Transform Calculus and its applications in Differential Equations
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Lecture – 21
Fourier Series of Functions having arbitrary period – I

In the last lecture, we have observed how to find out the Fourier series of a function $f(x)$, and also how to find out the Fourier series of an even function or of an odd function. As we have seen, for the Fourier series of even function, we have only the terms of cosine and if the function $f(x)$ is an odd function, then $f(x)$ can be expanded in terms of sine terms only.

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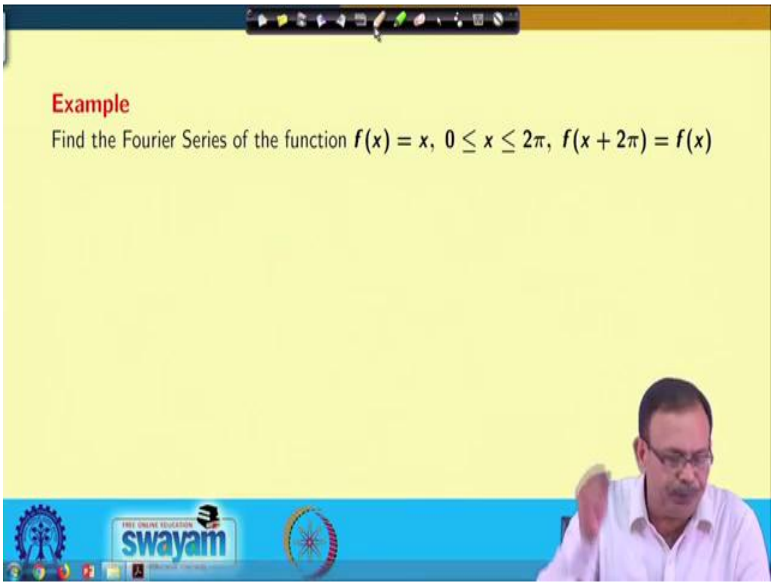

Now, let us take one example to check how we can find out the Fourier series of a function. Let us take a function $f(x) = x,\ 0 \leq x \leq 2\pi$ such that $f(x)$ is a periodic function with period 2π that is $f(x+2\pi) = f(x)$.

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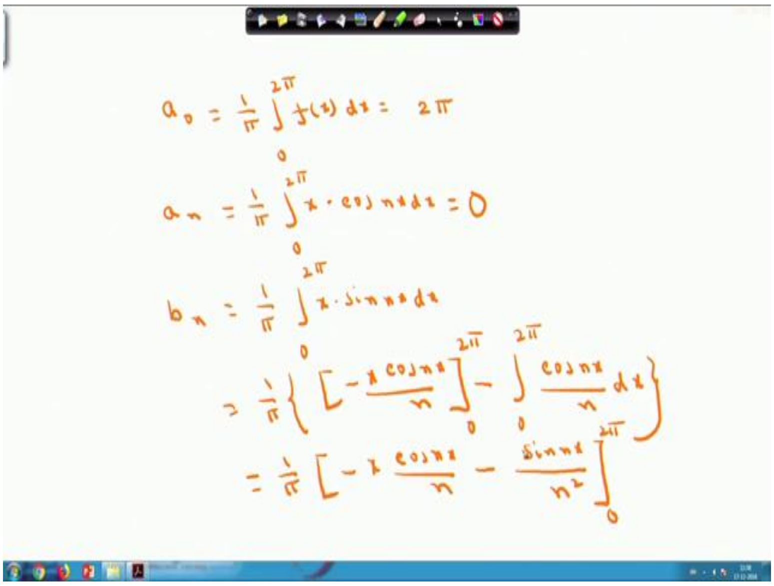

So, here,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

and if we calculate the value, we get

$$a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$
$$= 2\pi.$$

Similarly,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$
$$= \frac{1}{\pi}\left[\frac{x}{n}\sin nx\right]_0^{2\pi} + \frac{1}{n^2\pi}[\cos nx]_0^{2\pi}$$
$$= 0$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\right]_0^{2\pi} + \frac{1}{n^2\pi}[\sin nx]_0^{2\pi}$$

$$= -\frac{2}{n}.$$

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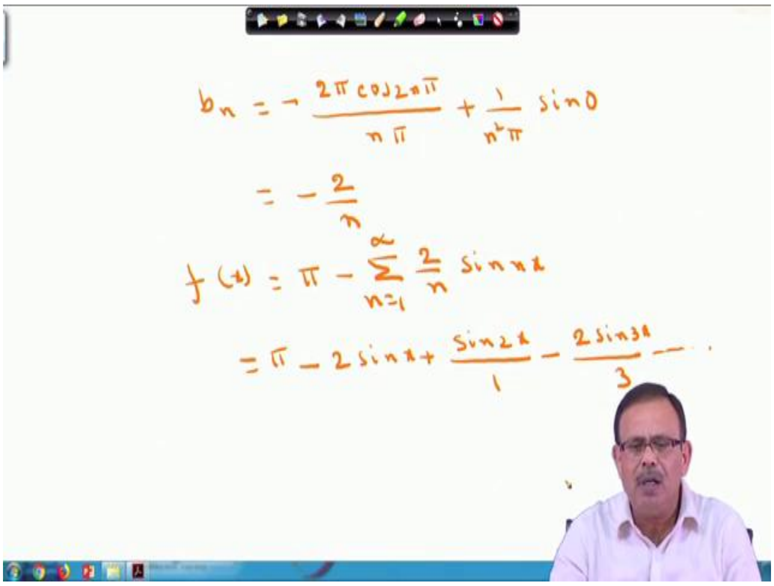

So that we can write

$$f(x) = \pi - \sum_{n=1}^{\infty} \frac{2}{n}\sin nx$$

$$= \pi - 2\sum_{n=1}^{\infty} \frac{1}{n}\sin nx.$$

If we expand it, we will get

$$f(x) = \pi - 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

$$= \pi - 2\sin x - \frac{\sin 2x}{1} - \frac{2\sin 3x}{3} - \cdots$$

So, once we know a_0, a_n and b_n, we can expand the function in the form of a series, as shown. So, if a function $f(x)$ is given to us, then we can easily find out the series for the function $f(x)$. Once we are representing $f(x)$ in terms of a Fourier series, that is in terms of sine and cosine series, then at any particular point where the function is continuous, we can find out the value of the function also.

