Transform Calculus and its Applications in Differential Equations

Prof. Adrijit Goswami
Department of Mathematics
Indian Institutes of Technology, Kharagpur

Lecture – 22

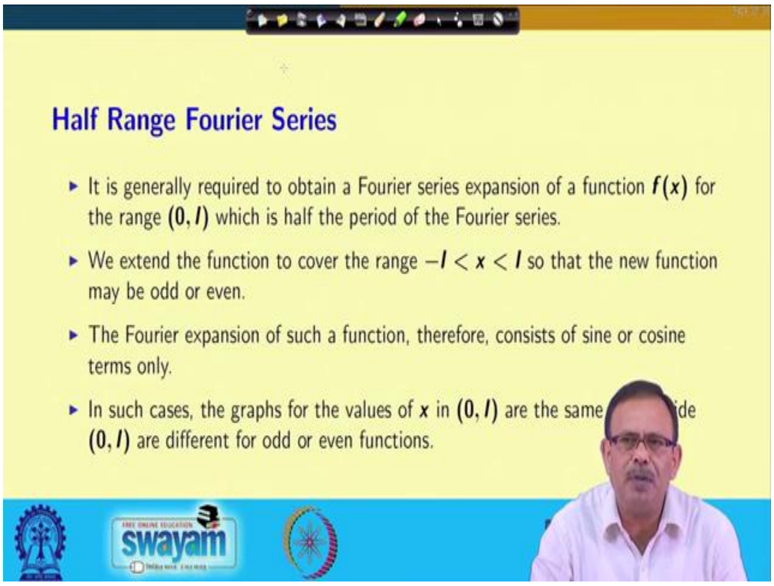
Fourier Series of Functions having arbitrary period – II

In this lecture, we will start with half range Fourier series.

(Refer Slide Time: 00:25)

So, earlier on Fourier series, whatever we have covered, is summarized as; the Fourier series of a generalized function, Fourier series for odd and even functions and Fourier series of a function which is defined in some arbitrary period.

(Refer Slide Time: 00:47)

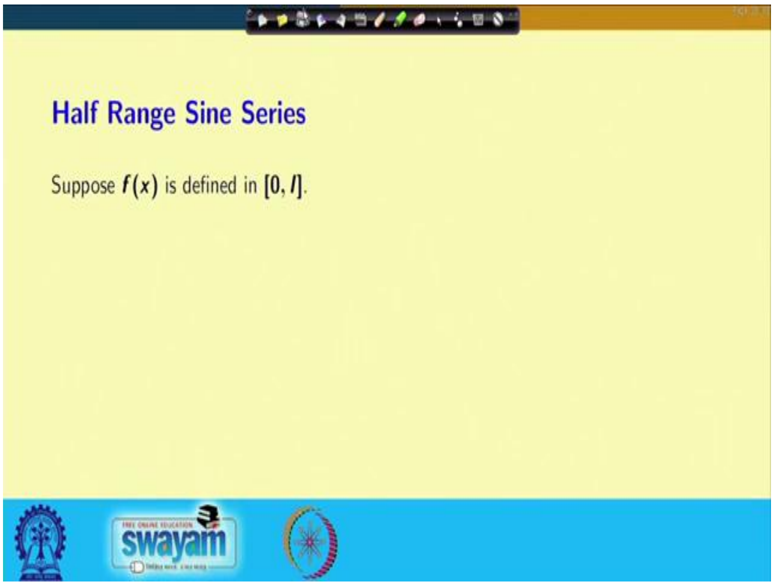


Next we come to half range Fourier series. It is generally required to obtain a Fourier series expansion of a function, for the range $(0, l)$ which is half the period of the Fourier series. In certain real life problems, we have observed that a function is defined in $(0, l)$ only and we want to express the function in terms of cosine or sine series only.

So, we want to study that if a function $f(x)$ is defined in $(0, l)$ or in the half range of that of a Fourier series, in that case how to find out the half range series of the function. So, what we can do is, we can extend the function to cover the range $(-l, l)$ so that the new function will be either even function or an odd function. So we can redefine the function in $(-l, 0)$ in such a way that the function may behave like an odd function or it may behave like an even function. So, the Fourier series expansion of such a function basically will be consisting of either sine terms or cosine terms only, because already we have observed that if a function is odd function or even function, in that case it will be consisting of sine terms or cosine terms only. So, the basic idea is whenever the function is defined in $(0, l)$, in that case, we will redefine the function in $(-l, 0)$ also so that the function is now defined in $(-l, l)$.

