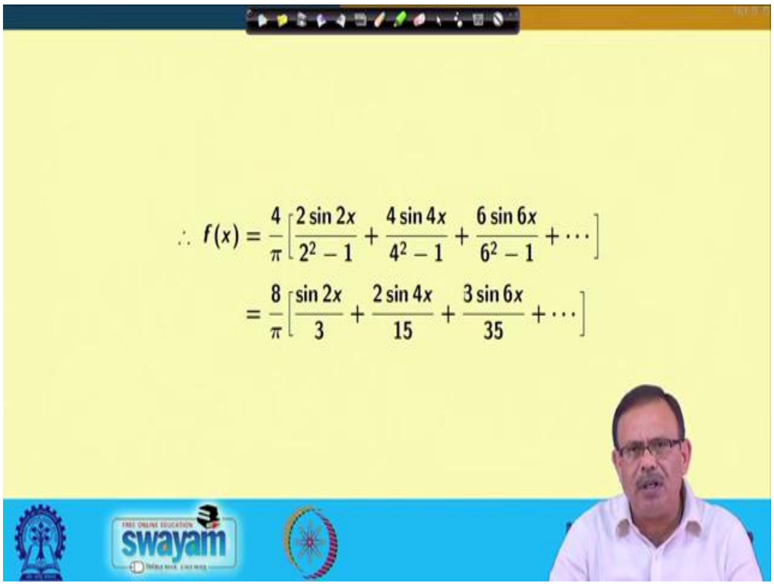

So therefore, if we are given a function, then we have seen how to express the function in terms of a half range sine or cosine series.

We can now find out the Fourier series of an odd function, Fourier series of an even function and half range Fourier cosine series or half range Fourier sine series of a function as well.

Thank you.