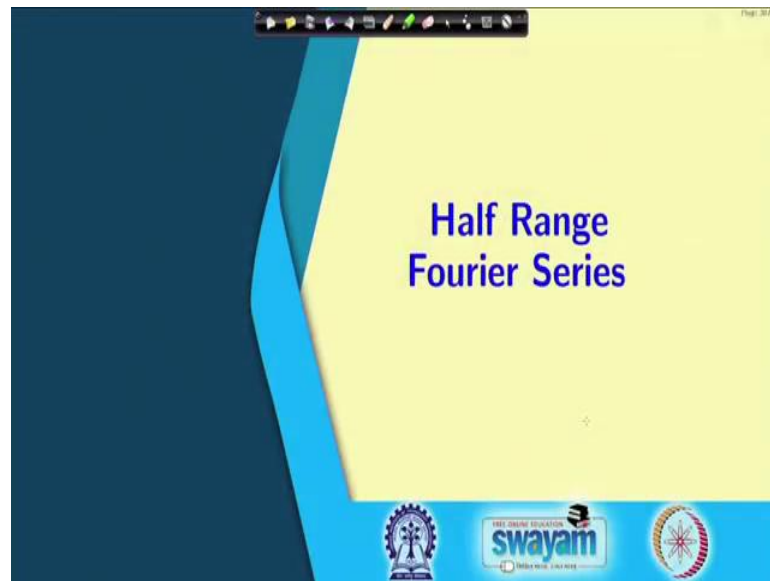


Transform Calculus and its Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institutes of Technology, Kharagpur

Lecture – 22
Fourier Series of Functions having arbitrary period – II

In this lecture, we will start with half range Fourier series.

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So, earlier on Fourier series, whatever we have covered, is summarized as; the Fourier series of a generalized function, Fourier series for odd and even functions and Fourier series of a function which is defined in some arbitrary period.

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Half Range Fourier Series

- ▶ It is generally required to obtain a Fourier series expansion of a function $f(x)$ for the range $(0, l)$ which is half the period of the Fourier series.
- ▶ We extend the function to cover the range $-l < x < l$ so that the new function may be odd or even.
- ▶ The Fourier expansion of such a function, therefore, consists of sine or cosine terms only.
- ▶ In such cases, the graphs for the values of x in $(0, l)$ are the same. The graphs in $(-l, 0)$ are different for odd or even functions.

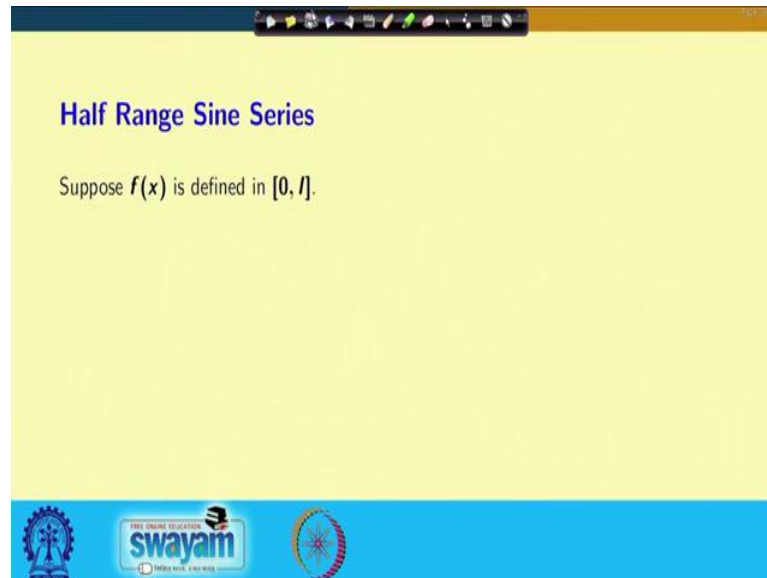
Next we come to half range Fourier series. It is generally required to obtain a Fourier series expansion of a function, for the range $(0, l)$ which is half the period of the Fourier series. In certain real life problems, we have observed that a function is defined in $(0, l)$ only and we want to express the function in terms of cosine or sine series only.

So, we want to study that if a function $f(x)$ is defined in $(0, l)$ or in the half range of that of a Fourier series, in that case how to find out the half range series of the function. So, what we can do is, we can extend the function to cover the range $(-l, l)$ so that the new function will be either even function or an odd function. So we can redefine the function in $(-l, 0)$ in such a way that the function may behave like an odd function or it may behave like an even function. So, the Fourier series expansion of such a function basically will be consisting of either sine terms or cosine terms only, because already we have observed that if a function is odd function or even function, in that case it will be consisting of sine terms or cosine terms only. So, the basic idea is whenever the function is defined in $(0, l)$, in that case, we will redefine the function in $(-l, 0)$ also so that the function is now defined in $(-l, l)$.

