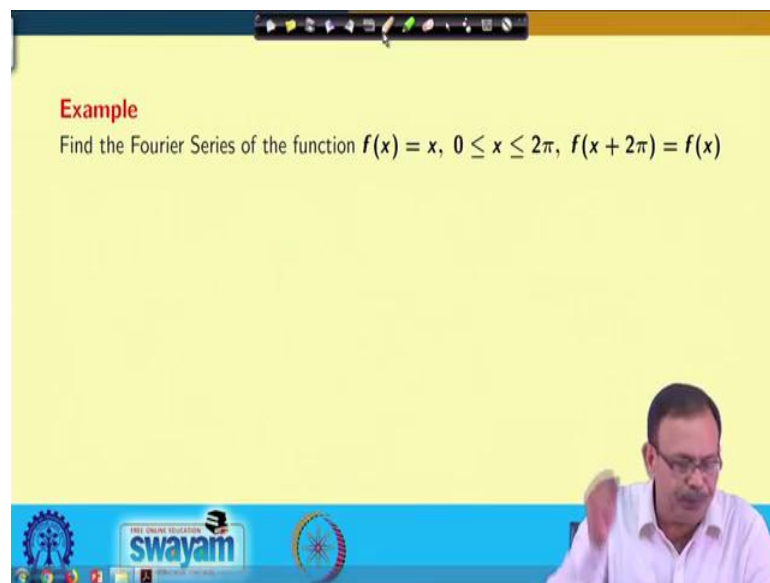


**Transform Calculus and its applications in Differential Equations**  
**Prof. Adrijit Goswami**  
**Department of Mathematics**  
**Indian Institute of Technology, Kharagpur**

**Lecture – 21**  
**Fourier Series of Functions having arbitrary period – I**

In the last lecture, we have observed how to find out the Fourier series of a function  $f(x)$ , and also how to find out the Fourier series of an even function or of an odd function. As we have seen, for the Fourier series of even function, we have only the terms of cosine and if the function  $f(x)$  is an odd function, then  $f(x)$  can be expanded in terms of sine terms only.

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The image shows a screenshot of a video lecture. The main content is a yellow slide with the following text:

**Example**  
Find the Fourier Series of the function  $f(x) = x, 0 \leq x \leq 2\pi, f(x + 2\pi) = f(x)$

At the bottom of the slide, there is a blue banner with the Swayam logo and the text "FREE ONLINE EDUCATION swayam". A small inset video of the professor is visible in the bottom right corner of the slide.

Now, let us take one example to check how we can find out the Fourier series of a function. Let us take a function  $f(x) = x, 0 \leq x \leq 2\pi$  such that  $f(x)$  is a periodic function with period  $2\pi$  that is  $f(x + 2\pi) = f(x)$ .

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The image shows handwritten mathematical derivations for the Fourier coefficients of a function  $f(x) = x$  over the interval  $[0, 2\pi]$ . The derivations are as follows:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = 2\pi$$
$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cdot \cos nx dx = 0$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin nx dx$$
$$= \frac{1}{\pi} \left\{ \left[ -\frac{x \cos nx}{n} \right]_0^{2\pi} - \int_0^{2\pi} \frac{\cos nx}{n} dx \right\}$$
$$= \frac{1}{\pi} \left[ -x \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_0^{2\pi}$$

So, here,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

and if we calculate the value, we get

$$a_0 = \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_0^{2\pi}$$
$$= 2\pi.$$

Similarly,

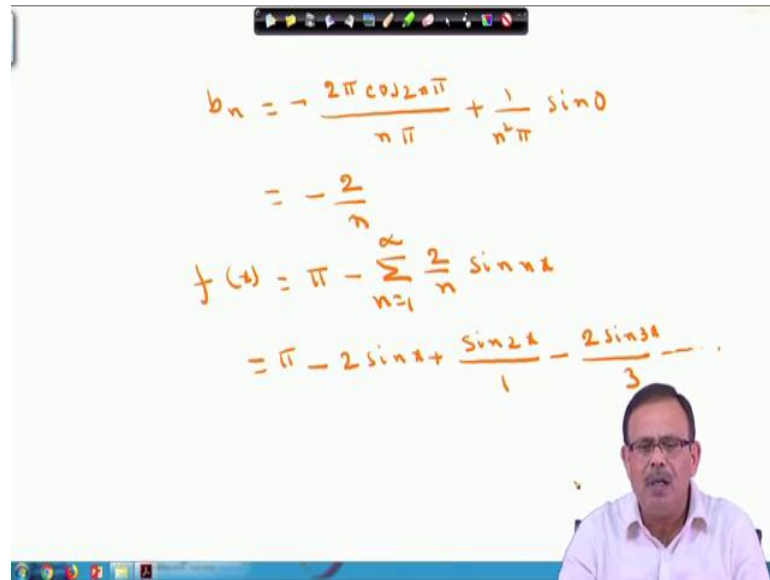
$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$
$$= \frac{1}{\pi} \left[ \frac{x}{n} \sin nx \right]_0^{2\pi} + \frac{1}{n^2 \pi} [\cos nx]_0^{2\pi}$$
$$= 0$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

$$\begin{aligned}\Rightarrow b_n &= \frac{1}{\pi} \left[ -\frac{x}{n} \cos nx \right]_0^{2\pi} + \frac{1}{n^2 \pi} [\sin nx]_0^{2\pi} \\ &= -\frac{2}{n}.\end{aligned}$$