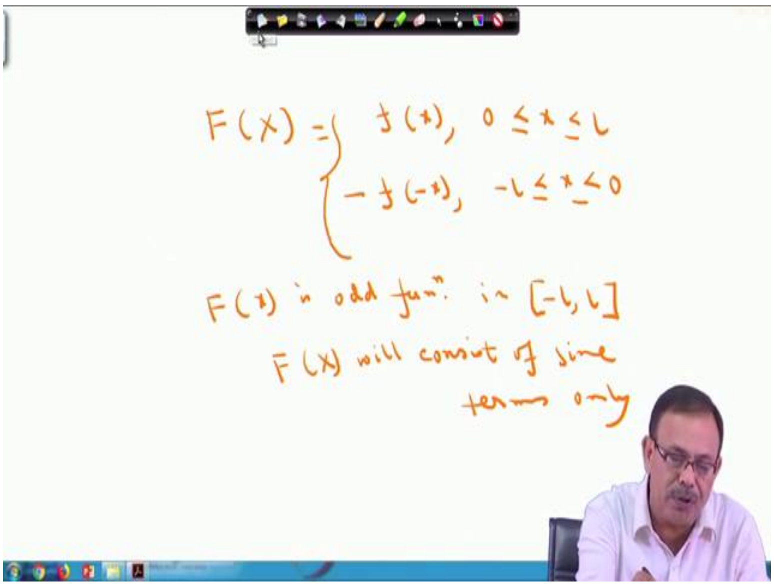
And while doing so, we will take care that $f(x)$ will be either even function or an odd function. And once $f(x)$ is defined in $(-l, l)$, we can always express $f(x)$ as a Fourier series. Now the nature of the function will remain same in $(0, l)$. But in $(-l, 0)$, the graph of the function will be varying depending upon whether it has been treated as odd function or it has been treated as even function.

(Refer Slide Time: 04:00)



First we will see the half range sine series. Suppose $f(x)$ is defined in $(0, l)$. So, we want to find out half range sine series of $f(x)$ in $(0, l)$.

(Refer Slide Time: 04:20)

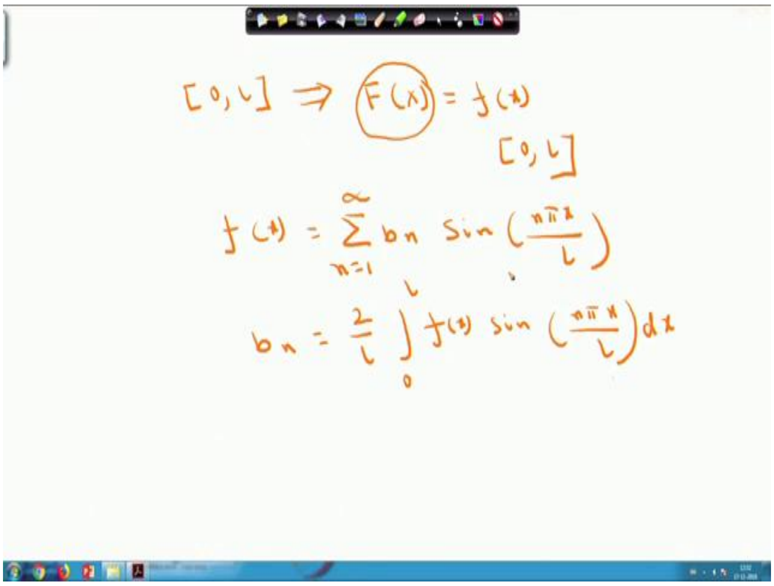


We are defining a new function $F(x)$ say, in such a fashion that

$$F(x) = \begin{cases} f(x), & 0 \le x \le l \\ -f(-x), & -l \le x \le 0 \end{cases}$$

Clearly, $F(x)$ is an odd function, in the interval $(-l, l)$. So, now, we know that the Fourier series expansion of $F(x)$ is consisting of only sine terms.

(Refer Slide Time: 05:52)



Further, in $(0, l)$, we have $F(x) = f(x)$. So, $F(x)$ gives the required sine series solution, of $f(x)$ in $(0, l)$. So that we can write down,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (1)$$

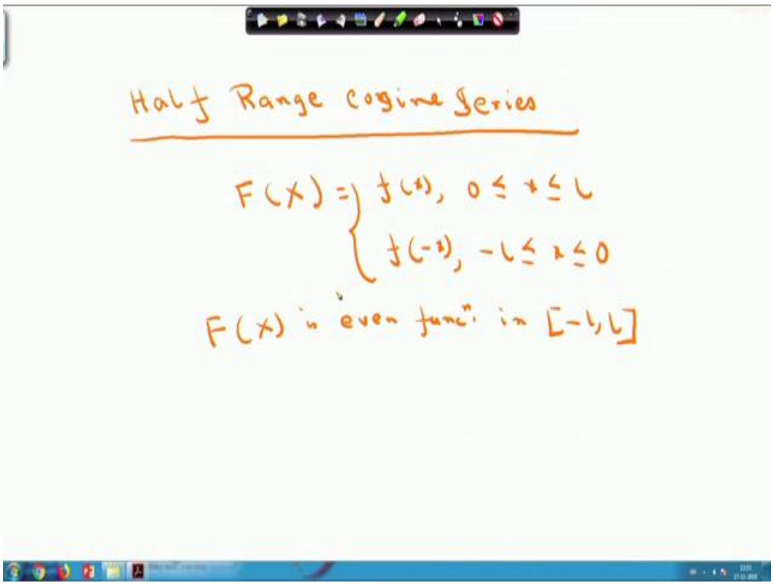
where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

Therefore, the half range sine series of a function $f(x)$ is given by (1).

Now, we come to half range cosine series.

(Refer Slide Time: 07:15)



Just like half range sine series, in half range cosine series also, we can define a new function $F(x)$ such that

$$F(x) = \begin{cases} f(x), & 0 \le x \le l \\ f(-x), & -l \le x \le 0 \end{cases}$$

Clearly, by this definition, $F(x)$ is an even function in $(-l, l)$.