And while doing so, we will take care that $f(x)$ will be either even function or an odd function. And once $f(x)$ is defined in $(-l, l)$, we can always express $f(x)$ as a Fourier series. Now the nature of the function will remain same in $(0, l)$. But in $(-l, 0)$, the graph

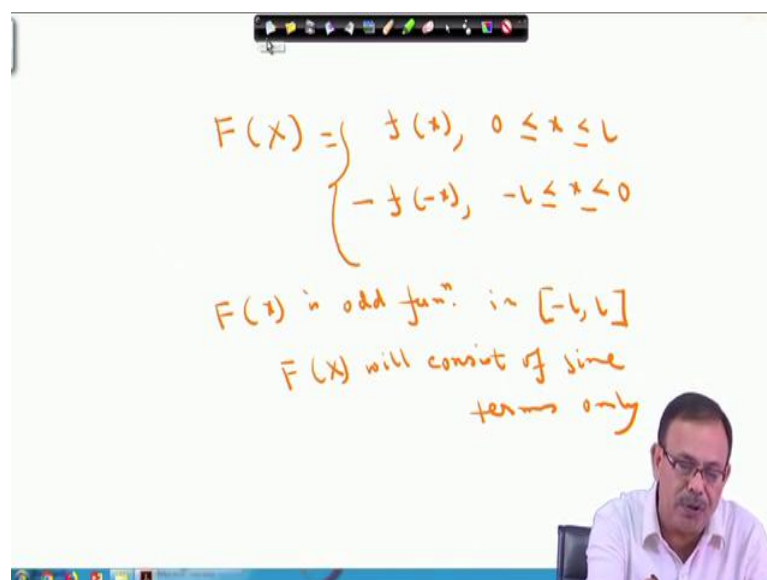
of the function will be varying depending upon whether it has been treated as odd function or it has been treated as even function.

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First we will see the half range sine series. Suppose $f(x)$ is defined in $(0, l)$. So, we want to find out half range sine series of $f(x)$ in $(0, l)$.

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We are defining a new function $F(x)$ say, in such a fashion that

$$F(x) = \begin{cases} f(x), & 0 \leq x \leq l \\ -f(-x), & -l \leq x \leq 0 \end{cases}$$

Clearly, $F(x)$ is an odd function, in the interval $(-l, l)$. So, now, we know that the Fourier series expansion of $F(x)$ is consisting of only sine terms.

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The image shows a whiteboard with handwritten mathematical notes. At the top, it says $[0, l] \Rightarrow F(x) = f(x)$ with $[0, l]$ written below $f(x)$. Below this, the function is expressed as a Fourier sine series: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$. The coefficient b_n is given by the integral formula: $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$.

Further, in $(0, l)$, we have $F(x) = f(x)$. So, $F(x)$ gives the required sine series solution, of $f(x)$ in $(0, l)$. So that we can write down,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad (1)$$

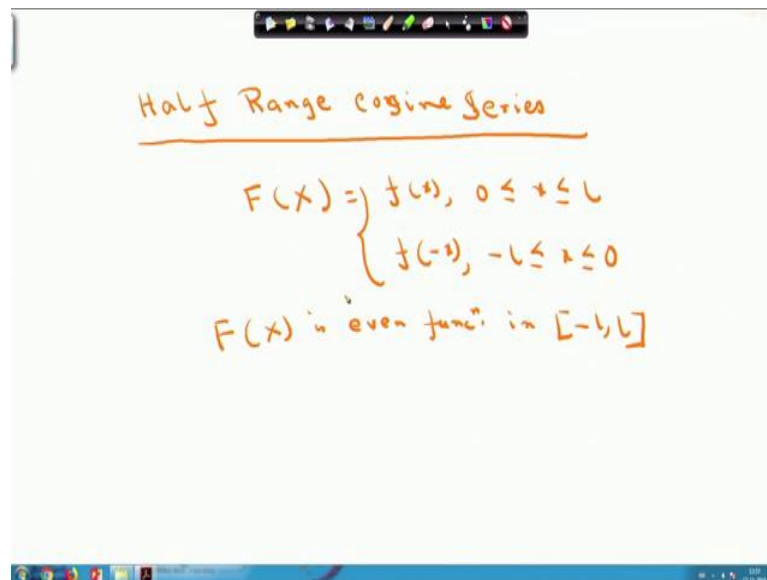
where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Therefore, the half range sine series of a function $f(x)$ is given by (1).