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So that we can write

$$\begin{aligned}f(x) &= \pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin nx \\ &= \pi - 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.\end{aligned}$$

If we expand it, we will get

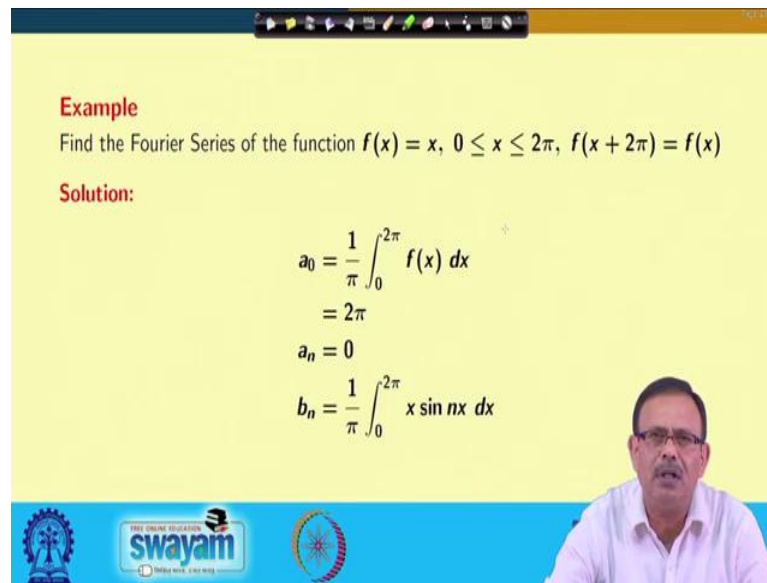
$$\begin{aligned}f(x) &= \pi - 2 \left[ \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right] \\ &= \pi - 2 \sin x - \frac{\sin 2x}{1} - \frac{2 \sin 3x}{3} - \dots\end{aligned}$$

So, once we know  $a_0$ ,  $a_n$  and  $b_n$ , we can expand the function in the form of a series, as shown. So, if a function  $f(x)$  is given to us, then we can easily find out the series for the function  $f(x)$ . Once we are representing  $f(x)$  in terms of a Fourier series, that is in terms of sine and cosine series, then at any particular point where the function is continuous, we can find out the value of the function also.

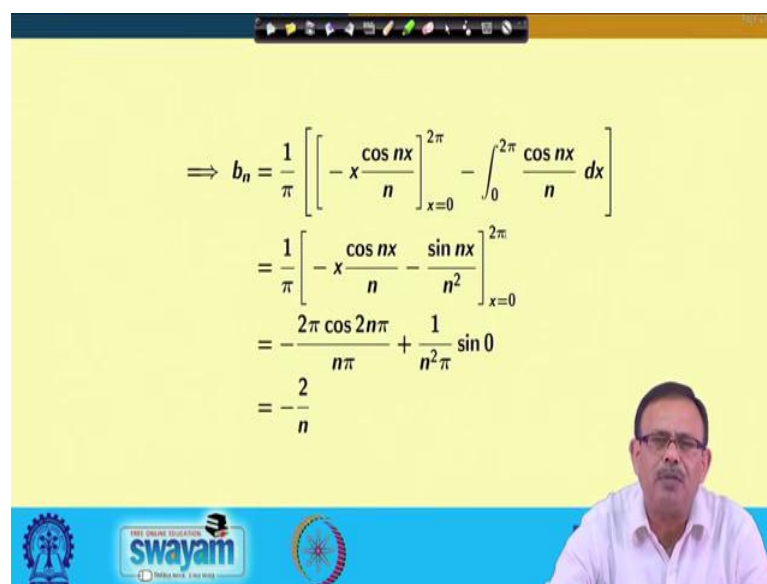
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**Example**  
Find the Fourier Series of the function  $f(x) = x$ ,  $0 \leq x \leq 2\pi$ ,  $f(x + 2\pi) = f(x)$

**Solution:**

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$
$$= 2\pi$$
$$a_n = 0$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx$$


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$$\Rightarrow b_n = \frac{1}{\pi} \left[ \left[ -x \frac{\cos nx}{n} \right]_{x=0}^{2\pi} - \int_0^{2\pi} \frac{\cos nx}{n} dx \right]$$
$$= \frac{1}{\pi} \left[ -x \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_{x=0}^{2\pi}$$
$$= -\frac{2\pi \cos 2n\pi}{n\pi} + \frac{1}{n^2\pi} \sin 0$$
$$= -\frac{2}{n}$$


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A slide from a video lecture showing a mathematical derivation. The slide has a yellow background and a blue footer with logos for 'swayam' and 'Free Online Education'. A man in a white shirt is visible in the bottom right corner. The text on the slide is:
$$\therefore f(x) = \pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$
$$= \pi - 2 \sin x - \frac{\sin 2x}{1} - \frac{2 \sin 3x}{3} - \dots$$

Now, we come to functions having arbitrary period.

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A slide from a video lecture with a yellow background and a blue footer. A man in a white shirt is visible in the bottom right corner. The text on the slide is:

**Functions having arbitrary periods**

So far we have dealt with Fourier Series expansions of functions having period  $2\pi$ .