So, we can find out the Fourier series of $F(x)$ in $(-l, l)$. And since $F(x)$ is an even function, therefore, it will be consisting of the cosine terms only.

Further, in $(0, l)$, we have $F(x) = f(x)$. So, $F(x)$ gives the required cosine series solution, of $f(x)$ in $(0, l)$. So that we can write down,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \tag{2}$$

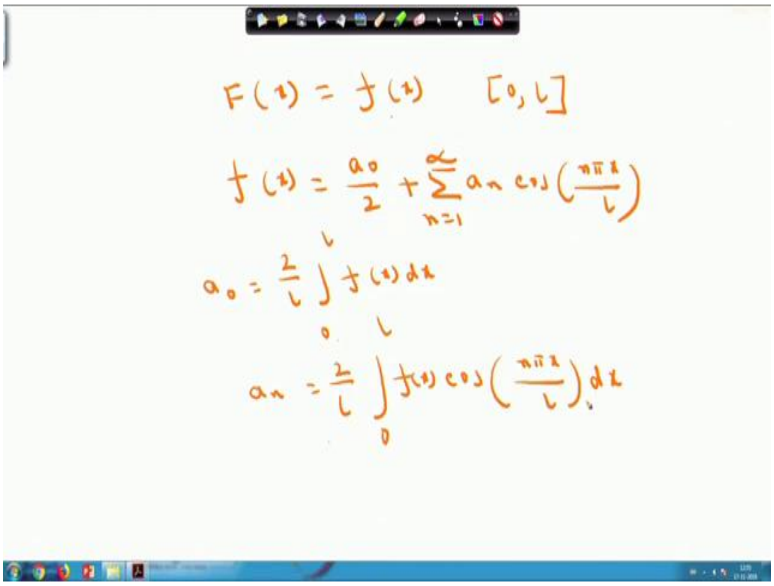
where

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

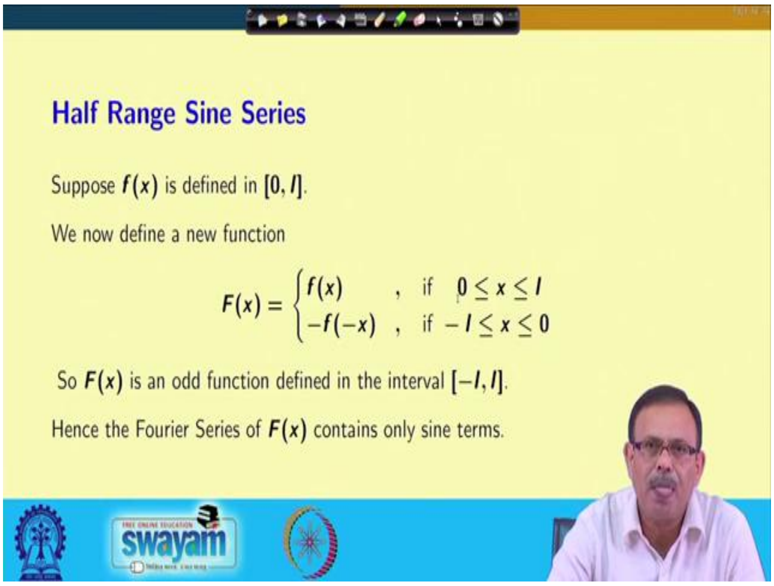
$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Therefore, the half range cosine series of $f(x)$ is given by (2).

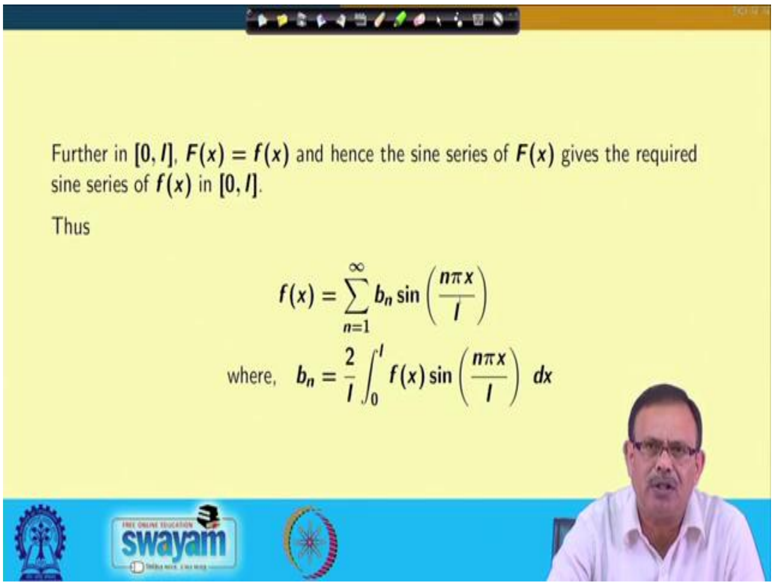
(Refer Slide Time: 08:41)