Now, we come to half range cosine series.

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Just like half range sine series, in half range cosine series also, we can define a new function $F(x)$ such that

$$F(x) = \begin{cases} f(x), & 0 \leq x \leq l \\ f(-x), & -l \leq x \leq 0 \end{cases}$$

Clearly, by this definition, $F(x)$ is an even function in $(-l, l)$.

So, we can find out the Fourier series of $F(x)$ in $(-l, l)$. And since $F(x)$ is an even function, therefore, it will be consisting of the cosine terms only.

Further, in $(0, l)$, we have $F(x) = f(x)$. So, $F(x)$ gives the required cosine series solution, of $f(x)$ in $(0, l)$. So that we can write down,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad (2)$$

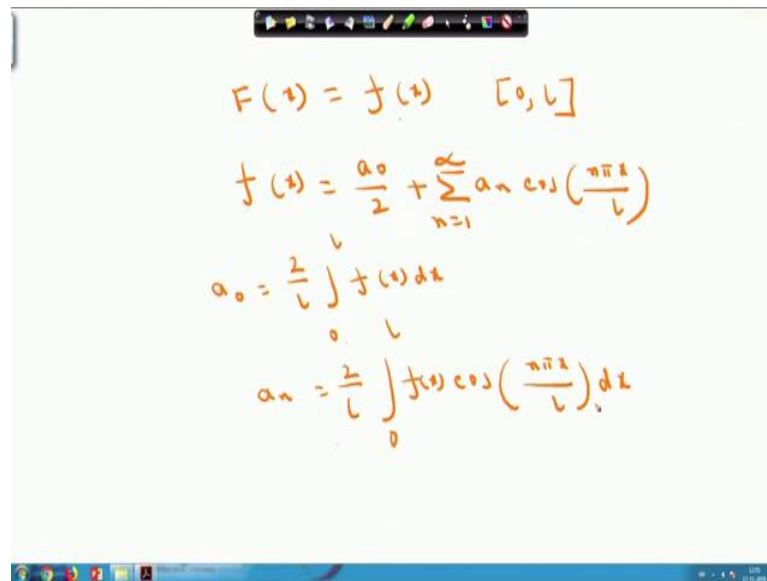
where

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Therefore, the half range cosine series of $f(x)$ is given by (2).

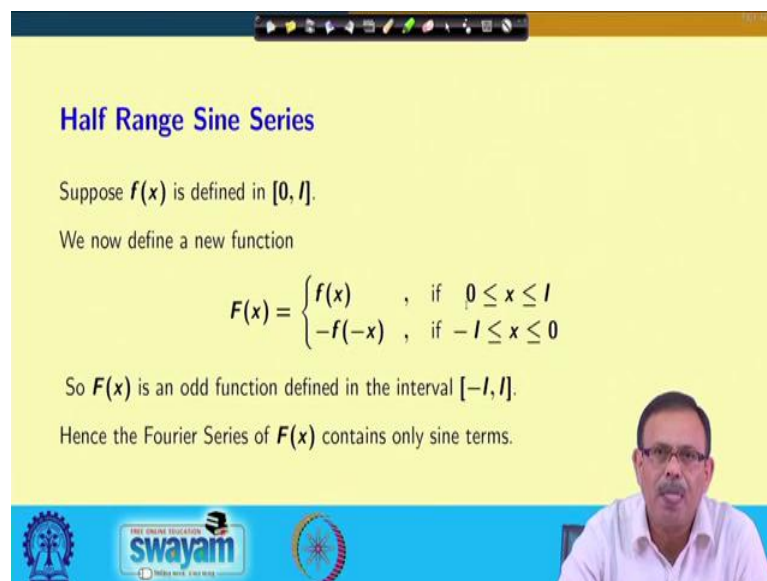
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The image shows a whiteboard with handwritten mathematical formulas in orange ink. The formulas are:

$$F(x) = f(x) \quad [0, L]$$
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$
$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$
$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

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Half Range Sine Series

Suppose $f(x)$ is defined in $[0, l]$.