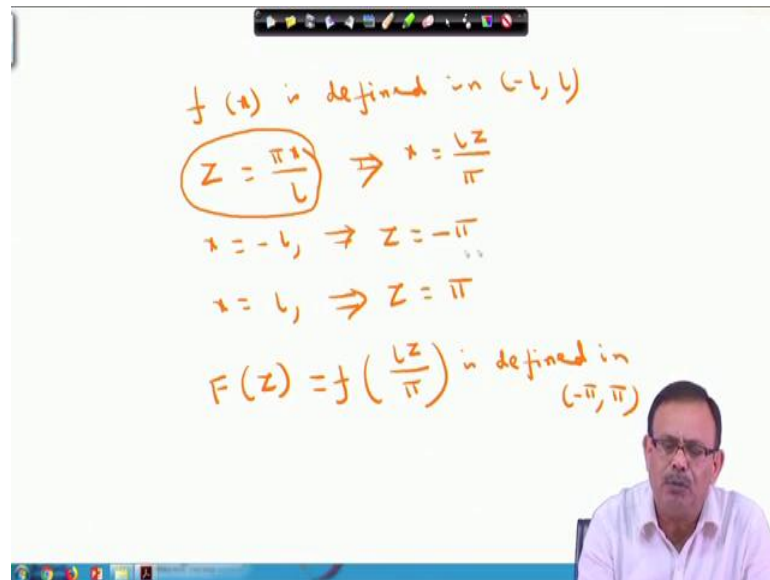
But in many of the problems, the functions may have arbitrary period (not necessarily  $2\pi$ ).

We now obtain Euler's formulae for Fourier coefficients for functions having period  $2l$  where  $l$  is any positive number.

So far we have dealt with Fourier series expansion of functions, where the function is periodic with period  $2\pi$ . But if it is of arbitrary period, then what will happen? In many practical problems and engineering problems, we have found that the function may have arbitrary period, not necessarily it will be  $2\pi$ . Or in other sense, we can tell, in general, that we can obtain Euler's formula for Fourier coefficients for the functions whose period

is  $2l$ . Earlier we have done it for period  $2\pi$ . Now, we want to check the effect of an arbitrary period i.e., a generalized one,  $2l$  say, where  $l$  is some positive number.

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So, in this case, suppose we have a function  $f(x)$  defined in  $(-l, l)$ . Since  $f(x)$  is defined in  $(-l, l)$  and it is a periodic function with period  $2l$ , so to match with the earlier things we are assuming

$$z = \frac{\pi x}{l}$$

so that from here, we can write down,

$$x = \frac{lz}{\pi}$$

So, as  $x = -l$ , we have  $z = -\pi$  and as  $x = l$ , we have  $z = \pi$ . So, basically we are making a substitution  $z = \frac{\pi x}{l}$ , and by this substitution, we are changing the interval or the range of the function from  $(-l, l)$  to  $(-\pi, \pi)$ . And we know the formulas for the range  $(-\pi, \pi)$  already. So, now, we can create a new function  $F(z)$  which we can define as  $F(z) = f\left(\frac{lz}{\pi}\right)$  defined in  $(-\pi, \pi)$ .

Once it is defined in  $(-\pi, \pi)$ , so we can expand  $F(z)$  in terms of Fourier series because we know the Fourier series expansion of a function  $f(x)$  in  $(-\pi, \pi)$ .

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Handwritten notes on a whiteboard showing the Fourier series expansion of  $F(z)$ . The notes include the general form  $F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$ , and the formulas for the coefficients:  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) dz$ ,  $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \cos nz dz$ , and  $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \sin nz dz$ .

$$\therefore F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) dz$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \cos nz dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \sin nz dz.$$

Now, we can replace  $F(z)$  by  $f\left(\frac{lz}{\pi}\right)$ .

$$\therefore f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) dz$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \cos nz dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \sin nz dz.$$

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Handwritten mathematical derivation on a whiteboard:

$$f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) dz$$

Substitution:  $x = \frac{lz}{\pi}$ ,  $dx = \frac{l}{\pi} dz$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \cos nz dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \sin nz dz$$

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Handwritten mathematical derivation on a whiteboard:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right]$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

Substitution:  $(c, c+2l)$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

A small video inset of a man speaking is visible in the bottom right corner of the slide.

We have,



$$x = \frac{lz}{\pi}.$$

So, by using this transformation and replacing  $dz$  by  $\frac{\pi}{l} dx$ , we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

where

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx.$$

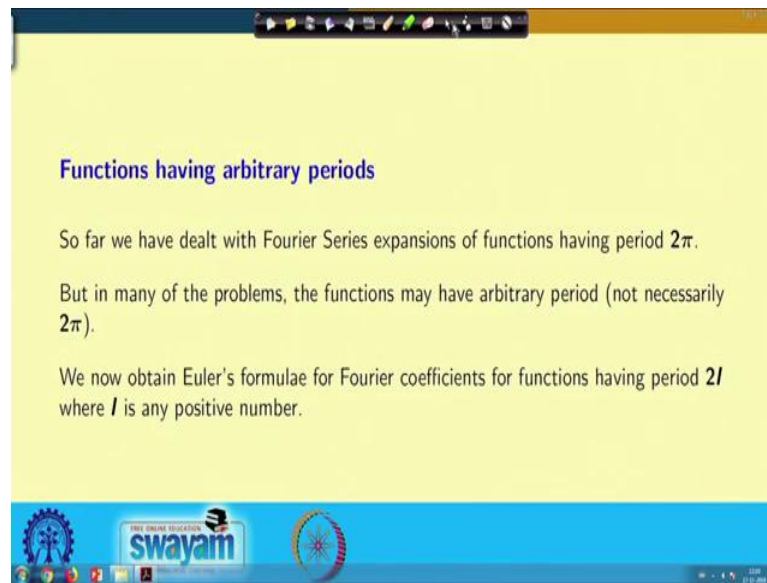
Please note that the earlier Fourier series whatever we have defined, that was for  $(-\pi, \pi)$ .