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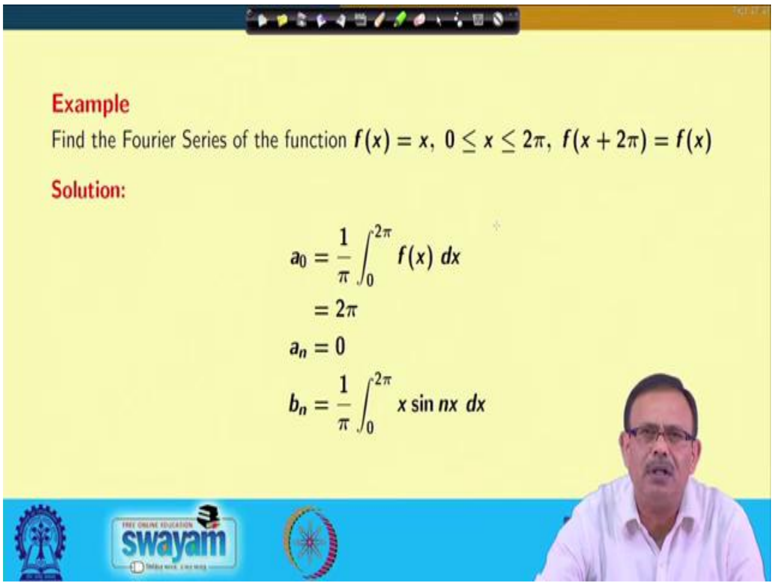

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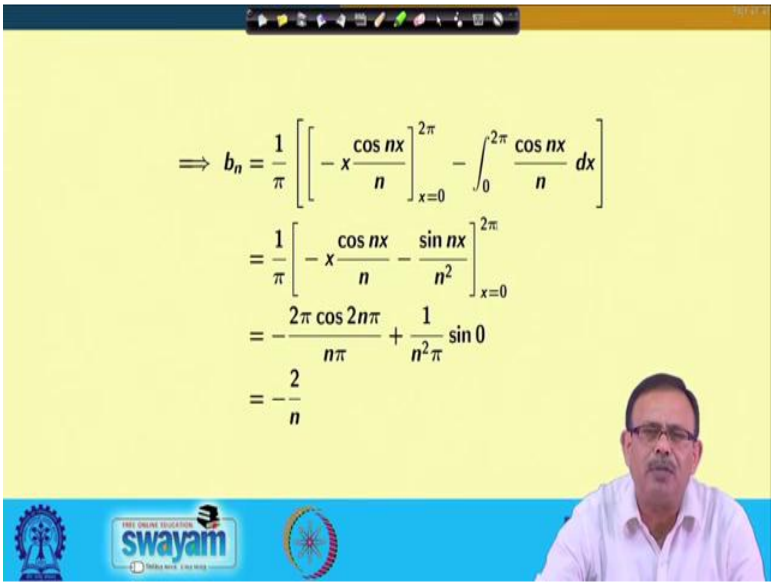

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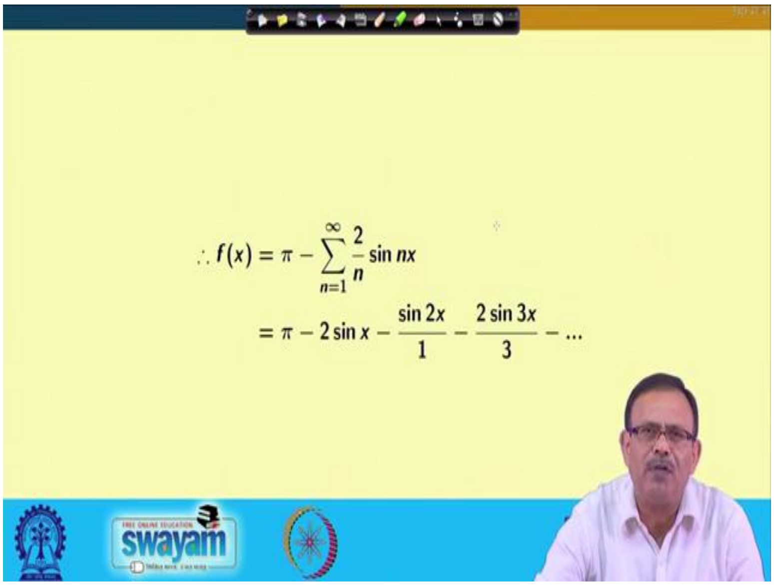

Now, we come to functions having arbitrary period.

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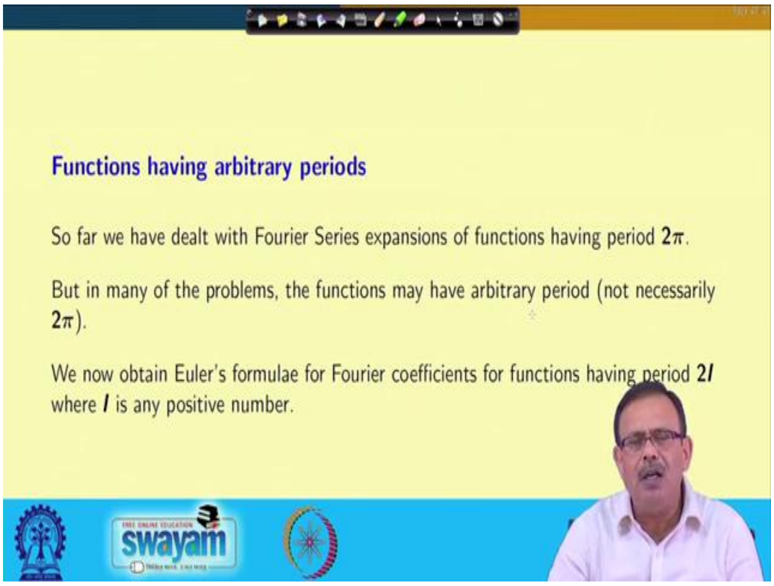

So far we have dealt with Fourier series expansion of functions, where the function is periodic with period 2π. But if it is of arbitrary period, then what will happen? In many practical problems and engineering problems, we have found that the function may have arbitrary period, not necessarily it will be 2π. Or in other sense, we can tell, in general, that we can obtain Euler's formula for Fourier coefficients for the functions whose period is $2l$. Earlier we have done it for period 2π. Now, we want to check the effect of an arbitrary period i.e., a generalized one, $2l$ say, where l is some positive number.