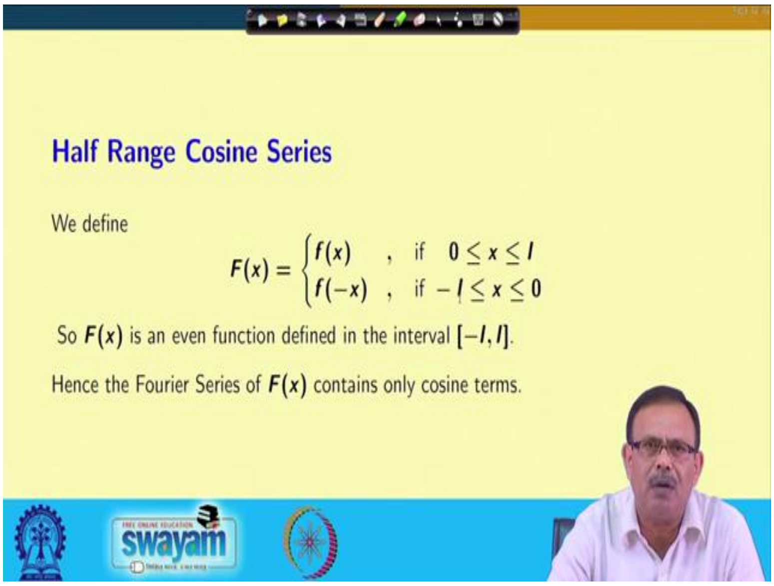
(Refer Slide Time: 10:13)



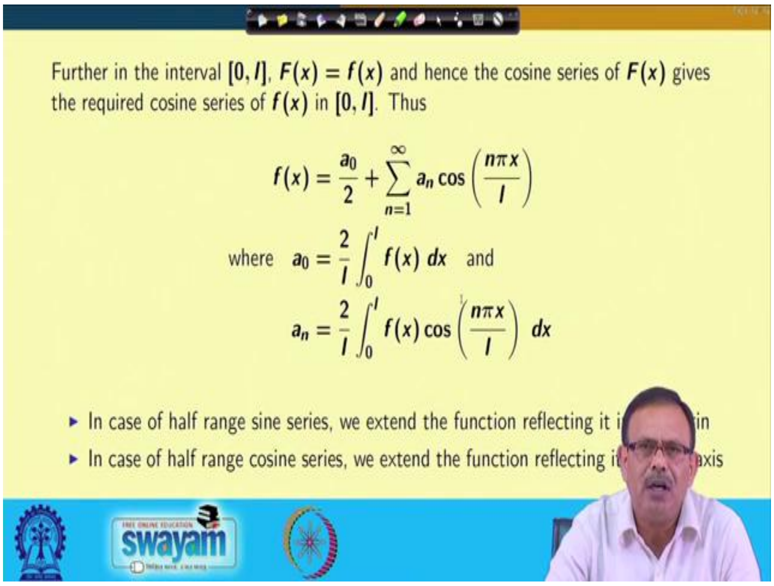
(Refer Slide Time: 10:44)



(Refer Slide Time: 11:14)

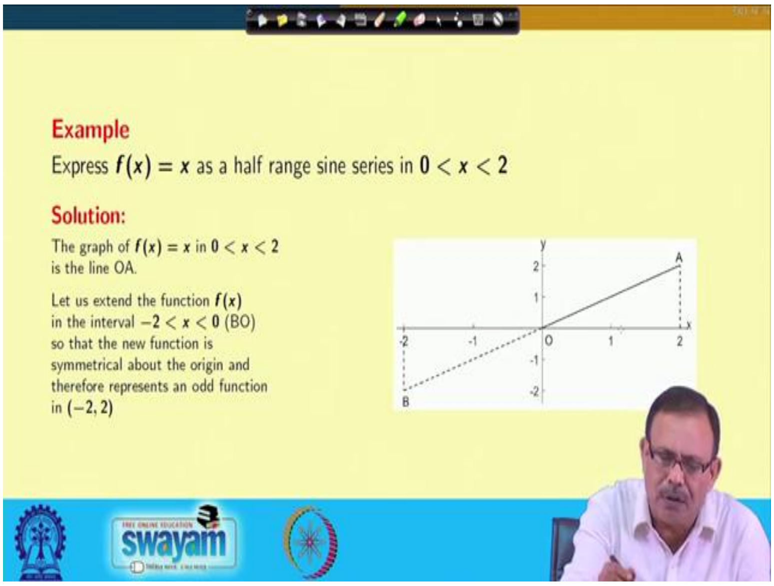


(Refer Slide Time: 11:38)



In case of half range sine series, we extend the function reflecting it in the origin, please note this one. And in case of half range cosine series, we extend the function reflecting it in the y axis. Let us see how it works with examples.