So, now, if a function  $f(x)$  is defined in  $(-l, l)$  i.e., for any arbitrary period  $2l$ , then also  $f(x)$  can be expanded in terms of Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right).$$

Therefore, this formula will be true whenever we consider any function  $f(x)$  which is defined in  $(c, c + 2l)$ , so that whenever  $c = -l$ , the range will be  $(-l, l)$ . We can make it  $(0, 2l)$  by making  $c = 0$ .

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**Functions having arbitrary periods**

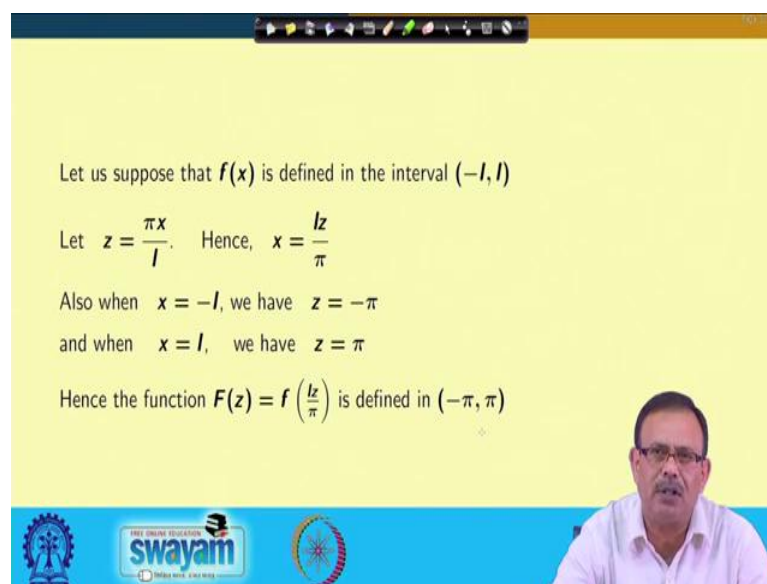
So far we have dealt with Fourier Series expansions of functions having period  $2\pi$ .

But in many of the problems, the functions may have arbitrary period (not necessarily  $2\pi$ ).

We now obtain Euler's formulae for Fourier coefficients for functions having period  $2l$  where  $l$  is any positive number.

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Let us suppose that  $f(x)$  is defined in the interval  $(-l, l)$

Let  $z = \frac{\pi x}{l}$ . Hence,  $x = \frac{lz}{\pi}$

Also when  $x = -l$ , we have  $z = -\pi$   
and when  $x = l$ , we have  $z = \pi$

Hence the function  $F(z) = f\left(\frac{lz}{\pi}\right)$  is defined in  $(-\pi, \pi)$

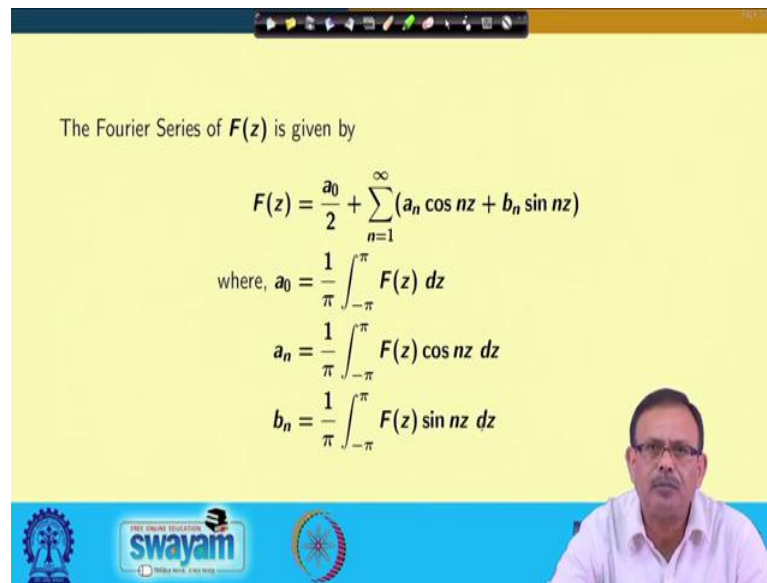
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The Fourier Series of  $F(z)$  is given by