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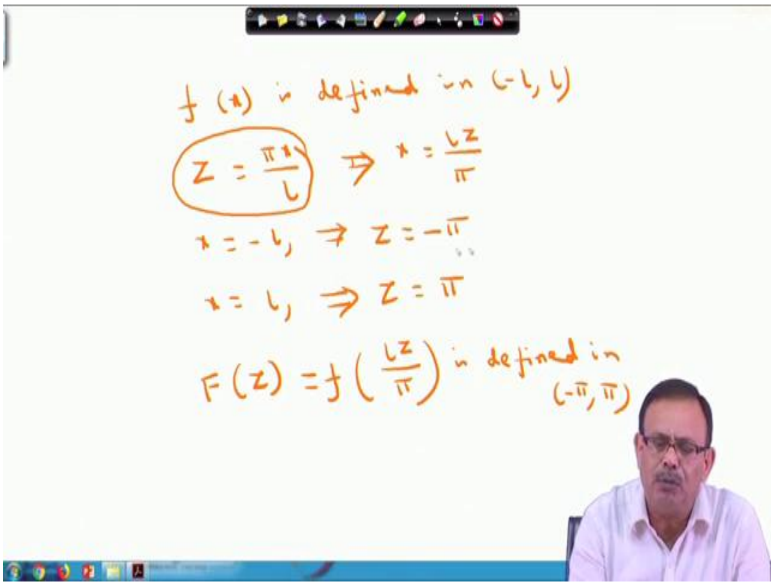

So, in this case, suppose we have a function $f(x)$ defined in $(-l, l)$. Since $f(x)$ is defined in $(-l, l)$ and it is a periodic function with period $2l$, so to match with the earlier things we are assuming

$$z = \frac{\pi x}{l}$$

so that from here, we can write down,

$$x = \frac{lz}{\pi}$$

So, as $x = -l$, we have $z = -\pi$ and as $x = l$, we have $z = \pi$. So, basically we are making a substitution $z = \frac{\pi x}{l}$, and by this substitution, we are changing the interval or the range of the function from $(-l, l)$ to $(-\pi, \pi)$. And we know the formulas for the range $(-\pi, \pi)$ already. So, now, we can create a new function $F(z)$ which we can define as $F(z) = f\left(\frac{lz}{\pi}\right)$ defined in $(-\pi, \pi)$.

Once it is defined in $(-\pi, \pi)$, so we can expand $F(z)$ in terms of Fourier series because we know the Fourier series expansion of a function $f(x)$ in $(-\pi, \pi)$.

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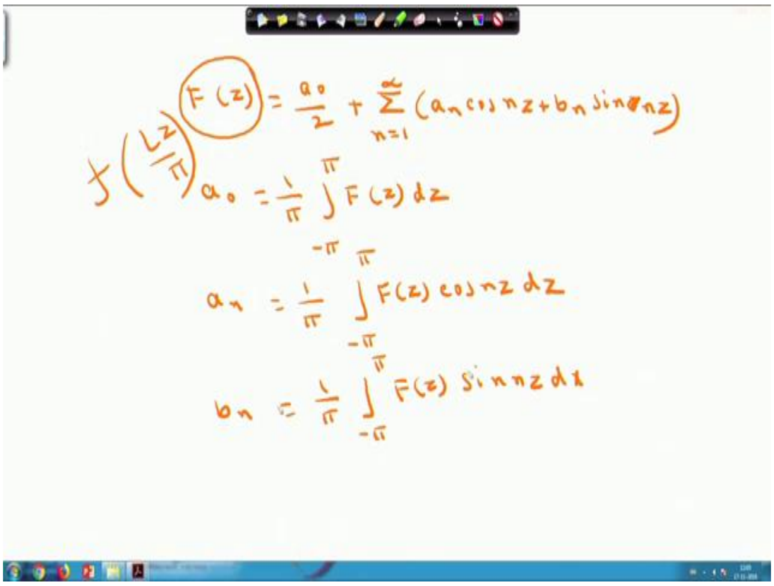

$$\therefore F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z)\, dz$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \cos nz\, dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \sin nz\, dz.$$

Now, we can replace $F(z)$ by $f\left(\frac{lz}{\pi}\right)$.

$$\therefore f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) dz$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \cos nz\, dz$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \sin nz\, dz.$$

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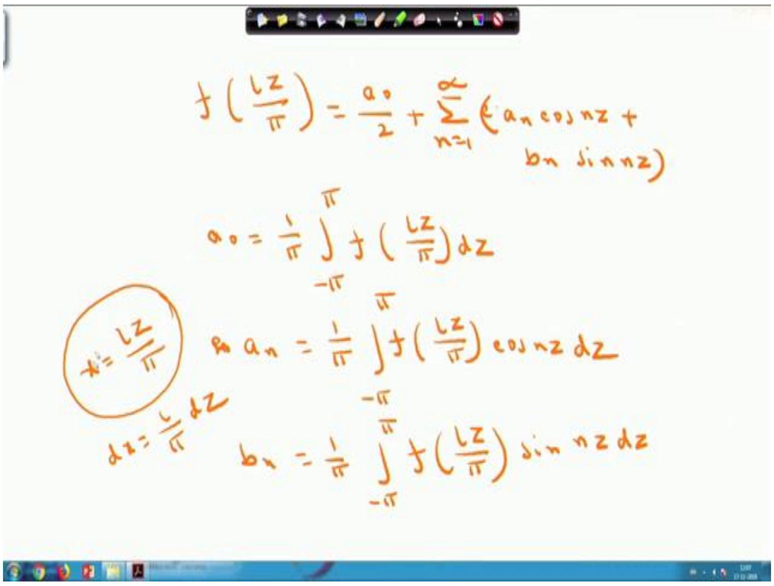

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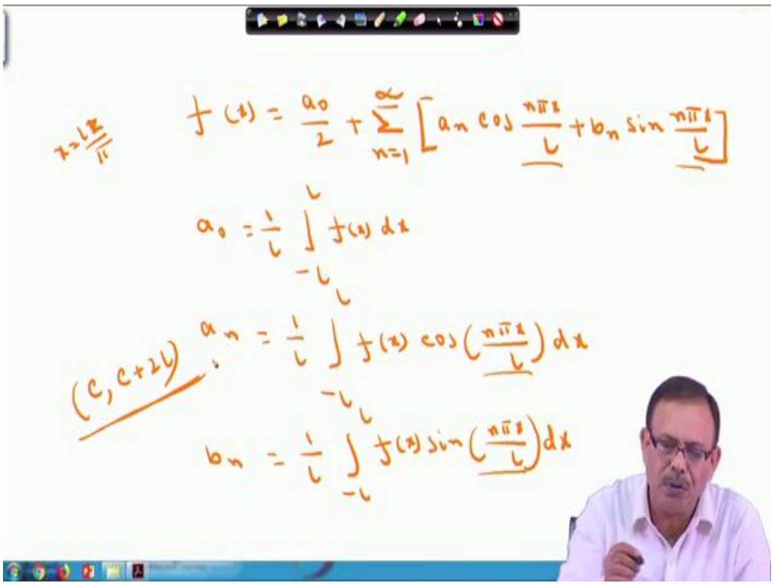

We have,

$$x = \frac{lz}{\pi}.$$

So, by using this transformation and replacing dz by $\frac{\pi}{l}dx$, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Please note that the earlier Fourier series whatever we have defined, that was for $(-\pi, \pi)$.

So, now, if a function $f(x)$ is defined in $(-l, l)$ i.e., for any arbitrary period $2l$, then also $f(x)$ can be expanded in terms of Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

Therefore, this formula will be true whenever we consider any function $f(x)$ which is defined in $(c, c + 2l)$, so that whenever $c = -l$, the range will be $(-l, l)$. We can make it $(0,2l)$ by making $c = 0$.