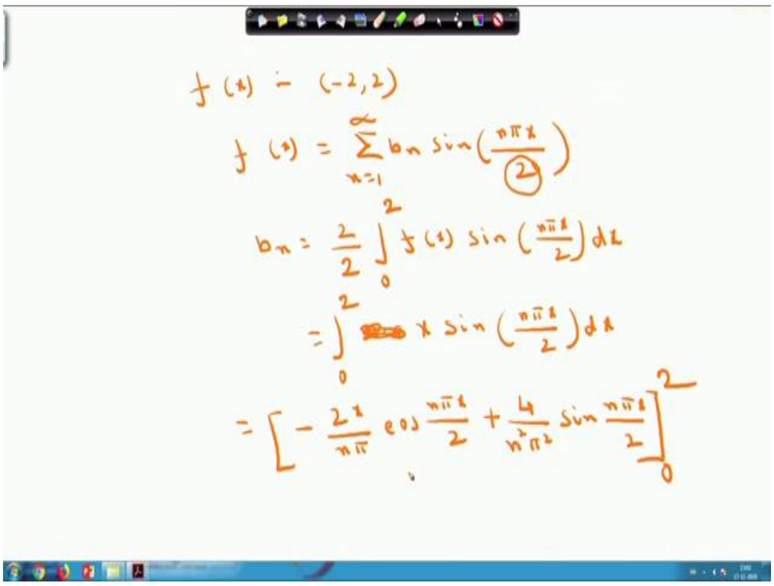
(Refer Slide Time: 12:46)



Suppose we want to find out the half range sine series for the function $f(x) = x$ when x lies between 0 and 2. For the same function, effectively afterwards we will see what is the half range cosine series. Now here if we see the graph of $f(x)$ in $(0,2)$, we have a straight line OA. Now we want to extend this function $f(x)$ in the interval $(-2,0)$ as OB so that the function is symmetric about the origin. Since we are finding out the half range sine series, so we will make it symmetric with respect to origin and therefore the function will represent an odd function in $(-2,2)$.

The basic idea is, whenever a function is given and we have to find out the half range series, whether it is cosine series or it is sine series, we will first draw the original graph in the given interval that is $(0, l)$. Then we extend it in $(-l, 0)$ in such a fashion that it will be symmetric either with respect to origin or with respect to the y axis depending upon whether we want to find the half range sine series or half range cosine series respectively. Once we have done this, then we can solve the problem using the normal process as we have done earlier.

(Refer Slide Time: 15:09)



So, now let us see, what is the next step. We have defined the function $f(x) = x$ in $(-2,2)$. And since the function is an odd function, so its Fourier series expansion will contain the terms of sine only. So, we can write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$b_n = \frac{2}{2}\int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

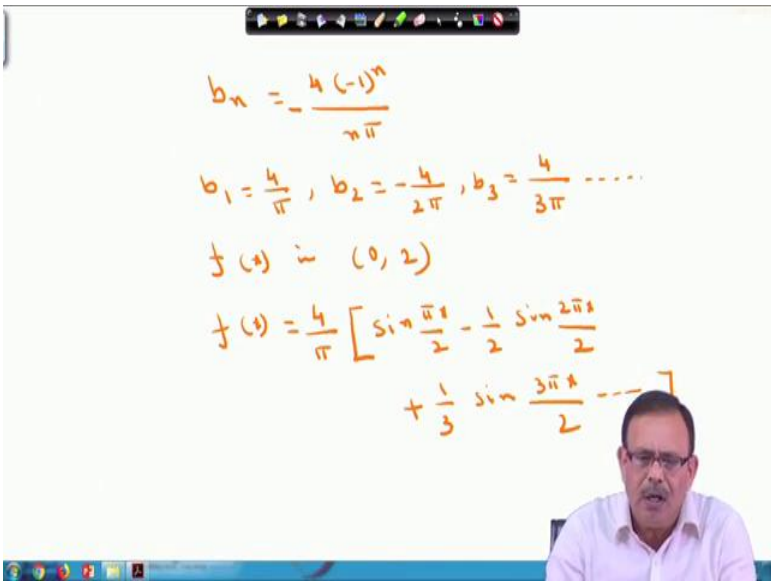
$$\Rightarrow b_n = \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \left(\frac{2}{n\pi}\right)^2 \left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -\frac{4}{n\pi}\cos n\pi$$

$$= -\frac{4(-1)^n}{n\pi}$$

(Refer Slide Time: 16:55)



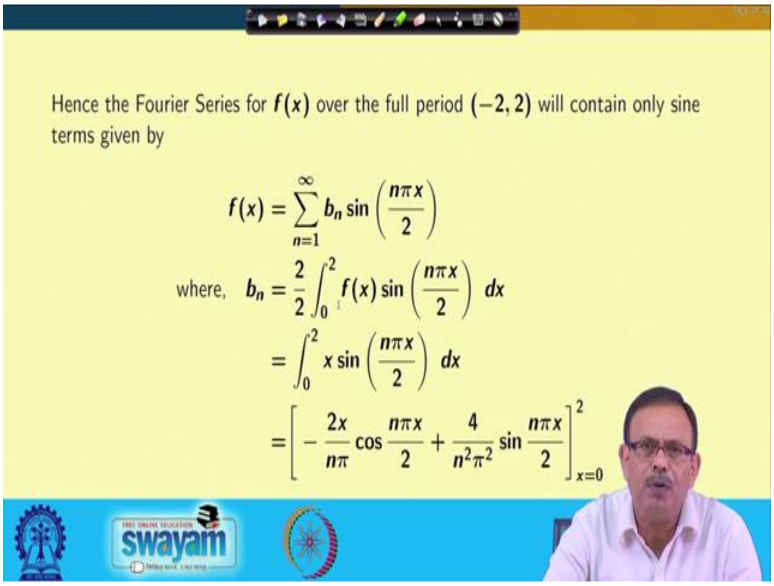
Therefore, the Fourier series expansion of $f(x)$ in the interval $(0,2)$ then becomes