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

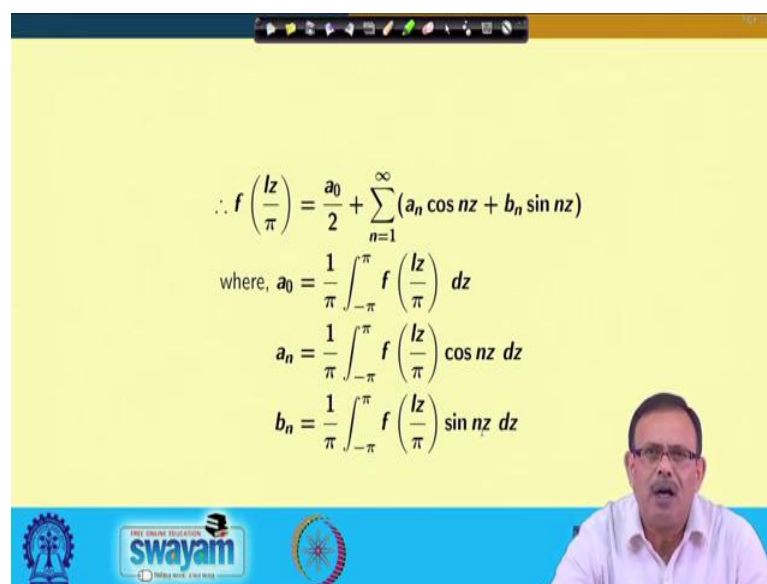
where,  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) dz$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \cos nz dz$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \sin nz dz$$


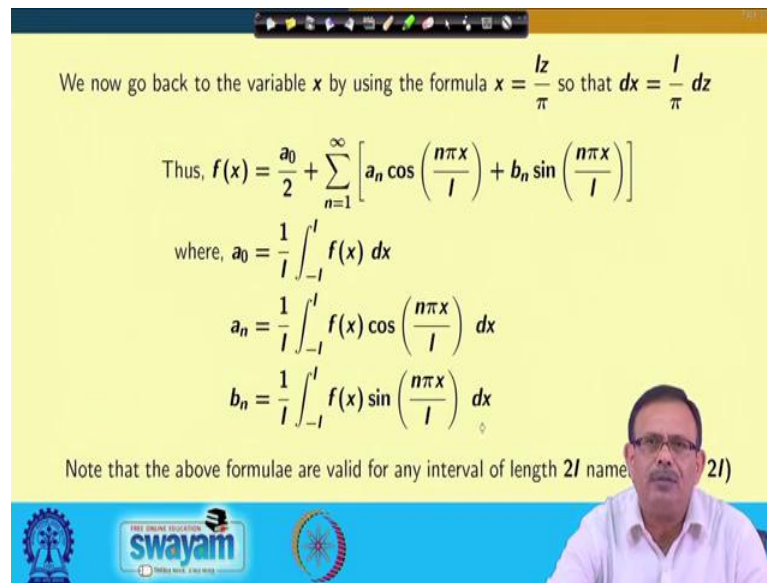
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$$\therefore f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

where,  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) dz$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \cos nz dz$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \sin nz dz$$


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We now go back to the variable  $x$  by using the formula  $x = \frac{lz}{\pi}$  so that  $dx = \frac{l}{\pi} dz$

$$\text{Thus, } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

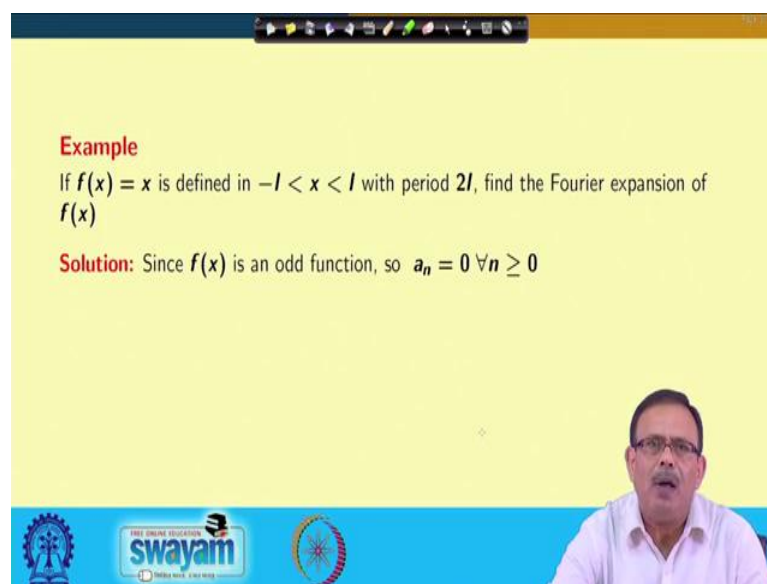
where,  $a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$
$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Note that the above formulae are valid for any interval of length  $2l$  name  $(2l)$

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**Example**  
If  $f(x) = x$  is defined in  $-l < x < l$  with period  $2l$ , find the Fourier expansion of  $f(x)$

**Solution:** Since  $f(x)$  is an odd function, so  $a_n = 0 \forall n \geq 0$

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Let us now see one example. Suppose we have a function  $f(x) = x$  which is defined in the range  $(-l, l)$  with a period  $2l$ . We want to find out the Fourier expansion of this function.