We now define a new function

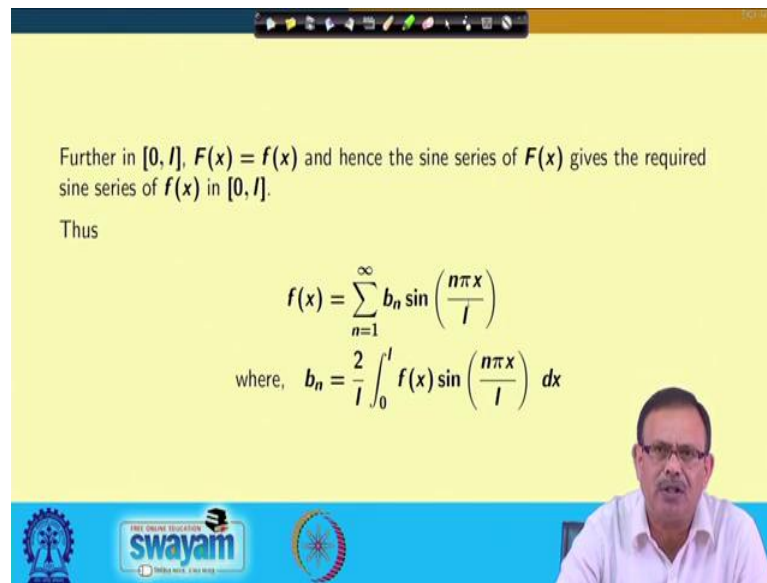
$$F(x) = \begin{cases} f(x) & , \text{ if } 0 \leq x \leq l \\ -f(-x) & , \text{ if } -l \leq x \leq 0 \end{cases}$$

So $F(x)$ is an odd function defined in the interval $[-l, l]$.

Hence the Fourier Series of $F(x)$ contains only sine terms.

The slide features a yellow background with a blue footer containing the Swamyam logo and a video feed of a presenter in the bottom right corner.

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Further in $[0, l]$, $F(x) = f(x)$ and hence the sine series of $F(x)$ gives the required sine series of $f(x)$ in $[0, l]$.

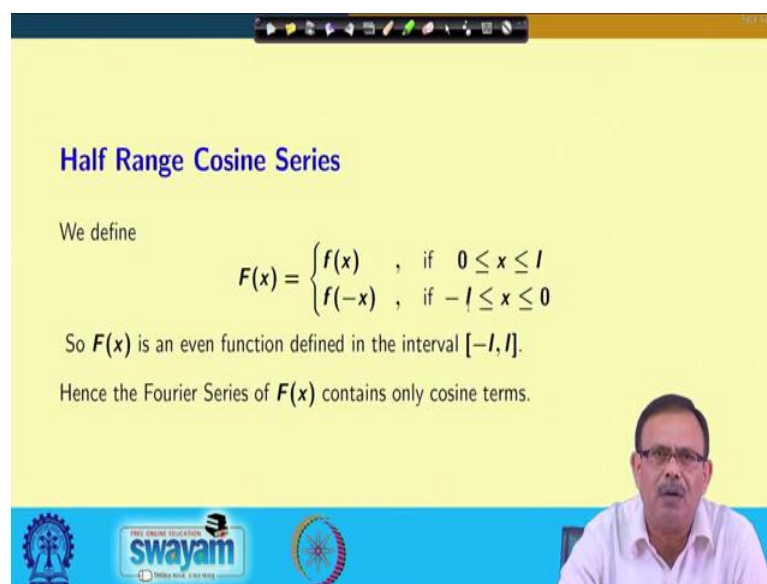
Thus

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where, $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

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Half Range Cosine Series

We define

$$F(x) = \begin{cases} f(x) & , \text{ if } 0 \leq x \leq l \\ f(-x) & , \text{ if } -l \leq x \leq 0 \end{cases}$$

So $F(x)$ is an even function defined in the interval $[-l, l]$.

Hence the Fourier Series of $F(x)$ contains only cosine terms.

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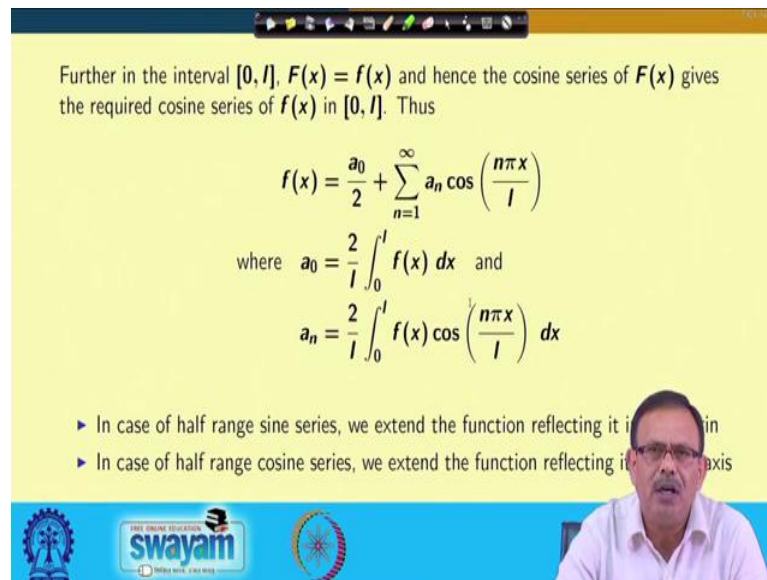
Further in the interval $[0, l]$, $F(x) = f(x)$ and hence the cosine series of $F(x)$ gives the required cosine series of $f(x)$ in $[0, l]$. Thus