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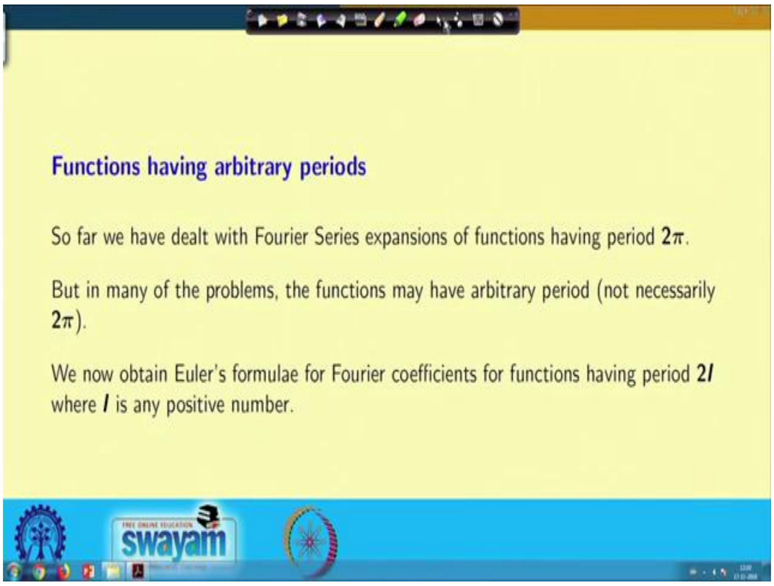

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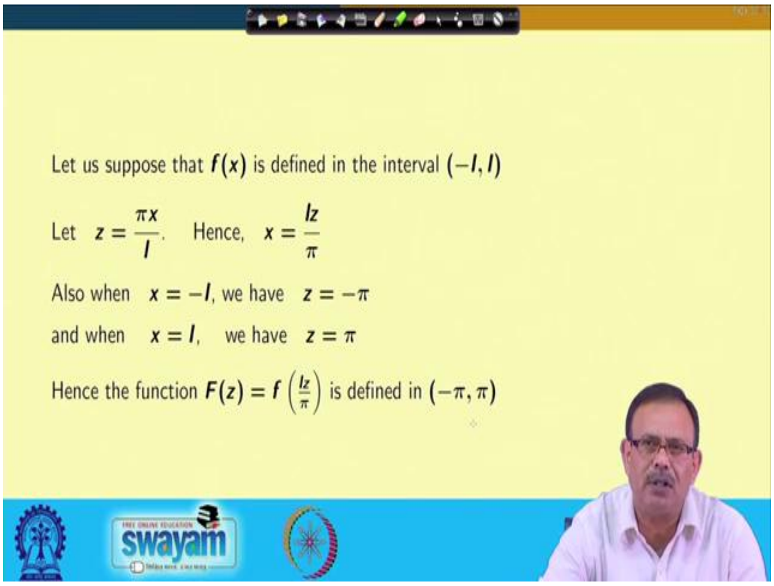

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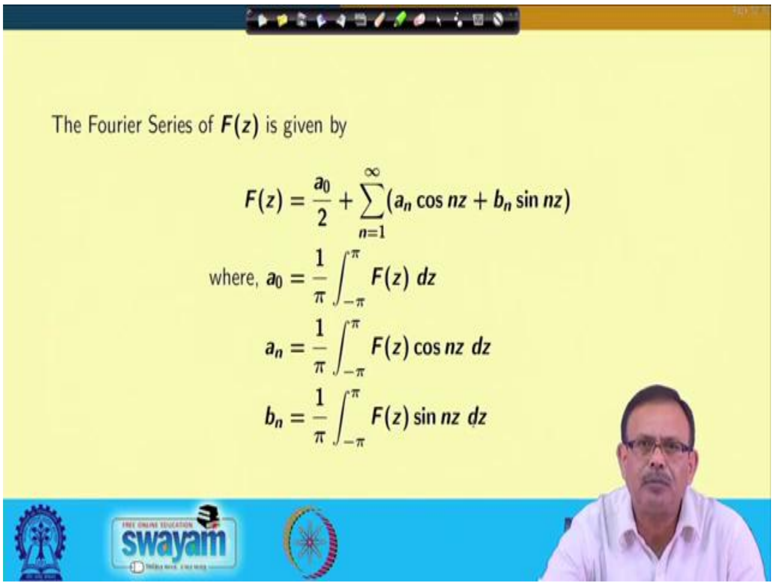

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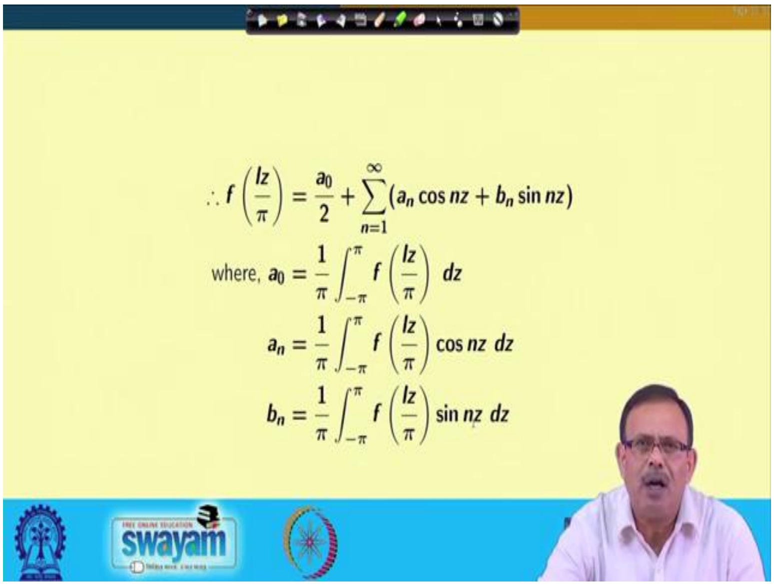

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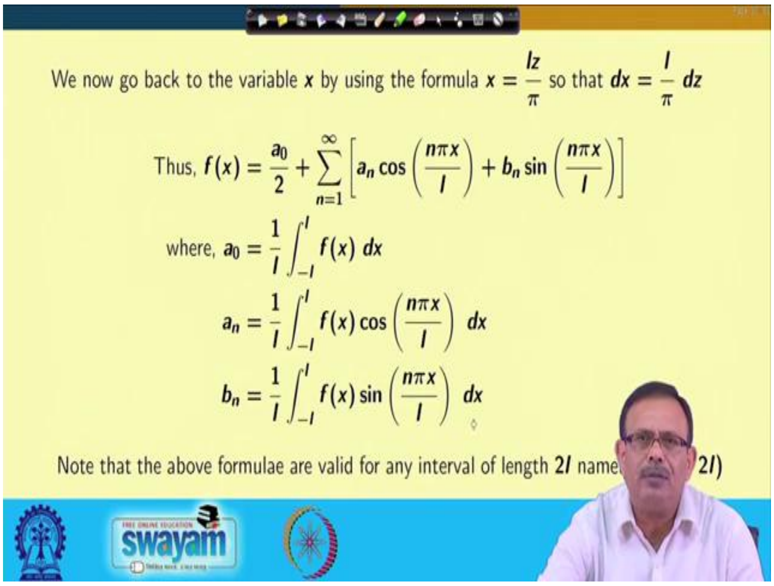