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$$\begin{aligned} f(x) &= x \text{ odd func.} \\ a_n &= 0 \quad + n \geq 0 \\ b_n &= \frac{2}{l} \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{l} \left[ -\frac{lx}{n\pi} \cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right) \right]_0^l \end{aligned}$$

Basically  $f(x)$  is nothing but an odd function. Therefore, directly we can tell

$$a_n = 0, \quad \forall n \geq 0.$$

But  $f(x) \sin \frac{n\pi x}{l}$  is an even function so that

$$\begin{aligned} b_n &= \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \\ &= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} dx \\ &= \frac{2}{l} \left[ -\frac{lx}{n\pi} \cos \frac{n\pi x}{l} \right]_0^l + \frac{2}{l} \frac{l^2}{n^2\pi^2} \left[ \sin \frac{n\pi x}{l} \right]_0^l \\ &= -\frac{2l}{n\pi} \cos n\pi \\ &= -\frac{2l}{n\pi} (-1)^n \\ &= \frac{2l}{n\pi} (-1)^{n+1}. \end{aligned}$$

(Refer Slide Time: 19:32)

The image shows a whiteboard with handwritten mathematical derivations. The first part shows the calculation of the coefficient  $b_n$ :

$$b_n = \frac{2}{l} \left[ -\frac{l^2 \cos n\pi}{n\pi} \right]$$
$$= -\frac{2l (-1)^n}{n\pi} = \frac{2(-1)^{n+1} \cdot l}{n\pi}$$

The second part shows the final Fourier series expansion for  $f(x)$ :

$$f(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n+1} \cdot l}{n} \sin\left(\frac{n\pi x}{l}\right) \right]$$

A small inset video of a man in a white shirt is visible in the bottom right corner of the slide.

Therefore, the Fourier series expansion of  $f(x)$  can be expressed as

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$
$$= \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}.$$

(Refer Slide Time: 21:05)

**Example**  
If  $f(x) = x$  is defined in  $-l < x < l$  with period  $2l$ , find the Fourier expansion of  $f(x)$

**Solution:** Since  $f(x)$  is an odd function, so  $a_n = 0 \forall n \geq 0$

$$b_n = \frac{2}{l} \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$$
$$= \frac{2}{l} \left[ -\frac{lx}{n\pi} \cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right) \right]_{x=0}^l$$

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$$\begin{aligned}\Rightarrow b_n &= \frac{2}{l} \left[ -\frac{l^2 \cos n\pi}{n\pi} \right] \\ &= -\frac{2l(-1)^n}{n\pi} \\ &= \frac{2(-1)^{n+1}l}{n\pi}\end{aligned}$$

$\therefore$  The Fourier Series is  $x = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n+1}l}{n} \sin \left( \frac{n\pi x}{l} \right) \right]$

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**Example**  
Find a Fourier Series to represent a periodic function  $x^2$  in the interval  $(-l, l)$

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Let us now move to the next example. In the earlier one, we wanted to find out the Fourier series expansion of the function  $f(x)$  when  $f(x) = x$ . Now, in this case, we are provided with  $f(x) = x^2$  defined in the interval  $(-l, l)$ .

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$$\begin{aligned} f(x) &= x^2 \text{ even} \\ f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \\ a_0 &= \frac{2}{l} \int_0^l x^2 dx = \frac{2}{l} \left[ \frac{x^3}{3} \right]_0^l = \frac{2l^2}{3} \\ a_n &= \int_0^l x^2 \cos\left(\frac{n\pi x}{l}\right) dx \end{aligned}$$

Clearly, since  $f(x) = x^2$  is an even function, so there will be only cosine terms in the Fourier Series expansion and  $b_n = 0$  for all  $n$ . So,  $f(x)$  can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where

$$\begin{aligned} a_0 &= \frac{1}{l} \int_{-l}^l f(x) dx \\ &= \frac{2}{l} \int_0^l x^2 dx \\ &= \frac{2}{l} \left[ \frac{x^3}{3} \right]_0^l \\ &= \frac{2l^2}{3} \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx \\ &= \frac{2}{l} \int_0^l x^2 \cos \frac{n\pi x}{l} dx \end{aligned}$$



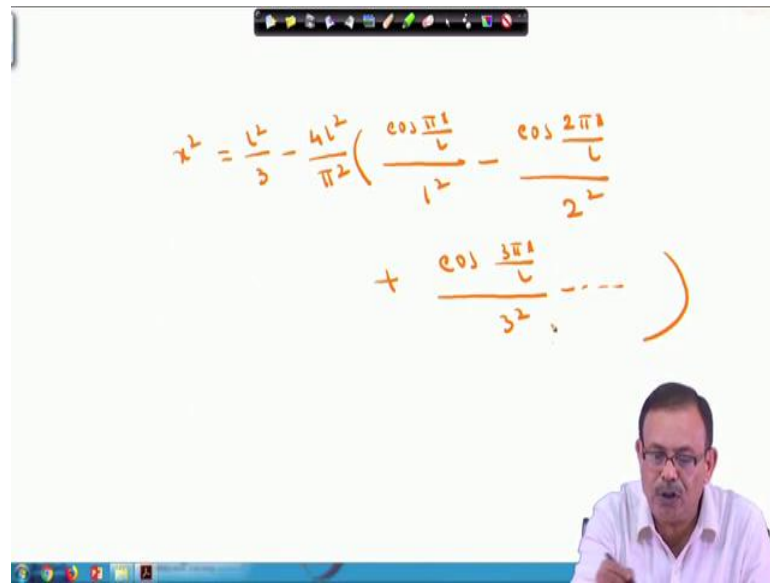
$$\begin{aligned}
\Rightarrow a_n &= \frac{2}{l} \left[ \frac{l}{n\pi} x^2 \sin \frac{n\pi x}{l} \right]_0^l - \frac{4}{l} \frac{l}{n\pi} \int_0^l x \sin \frac{n\pi x}{l} dx \\
&= -\frac{4}{n\pi} \left[ -\frac{lx}{n\pi} \cos \frac{n\pi x}{l} \right]_0^l + \frac{4}{n\pi} \frac{l^2}{n^2\pi^2} \left[ \sin \frac{n\pi x}{l} \right]_0^l \\
&= \frac{4l^2}{n^2\pi^2} \cos n\pi \\
&= \frac{4l^2}{n^2\pi^2} (-1)^n.
\end{aligned}$$