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l} \int_0^l f(x) dx$ and

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

- ▶ In case of half range sine series, we extend the function reflecting it in the origin
- ▶ In case of half range cosine series, we extend the function reflecting it in the y-axis



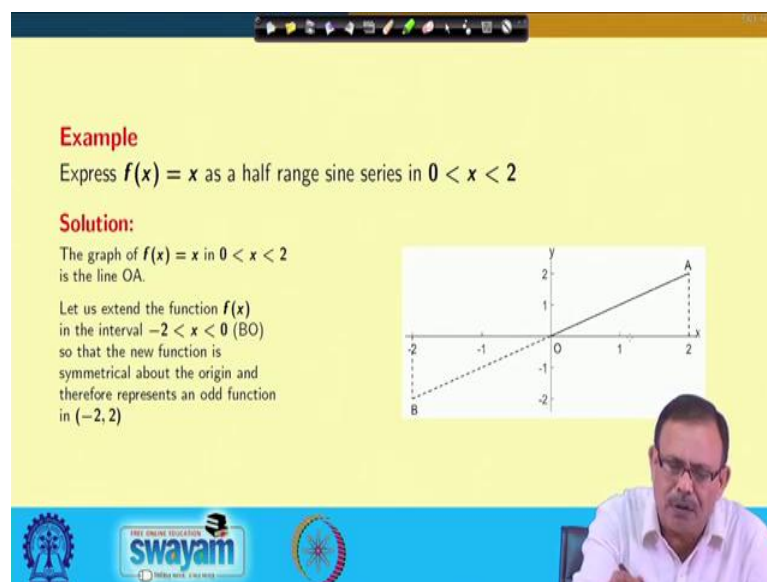
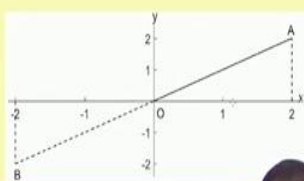
In case of half range sine series, we extend the function reflecting it in the origin, please note this one. And in case of half range cosine series, we extend the function reflecting it in the y axis. Let us see how it works with examples.

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Example
Express $f(x) = x$ as a half range sine series in $0 < x < 2$

Solution:
The graph of $f(x) = x$ in $0 < x < 2$ is the line OA.

Let us extend the function $f(x)$ in the interval $-2 < x < 0$ (BO) so that the new function is symmetrical about the origin and therefore represents an odd function in $(-2, 2)$



Suppose we want to find out the half range sine series for the function $f(x) = x$ when x lies between 0 and 2. For the same function, effectively afterwards we will see what is the half range cosine series. Now here if we see the graph of $f(x)$ in $(0,2)$, we have a straight line OA. Now we want to extend this function $f(x)$ in the interval $(-2,0)$ as OB so that

the function is symmetric about the origin. Since we are finding out the half range sine series, so we will make it symmetric with respect to origin and therefore the function will represent an odd function in $(-2,2)$.

The basic idea is, whenever a function is given and we have to find out the half range series, whether it is cosine series or it is sine series, we will first draw the original graph in the given interval that is $(0, l)$. Then we extend it in $(-l, 0)$ in such a fashion that it will be symmetric either with respect to origin or with respect to the y axis depending upon whether we want to find the half range sine series or half range cosine series respectively. Once we have done this, then we can solve the problem using the normal process as we have done earlier.