$$f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)$$

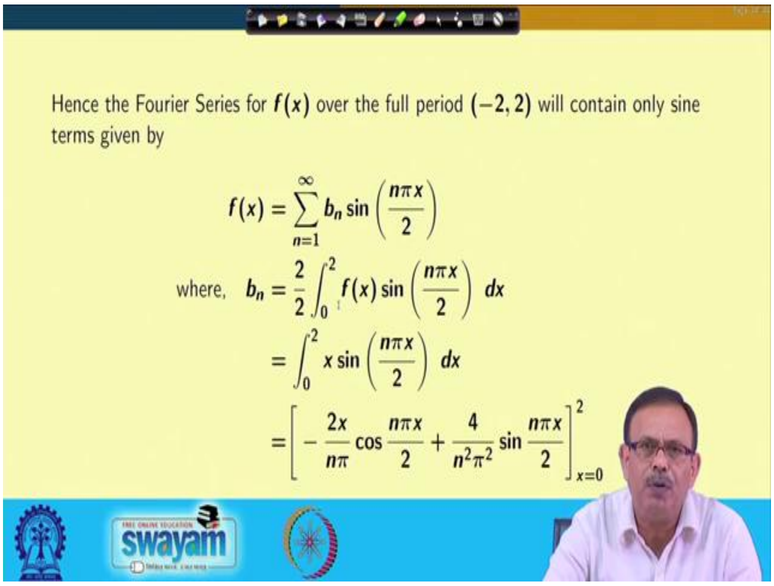
$$= \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) - \cdots\right]$$

which represents the half range sine series of $f(x)$. So, whenever a function is given to us and we have to find out the half range sine series, in that case we extend the function in such a fashion that it is symmetric about the origin. So in this case, we are redefining the function $f(x)$ in $(-2,2)$ which becomes an odd function. Then using the normal process, the series of $f(x)$ contains only the sine terms, i.e., there will be only the Fourier coefficient b_n and $a_n = 0$.

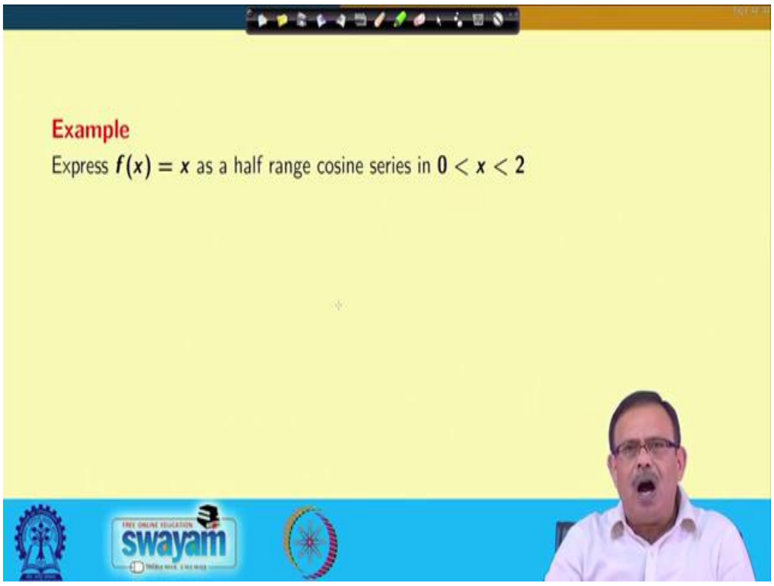
(Refer Slide Time: 18:57)



(Refer Slide Time: 19:13)

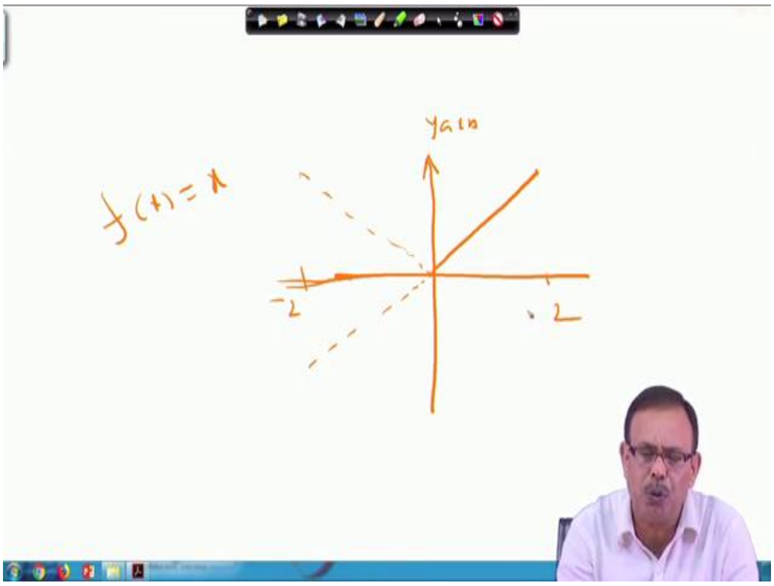


(Refer Slide Time: 19:44)