Therefore, for  $n = 1$ , we have  $a_1 = -\frac{4l^2}{\pi^2}$ , for  $n = 2$ , we have  $a_2 = \frac{4l^2}{2^2\pi^2}$ , for  $n = 3$ , we have  $a_3 = -\frac{4l^2}{3^2\pi^2}$  and so on.

(Refer Slide Time: 23:26)

$$\begin{aligned}
a_n &= \frac{2}{l} \left[ \frac{x^2 \sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} - 2x \left( -\frac{\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right. \\
&\quad \left. + 2 \left( -\frac{\sin \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) \right]_0^l \\
&= \frac{4l^2(-1)^n}{n^2\pi^2} \\
a_1 &= -\frac{4l^2}{\pi^2}, a_2 = \frac{4l^2}{2^2\pi^2}, a_3 = -\frac{4l^2}{3^2\pi^2}
\end{aligned}$$

(Refer Slide Time: 25:14)



The whiteboard contains the following handwritten equations:

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2} \left( \frac{\cos \frac{\pi x}{l}}{1^2} - \frac{\cos \frac{2\pi x}{l}}{2^2} + \frac{\cos \frac{3\pi x}{l}}{3^2} - \dots \right)$$

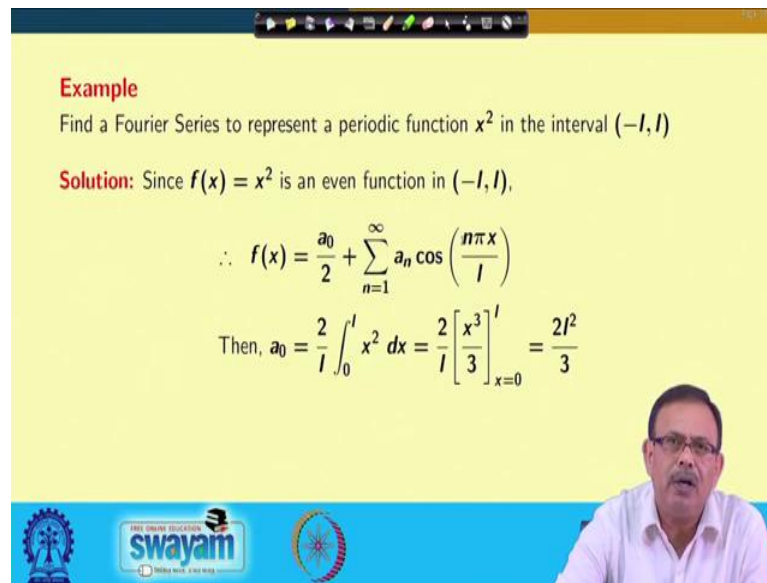
So we can write down the Fourier Series expansion of  $f(x)$  as

$$\begin{aligned} x^2 &= \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} (-1)^n \cos \frac{n\pi x}{l} \\ &= \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{l} \\ &= \frac{l^2}{3} - \frac{4l^2}{\pi^2} \left[ \frac{\cos \frac{\pi x}{l}}{1^2} - \frac{\cos \frac{2\pi x}{l}}{2^2} + \frac{\cos \frac{3\pi x}{l}}{3^2} - \dots \right]. \end{aligned}$$

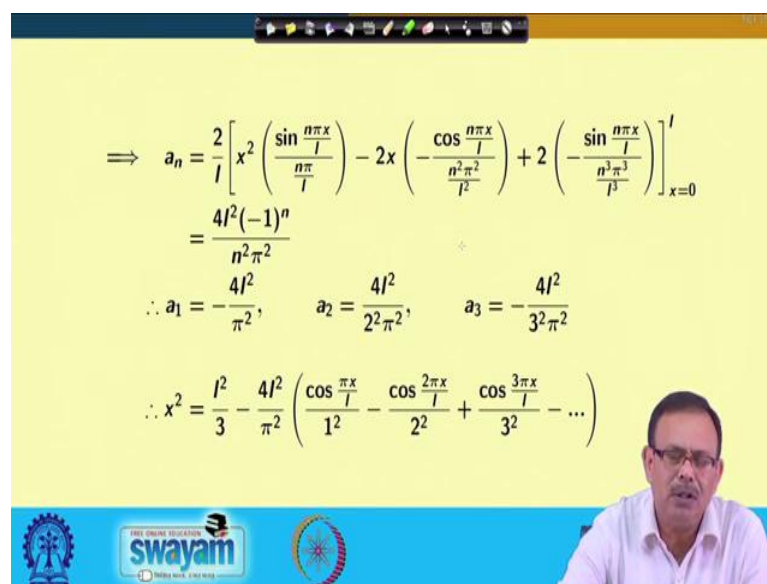
(Refer Slide Time: 26:03)