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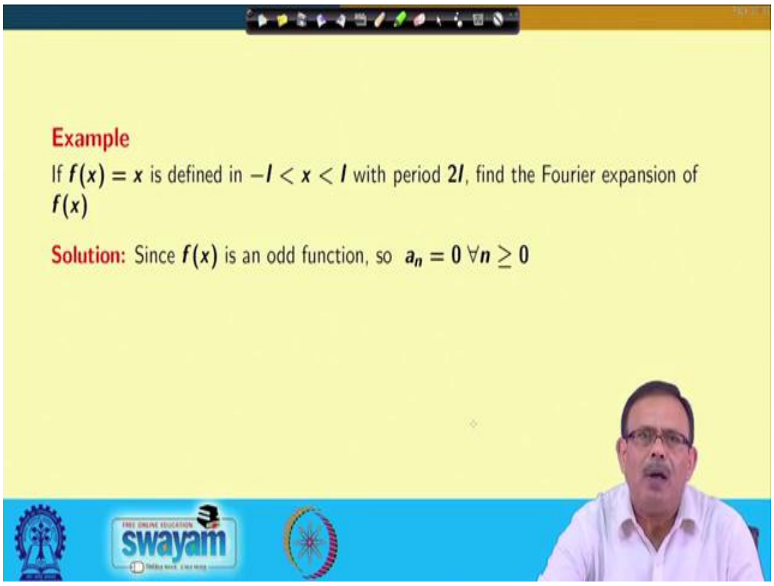

Let us now see one example. Suppose we have a function $f(x) = x$ which is defined in the range $(-l, l)$ with a period $2l$. We want to find out the Fourier expansion of this function.

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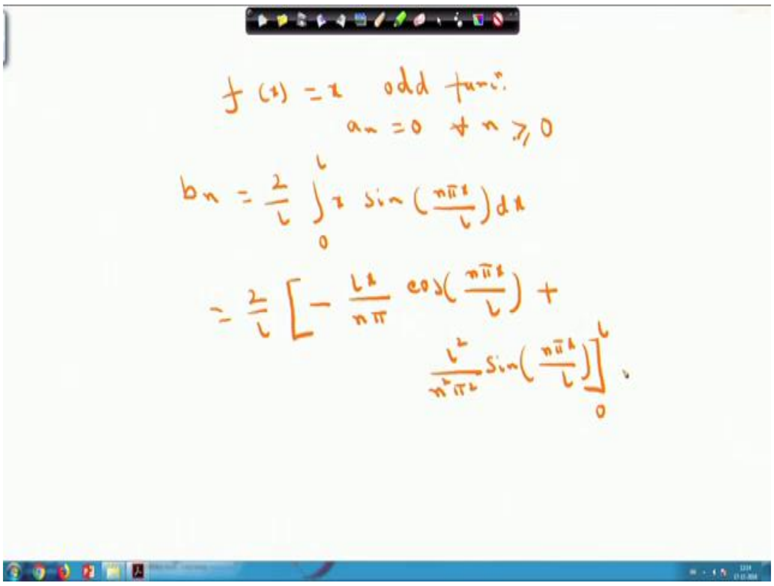

Basically $f(x)$ is nothing but an odd function. Therefore, directly we can tell

$$a_n = 0, \quad \forall n \geq 0.$$

But $f(x)\sin\frac{n\pi x}{l}$ is an even function so that

$$\begin{aligned}
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2}{l}\int_{0}^{l} x\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2}{l}\left[-\frac{lx}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l + \frac{2}{l}\frac{l^2}{n^2\pi^2}\left[\sin\frac{n\pi x}{l}\right]_0^l \\
&= -\frac{2l}{n\pi}\cos n\pi \\
&= -\frac{2l}{n\pi}(-1)^n \\
&= \frac{2l}{n\pi}(-1)^{n+1}.
\end{aligned}$$

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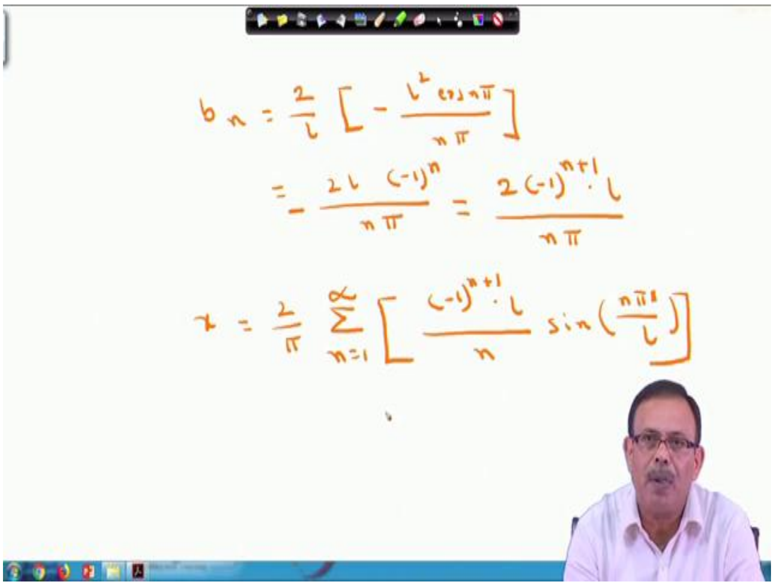

Therefore, the Fourier series expansion of $f(x)$ can be expressed as

$$x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{2l}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l}.$$

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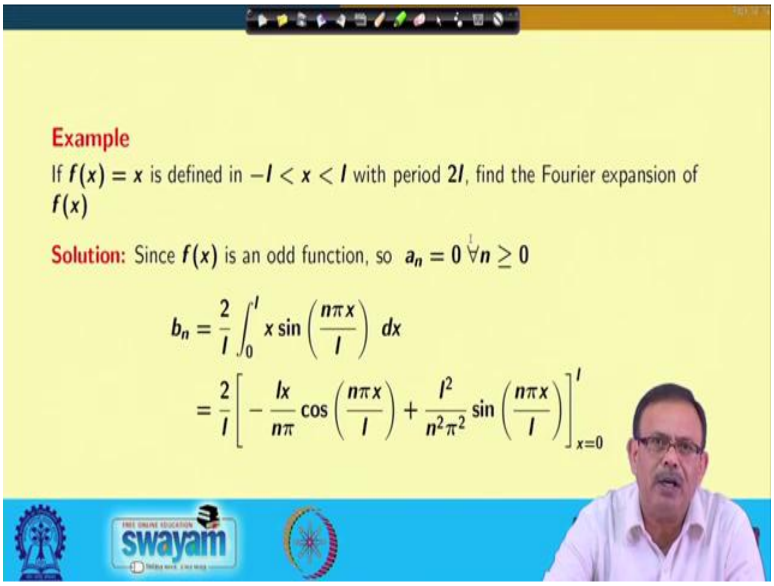