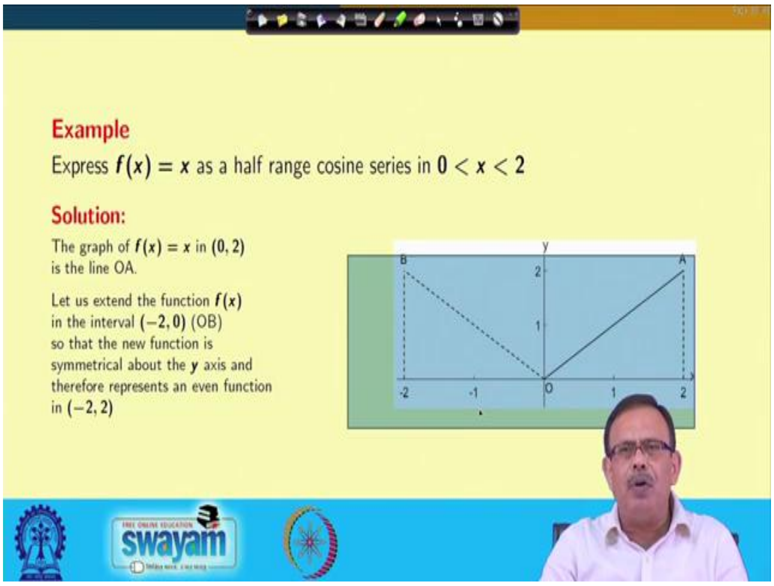
Now, let us take the same example, $f(x) = x$ and now we want to express it as a half range cosine series in $(0,2)$.

(Refer Slide Time: 20:00)



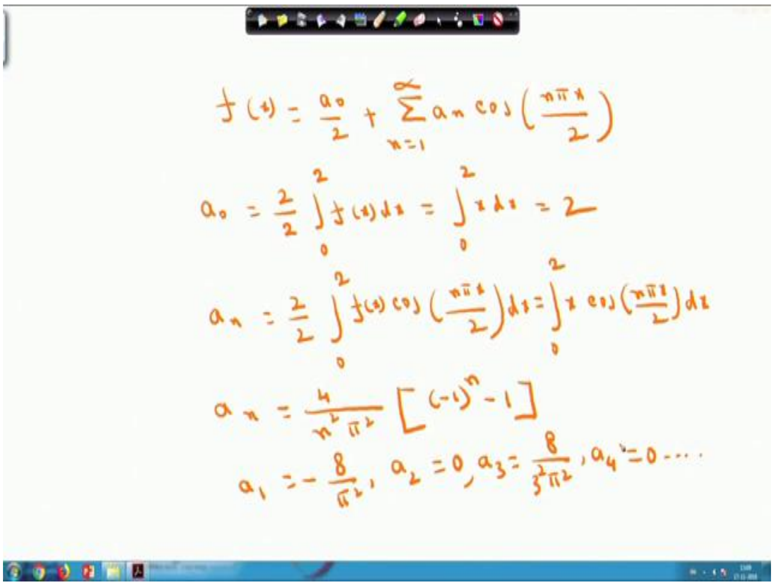
In the earlier case, we had to obtain the sine series, so we extended the function in such a way that it is symmetric about the origin. Now for the cosine series of the function $f(x)$, we have to extend the function in such a fashion in the interval $(-2,0)$ so that it will be symmetric with respect to y axis.

(Refer Slide Time: 21:21)



If we see the graph of the function in the interval $(-2,2)$, we visualize that the original function remains unchanged in $(0,2)$. It is extended in $(-2,0)$ in such a fashion that it is symmetric with respect to the y axis. The function now clearly represents an even function.

(Refer Slide Time: 22:39)



So its Fourier series expansion will contain cosine terms only. So, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

where

$$
\begin{aligned}
a_0 &= \frac{2}{2}\int_0^2 f(x)dx \\
&= \int_0^2 x dx \\
&= 2
\end{aligned}
$$

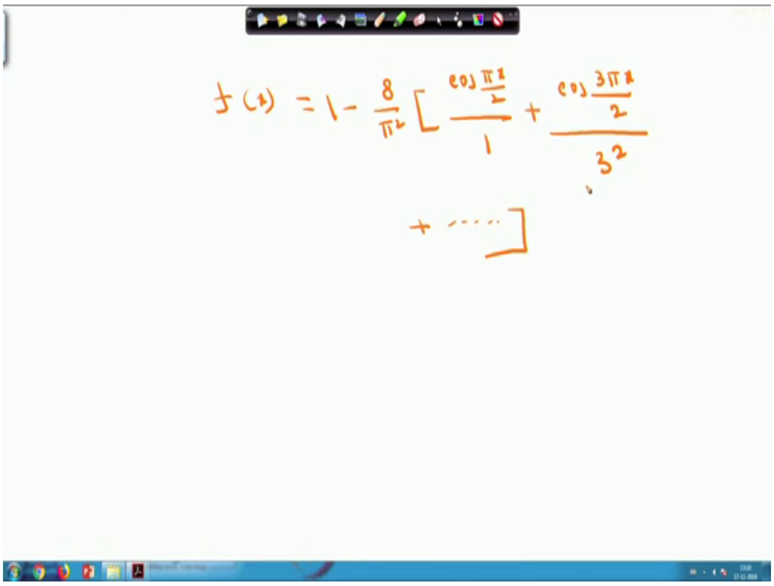
$$
\begin{aligned}
a_n &= \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 + \left(\frac{2}{n\pi}\right)^2\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 \\
&= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \\
&= \frac{4[(-1)^n - 1]}{n^2\pi^2}
\end{aligned}
$$