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$$\begin{aligned}
 f(x) &= (-2, 2) \\
 f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \\
 b_n &= \frac{2}{2} \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= \left[-\frac{2x}{n\pi} \cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin\frac{n\pi x}{2} \right]_0^2
 \end{aligned}$$

So, now let us see, what is the next step. We have defined the function $f(x) = x$ in $(-2,2)$. And since the function is an odd function, so its Fourier series expansion will contain the terms of sine only. So, we can write

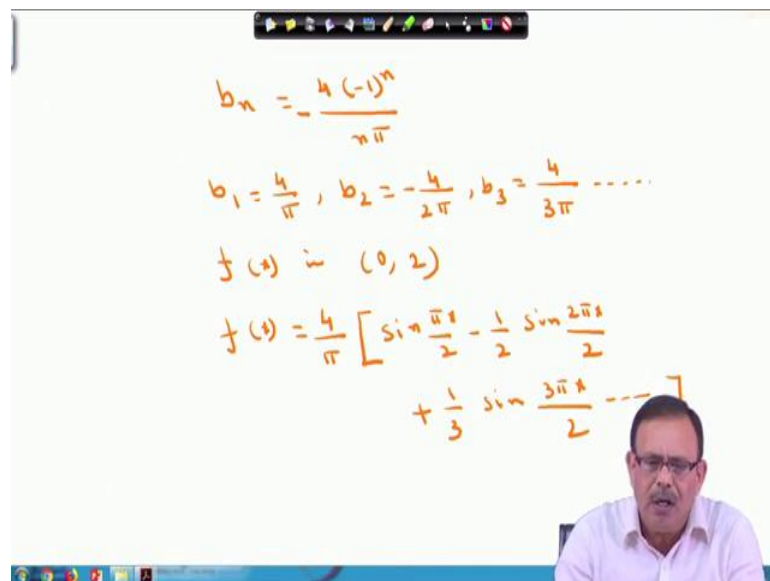
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\begin{aligned}
\Rightarrow b_n &= \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx \\
&= \left[-\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \left(\frac{2}{n\pi}\right)^2 \left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 \\
&= -\frac{4}{n\pi} \cos n\pi \\
&= -\frac{4(-1)^n}{n\pi}
\end{aligned}$$

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Therefore, the Fourier series expansion of $f(x)$ in the interval $(0,2)$ then becomes

$$\begin{aligned}
f(x) &= -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi x}{2}\right) \\
&= \frac{4}{\pi} \left[\sin\left(\frac{\pi x}{2}\right) - \frac{1}{2} \sin\left(\frac{2\pi x}{2}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{2}\right) - \dots \right]
\end{aligned}$$

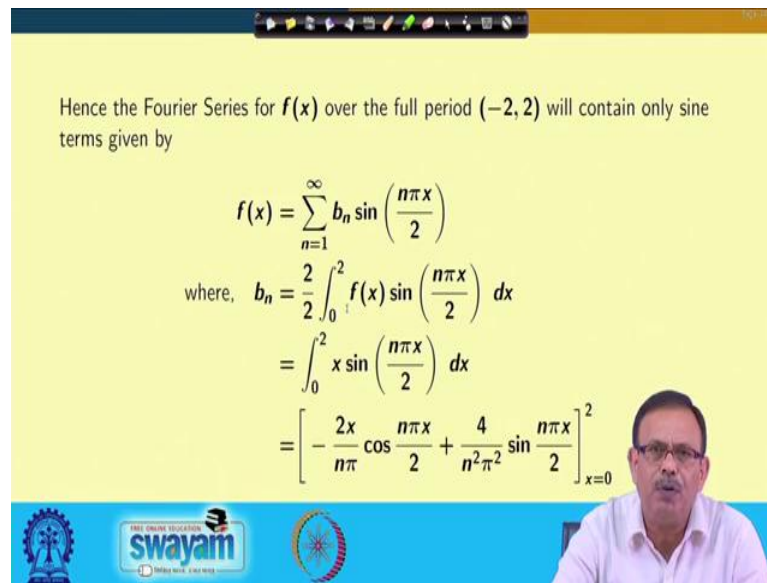
which represents the half range sine series of $f(x)$. So, whenever a function is given to us and we have to find out the half range sine series, in that case we extend the function in such a fashion that it is symmetric about the origin. So in this case, we are redefining the function $f(x)$ in $(-2,2)$ which becomes an odd function. Then using the normal process, the series of $f(x)$ contains only the sine terms, i.e., there will be only the Fourier coefficient b_n and $a_n = 0$.