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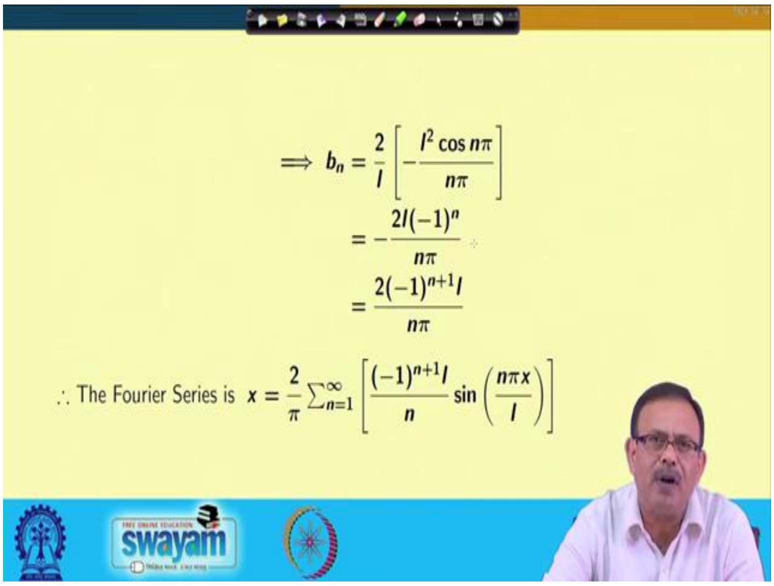

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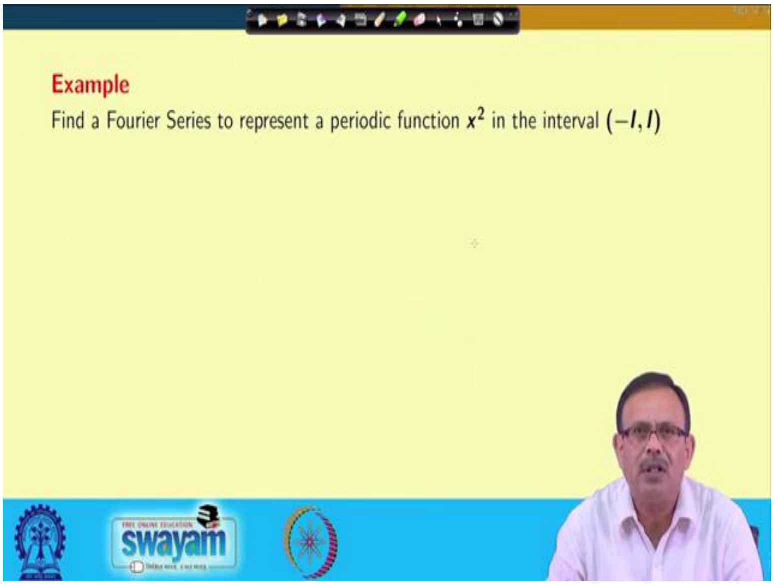

Let us now move to the next example. In the earlier one, we wanted to find out the Fourier series expansion of the function $f(x)$ when $f(x) = x$. Now, in this case, we are provided with $f(x) = x^2$ defined in the interval $(-l, l)$.

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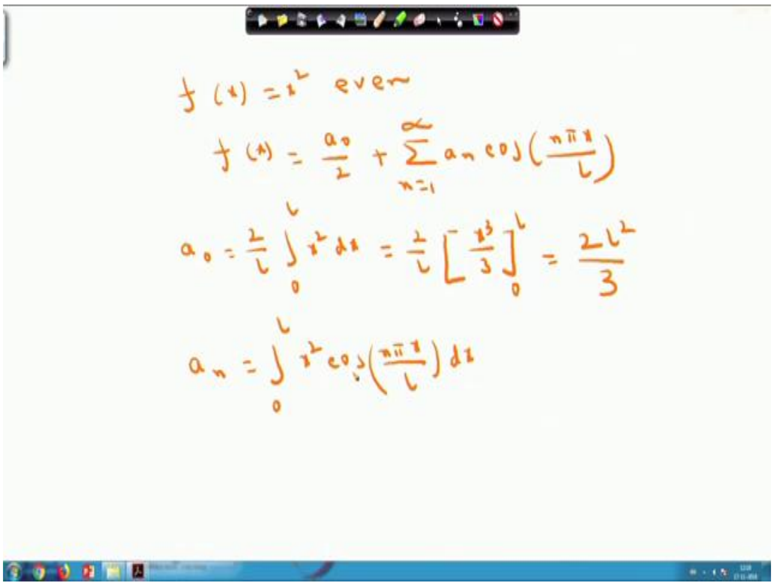

Clearly, since $f(x) = x^2$ is an even function, so there will be only cosine terms in the Fourier Series expansion and $b_n = 0$ for all n. So, $f(x)$ can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where

$$\begin{aligned} a_0 &= \frac{1}{l}\int_{-l}^{l} f(x)\,dx \\ &= \frac{2}{l}\int_{0}^{l} x^2\,dx \\ &= \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l \\ &= \frac{2l^2}{3} \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \\ &= \frac{2}{l}\int_{0}^{l} x^2\cos\frac{n\pi x}{l}\,dx \end{aligned}$$

$$\Rightarrow a_n = \frac{2}{l}\left[\frac{l}{n\pi}x^2 \sin\frac{n\pi x}{l}\right]_0^l - \frac{4}{l}\frac{l}{n\pi}\int_0^l x \sin\frac{n\pi x}{l}\,dx$$

$$= -\frac{4}{n\pi}\left[-\frac{lx}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l + \frac{4}{n\pi}\frac{l^2}{n^2\pi^2}\left[\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4l^2}{n^2\pi^2}\cos n\pi$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n.$$

Therefore, for $n = 1$, we have $a_1 = -\frac{4l^2}{\pi^2}$, for $n = 2$, we have $a_2 = \frac{4l^2}{2^2\pi^2}$, for $n = 3$, we have $a_3 = -\frac{4l^2}{3^2\pi^2}$ and so on.