Therefore, the Fourier series expansion of $f(x)$ in the interval $(0,2)$ then becomes

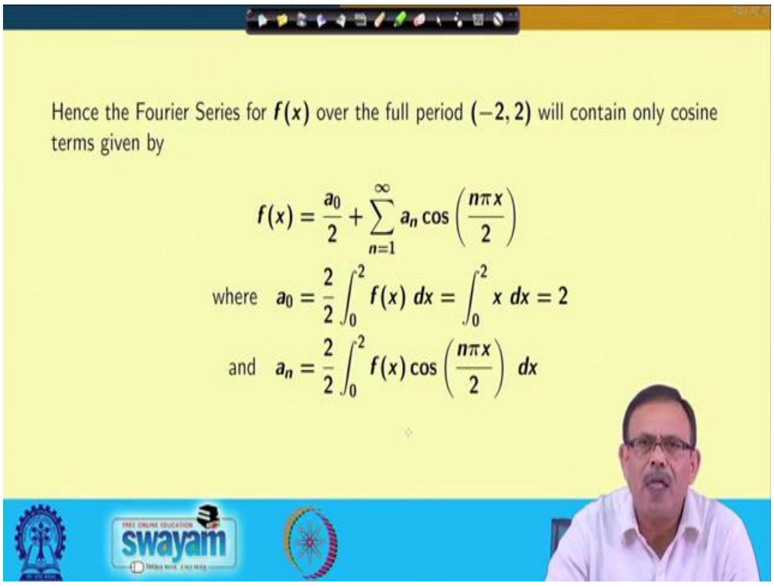
$$
\begin{aligned}
f(x) &= \frac{2}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos\left(\frac{n\pi x}{2}\right) \\
&= 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right]
\end{aligned}
$$

which represents the half range cosine series of $f(x)$.

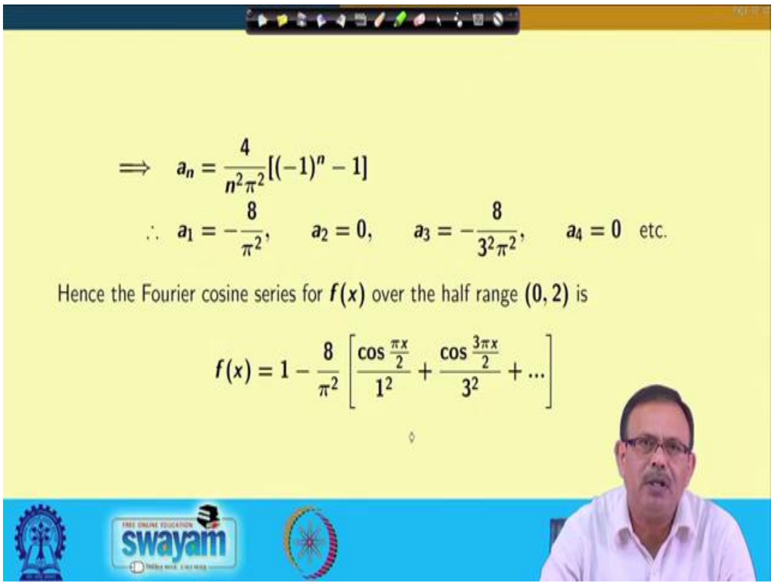
(Refer Slide Time: 24:32)



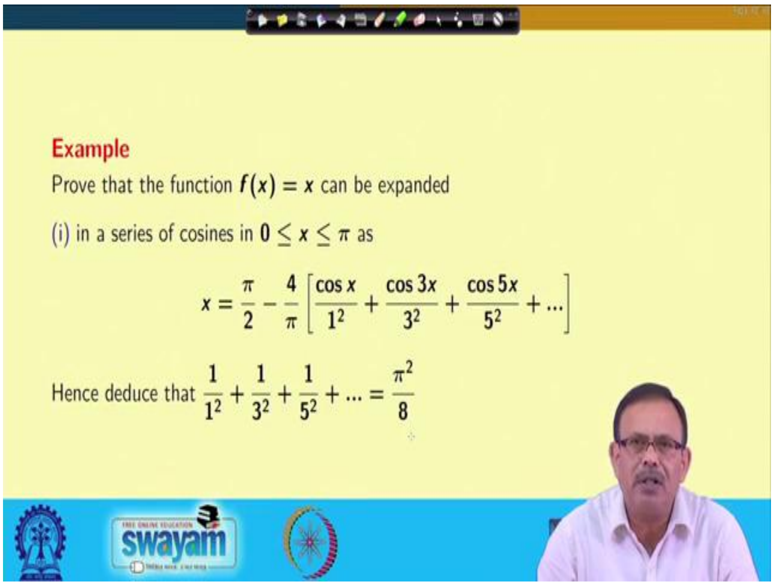
(Refer Slide Time: 25:23)



(Refer Slide Time: 25:45)



(Refer Slide Time: 26:22)



The next problem is to prove that $f(x) = x$ can be expanded in a series of cosines in $[0, \pi]$. Since this will take some time so we will start this example in the next lecture. Thank you.