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Hence the Fourier Series for $f(x)$ over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where, $b_n = \frac{2}{2} \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$

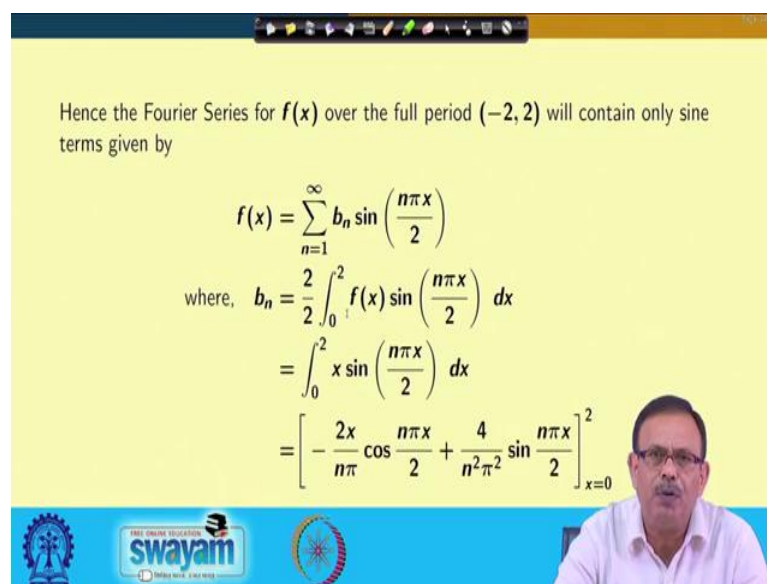
$$= \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$
$$= \left[-\frac{2x}{n\pi} \cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin\frac{n\pi x}{2} \right]_{x=0}^2$$


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Hence the Fourier Series for $f(x)$ over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where, $b_n = \frac{2}{2} \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$

$$= \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$
$$= \left[-\frac{2x}{n\pi} \cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin\frac{n\pi x}{2} \right]_{x=0}^2$$


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Example
Express $f(x) = x$ as a half range cosine series in $0 < x < 2$

The slide features a yellow background with a blue header and footer. The footer contains the Swayam logo and the text 'FREE ONLINE EDUCATION swayam'. A small video inset in the bottom right corner shows a man speaking.

Now, let us take the same example, $f(x) = x$ and now we want to express it as a half range cosine series in $(0,2)$.

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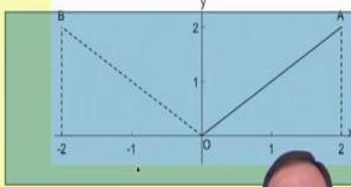

The graph shows a coordinate system with a vertical y-axis and a horizontal x-axis. The x-axis is labeled with -2 and 2. A solid line represents the function $f(x) = x$ for $0 < x < 2$. This line is extended as a dashed line into the interval $(-2, 0)$ such that the resulting function is symmetric about the y-axis, forming a V-shape. The handwritten text $f(x) = x$ is written in the upper left quadrant. The video inset shows the same man from the previous slide.

In the earlier case, we had to obtain the sine series, so we extended the function in such a way that it is symmetric about the origin. Now for the cosine series of the function $f(x)$, we have to extend the function in such a fashion in the interval $(-2,0)$ so that it will be symmetric with respect to y axis.

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Example
Express $f(x) = x$ as a half range cosine series in $0 < x < 2$

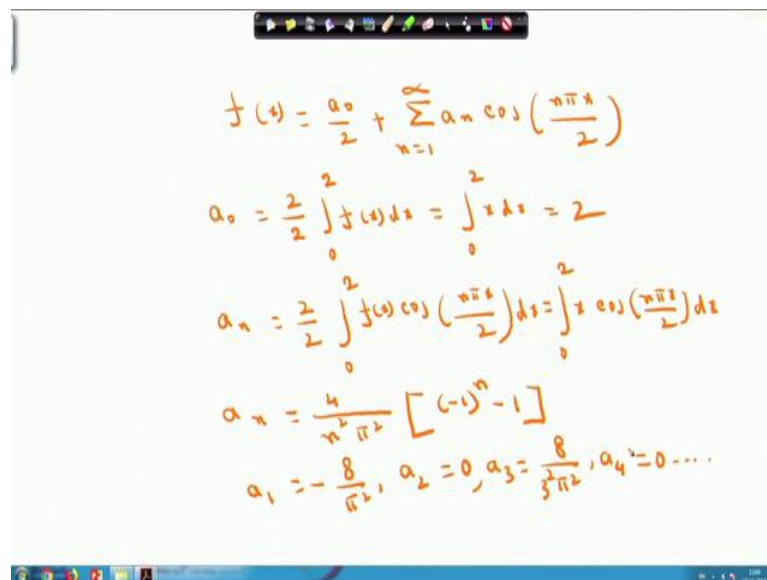
Solution:
The graph of $f(x) = x$ in $(0, 2)$ is the line OA.
Let us extend the function $f(x)$ in the interval $(-2, 0)$ (OB) so that the new function is symmetrical about the y axis and therefore represents an even function in $(-2, 2)$

Logos for Swamyam and other educational institutions are visible at the bottom of the slide.