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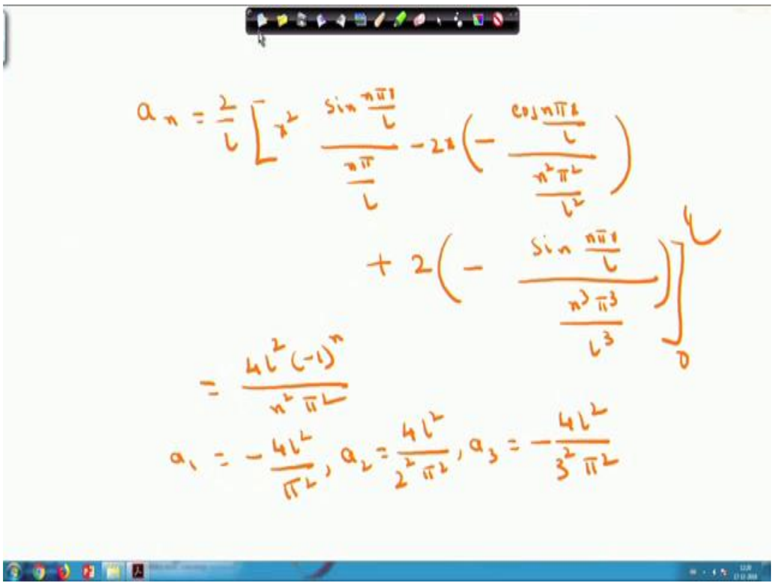

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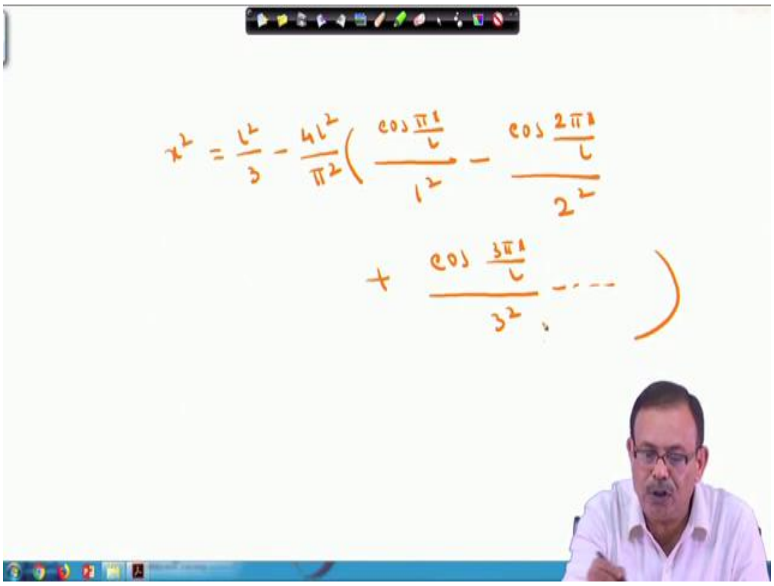

So we can write down the Fourier Series expansion of $f(x)$ as

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos\frac{\pi x}{l}}{1^2} - \frac{\cos\frac{2\pi x}{l}}{2^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} - \cdots\right].$$

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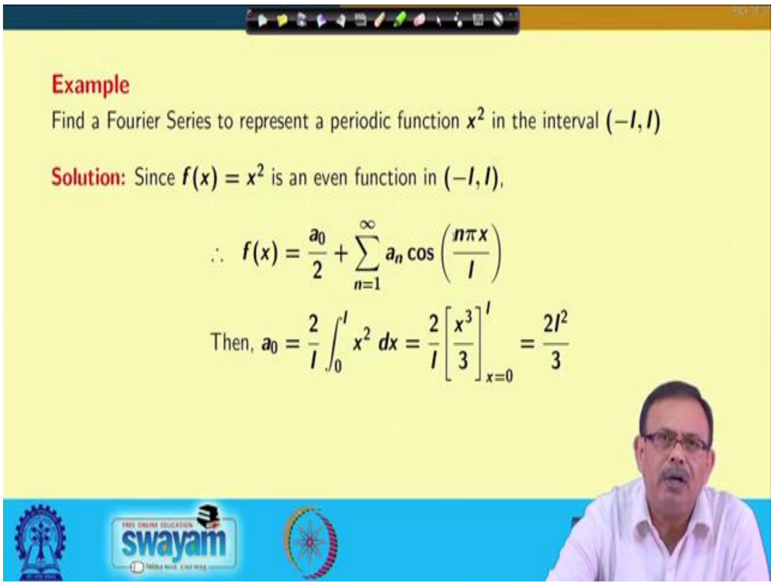

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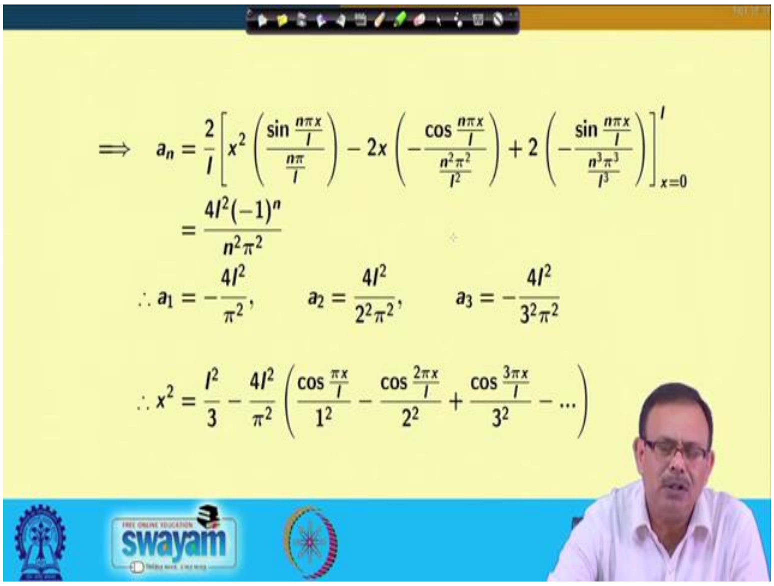

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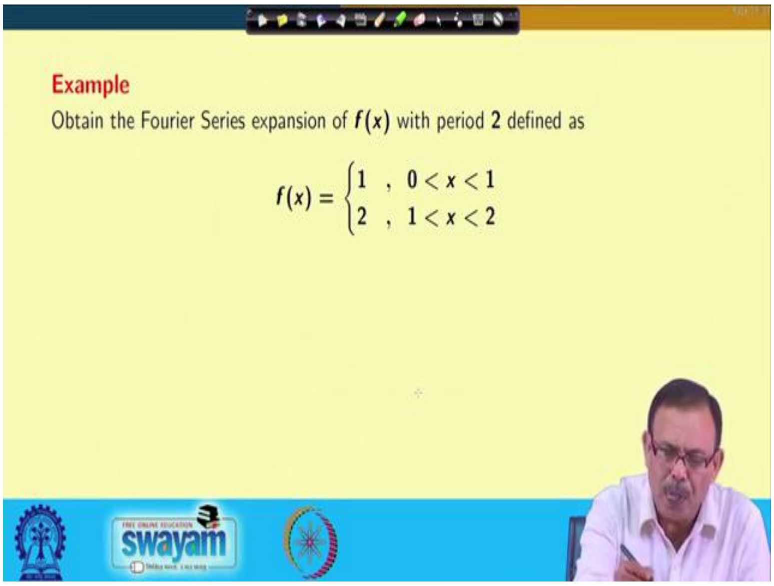