**Example**  
Find a Fourier Series to represent a periodic function  $x^2$  in the interval  $(-l, l)$

**Solution:** Since  $f(x) = x^2$  is an even function in  $(-l, l)$ ,

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$
$$\text{Then, } a_0 = \frac{2}{l} \int_0^l x^2 dx = \frac{2}{l} \left[ \frac{x^3}{3} \right]_{x=0}^l = \frac{2l^2}{3}$$


(Refer Slide Time: 26:17)

$$\Rightarrow a_n = \frac{2}{l} \left[ x^2 \left( \frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - 2x \left( -\frac{\cos \frac{n\pi x}{l}}{n^2 \pi^2} \right) + 2 \left( -\frac{\sin \frac{n\pi x}{l}}{n^3 \pi^3} \right) \right]_{x=0}^l$$
$$= \frac{4l^2 (-1)^n}{n^2 \pi^2}$$
$$\therefore a_1 = -\frac{4l^2}{\pi^2}, \quad a_2 = \frac{4l^2}{2^2 \pi^2}, \quad a_3 = -\frac{4l^2}{3^2 \pi^2}$$
$$\therefore x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2} \left( \frac{\cos \frac{\pi x}{l}}{1^2} - \frac{\cos \frac{2\pi x}{l}}{2^2} + \frac{\cos \frac{3\pi x}{l}}{3^2} - \dots \right)$$


(Refer Slide Time: 26:33)

**Example**  
Obtain the Fourier Series expansion of  $f(x)$  with period 2 defined as

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$$

The slide also features a video feed of a lecturer in the bottom right corner and logos for Swamyam and other educational institutions at the bottom.

In the next problem, a function  $f(x)$  is given with period 2, where  $f(x)$  is defined by

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$$

(Refer Slide Time: 26:49)

Handwritten derivation for the Fourier series coefficients of the function  $f(x)$  with period 2:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$
$$a_0 = \int_0^2 f(x) dx = \int_0^1 1 \cdot dx + \int_1^2 2 \cdot dx = 3$$
$$a_n = \int_0^2 f(x) \cos n\pi x dx = \int_0^1 \cos n\pi x dx + \int_1^2 2 \cos n\pi x dx = 0$$

The handwritten work includes a circled 'U=1' and a range '0 ≤ x ≤ 2' at the top left.

Looking at  $f(x)$ , we are not able to tell whether it is an even or an odd function, so that  $f(x)$  can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x) \quad [\because l = 1]$$

where

$$\begin{aligned} a_0 &= \frac{2}{2} \int_0^2 f(x) dx \\ &= \int_0^1 1 \cdot dx + \int_1^2 2 \cdot dx \\ &= 3 \end{aligned}$$

$$\begin{aligned} a_n &= \int_0^2 f(x) \cos n\pi x dx \\ &= \int_0^1 \cos n\pi x dx + \int_1^2 2 \cos n\pi x dx \\ &= \left[ \frac{1}{n\pi} \sin n\pi x \right]_0^1 + 2 \left[ \frac{1}{n\pi} \sin n\pi x \right]_1^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \int_0^2 f(x) \sin n\pi x dx \\ &= \int_0^1 \sin n\pi x dx + \int_1^2 2 \sin n\pi x dx \\ &= \left[ -\frac{1}{n\pi} \cos n\pi x \right]_0^1 + 2 \left[ -\frac{1}{n\pi} \cos n\pi x \right]_1^2 \\ &= \frac{\cos n\pi - 1}{n\pi} \\ &= \frac{(-1)^n - 1}{n\pi} \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{2}{n\pi}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

(Refer Slide Time: 28:33)

The image shows a whiteboard with handwritten mathematical steps for finding the Fourier coefficient  $b_n$ . The steps are as follows:

$$b_n = \int_0^2 f(x) \sin n\pi x dx = \int_0^1 \sin n\pi x dx + \int_1^2 2 \sin n\pi x dx$$
$$= \left[ -\frac{\cos n\pi x}{n\pi} \right]_0^1 + 2 \left[ -\frac{\cos n\pi x}{n\pi} \right]_1^2$$
$$= \frac{1}{n\pi} [(-1)^n - 1]$$

Then, the final result is given as:

$$b_n = 0, \quad n \text{ - even}$$
$$= -\frac{2}{n\pi}, \quad n \text{ - odd}$$

(Refer Slide Time: 30:05)

The image shows a whiteboard with handwritten mathematical steps for the Fourier series expansion of  $f(x)$ . The steps are as follows:

$$f(x) = \frac{3}{2} - \frac{2}{\pi} \left[ \frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \dots \right]$$

In the bottom right corner of the whiteboard, there is a small video inset showing a man in a white shirt and glasses.

So, we can now write down the Fourier Series expansion of  $f(x)$  as

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} b_n \sin n\pi x$$
$$= \frac{3}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} \sin n\pi x$$
$$= \frac{3}{2} - \frac{2}{\pi} \left[ \frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \dots \right].$$

Therefore, if a function  $f(x)$  is defined either in  $(-\pi, \pi)$  or in  $(-l, l)$ ; whether it is an odd function or an even function, we can find out the Fourier series expansion of  $f(x)$  in a similar fashion. Thank you.