If we see the graph of the function in the interval $(-2,2)$, we visualize that the original function remains unchanged in $(0,2)$. It is extended in $(-2,0)$ in such a fashion that it is symmetric with respect to the y axis. The function now clearly represents an even function.

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So its Fourier series expansion will contain cosine terms only. So, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

where

$$\begin{aligned}a_0 &= \frac{2}{2} \int_0^2 f(x) dx \\ &= \int_0^2 x dx \\ &= 2\end{aligned}$$

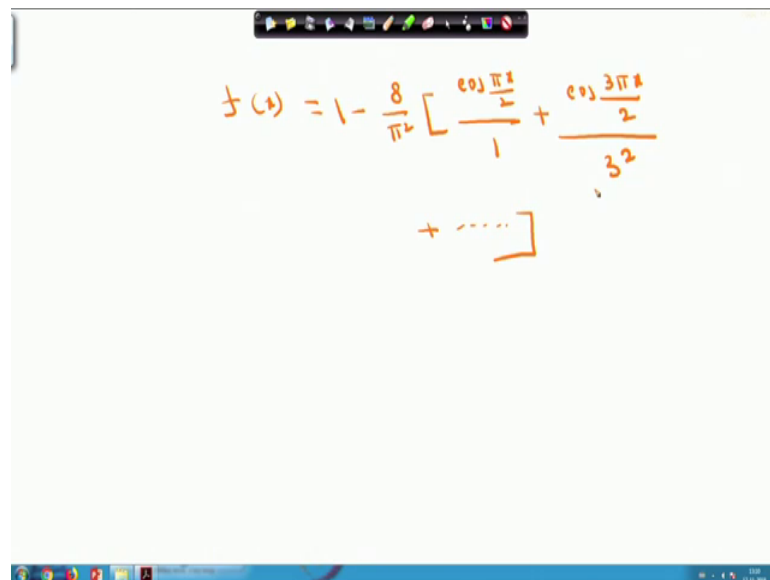
$$\begin{aligned}a_n &= \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \right]_0^2 + \left(\frac{2}{n\pi}\right)^2 \left[\cos\left(\frac{n\pi x}{2}\right) \right]_0^2 \\ &= \frac{4}{n^2\pi^2} (\cos n\pi - 1) \\ &= \frac{4[(-1)^n - 1]}{n^2\pi^2}\end{aligned}$$

Therefore, the Fourier series expansion of $f(x)$ in the interval $(0,2)$ then becomes

$$\begin{aligned}f(x) &= \frac{2}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos\left(\frac{n\pi x}{2}\right) \\ &= 1 - \frac{8}{\pi^2} \left[\frac{1}{1^2} \cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2} \cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2} \cos\left(\frac{5\pi x}{2}\right) + \dots \right]\end{aligned}$$

which represents the half range cosine series of $f(x)$.

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A screenshot of a whiteboard showing a handwritten mathematical expression for a function $f(x)$. The expression is:

$$f(x) = 1 - \frac{8}{\pi^2} \left[\frac{\cos \frac{\pi x}{2}}{1} + \frac{\cos \frac{3\pi x}{2}}{3^2} + \dots \right]$$

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Hence the Fourier Series for $f(x)$ over the full period $(-2, 2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

where $a_0 = \frac{2}{2} \int_0^2 f(x) dx = \int_0^2 x dx = 2$

and $a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$

◇



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$$\Rightarrow a_n = \frac{4}{n^2\pi^2} [(-1)^n - 1]$$
$$\therefore a_1 = -\frac{8}{\pi^2}, \quad a_2 = 0, \quad a_3 = -\frac{8}{3^2\pi^2}, \quad a_4 = 0 \text{ etc.}$$

Hence the Fourier cosine series for $f(x)$ over the half range $(0, 2)$ is

$$f(x) = 1 - \frac{8}{\pi^2} \left[\frac{\cos \frac{\pi x}{2}}{1^2} + \frac{\cos \frac{3\pi x}{2}}{3^2} + \dots \right]$$

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Example

Prove that the function $f(x) = x$ can be expanded

(i) in a series of cosines in $0 \leq x \leq \pi$ as

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

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The next problem is to prove that $f(x) = x$ can be expanded in a series of cosines in $[0, \pi]$. Since this will take some time so we will start this example in the next lecture. Thank you.