In the next problem, a function $f(x)$ is given with period 2, where $f(x)$ is defined by

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$$

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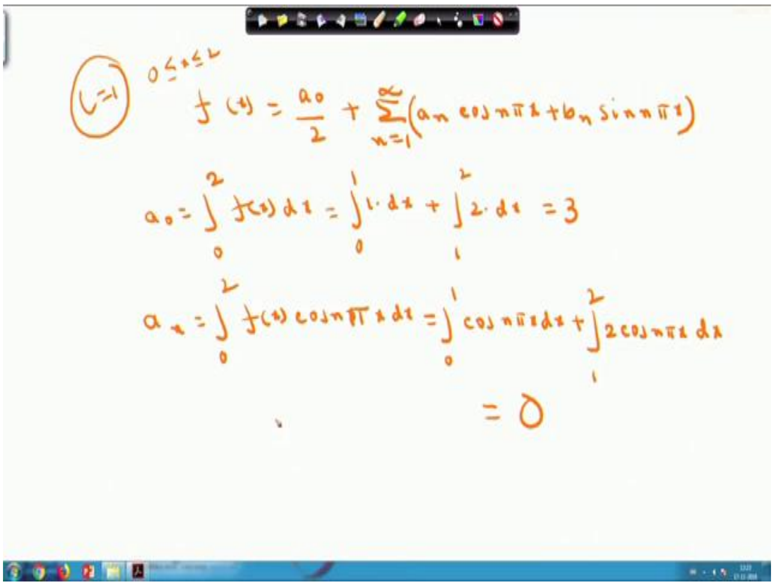

Looking at $f(x)$, we are not able to tell whether it is an even or an odd function, so that $f(x)$ can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x) \quad [\because l = 1]$$

where

$$\begin{aligned} a_0 &= \frac{2}{2}\int_0^2 f(x)\,dx \\ &= \int_0^1 1.\,dx + \int_1^2 2.\,dx \\ &= 3 \end{aligned}$$

$$\begin{aligned} a_n &= \int_0^2 f(x)\cos n\pi x\,dx \\ &= \int_0^1 \cos n\pi x\,dx + \int_1^2 2\cos n\pi x\,dx \\ &= \left[\frac{1}{n\pi}\sin n\pi x\right]_0^1 + 2\left[\frac{1}{n\pi}\sin n\pi x\right]_1^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \int_0^2 f(x)\sin n\pi x\,dx \\ &= \int_0^1 \sin n\pi x\,dx + \int_1^2 2\sin n\pi x\,dx \\ &= \left[-\frac{1}{n\pi}\cos n\pi x\right]_0^1 + 2\left[-\frac{1}{n\pi}\cos n\pi x\right]_1^2 \\ &= \frac{\cos n\pi - 1}{n\pi} \\ &= \frac{(-1)^n - 1}{n\pi} \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{2}{n\pi}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

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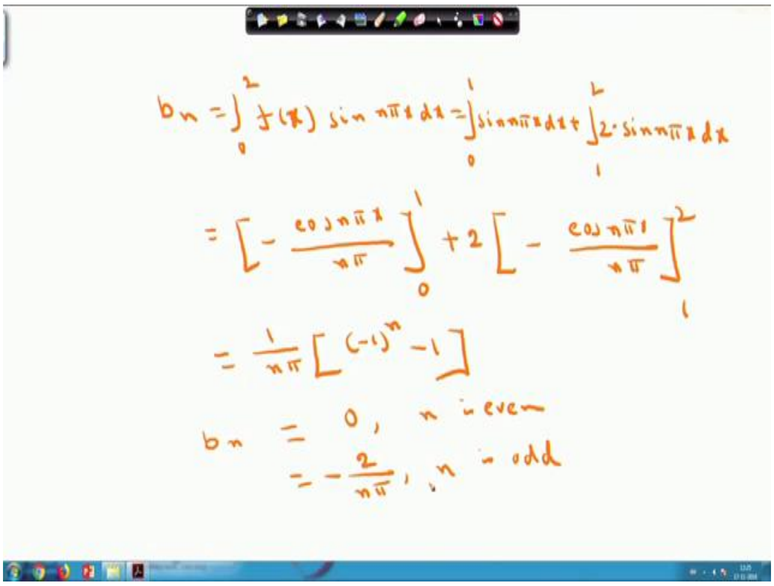

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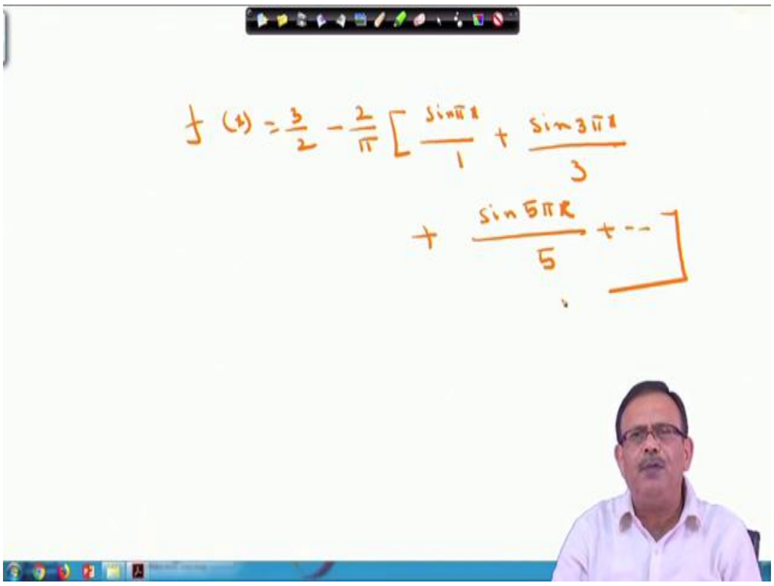

So, we can now write down the Fourier Series expansion of $f(x)$ as

$$
\begin{aligned}
f(x) &= \frac{3}{2} + \sum_{n=1}^{\infty} b_n \sin n\pi x \\
&= \frac{3}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} \sin n\pi x \\
&= \frac{3}{2} - \frac{2}{\pi} \left[\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \cdots \right].
\end{aligned}
$$

Therefore, if a function $f(x)$ is defined either in $(-\pi, \pi)$ or in $(-l, l)$; whether it is an odd function or an even function, we can find out the Fourier series expansion of $f(x)$ in a similar fashion. Thank you.