

Transform Calculus and its Applications in Differential Equations
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Lecture - 20
Fourier Series for Even and Odd Functions

So in the last lecture, we have started the Fourier series. In this lecture, we will start with the convergence of Fourier series of a function, to check whether a function can be expressed in terms of Fourier series or not. We will study and discuss the conditions under which a function can be expanded as a Fourier series.

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Convergence of Fourier Series

Theorem
Any function $f(x)$ can be developed as a Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are constants provided

- (i) $f(x)$ is periodic, single-valued and finite
- (ii) $f(x)$ has a finite number of discontinuities in any one period
- (iii) $f(x)$ has at most a finite number of maxima and minima

Let us go through a theorem which states the convergence of Fourier series. Any function $f(x)$ can be developed as a Fourier series given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where the coefficients a_0, a_n and b_n are constants (whose values have been derived in the previous lecture), provided a few conditions are satisfied by the function $f(x)$. Firstly, $f(x)$ should be periodic, single-valued function and finite. Secondly, $f(x)$ should have a finite number of discontinuities in any one period. Thirdly, $f(x)$ should have at most a finite number of maxima and minima.

So, if a function $f(x)$ satisfies these 3 conditions, then only we can expand it in terms of a Fourier series i.e., in terms of sine and cosine series.

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i.e., if (i), (ii), (iii) are satisfied, then the Fourier Series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous.

Also the series converges to the average of the left limit and right limit of $f(x)$ at each point where $f(x)$ is discontinuous.

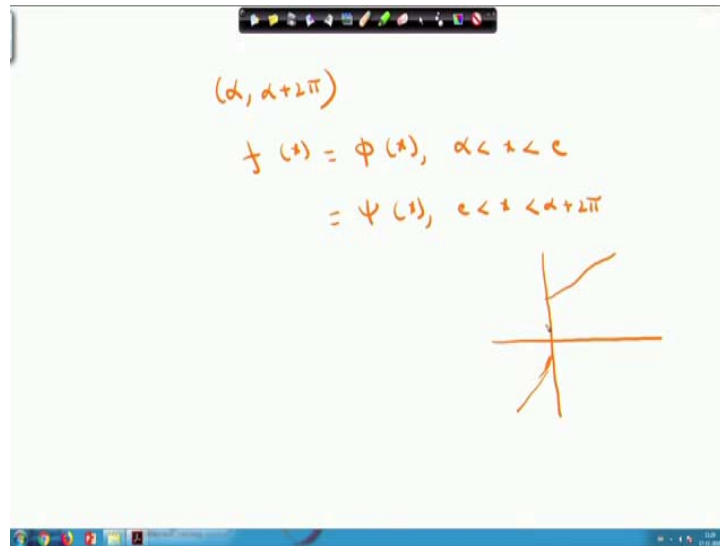
For instance, if in the interval $(\alpha, \alpha + 2\pi)$, $f(x)$ is defined by

$$\begin{aligned} f(x) &= \phi(x), & \alpha < x < c \\ &= \psi(x), & c < x < \alpha + 2\pi \end{aligned}$$

(c being a point of discontinuity)

We say, if the given conditions are satisfied, then the Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous. Also the series converges to the average of the left limit and right limit of $f(x)$ at each point where $f(x)$ is discontinuous. So, from the first one, if $f(x)$ is continuous at a point, $x = x_0$, then the series can be convergent to $f(x_0)$. But if there is a point of discontinuity, in that case the value of the series will be equal to average of the left hand limit and the right hand limit of $f(x)$ at each of the points where the function is discontinuous.

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Suppose our function $f(x)$ is defined in the interval α to $\alpha + 2\pi$ as

$$f(x) = \begin{cases} \phi(x), & \alpha < x < c \\ \psi(x), & c < x < \alpha + 2\pi \end{cases}$$

So, we see that c is the point of discontinuity in this case.

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Handwritten formulas for the Fourier coefficients a_0 , a_n , and b_n of the piecewise function $f(x)$. The formulas are:

$$a_0 = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) dx + \int_c^{\alpha + 2\pi} \psi(x) dx \right]$$
$$a_n = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) \cos nx dx + \int_c^{\alpha + 2\pi} \psi(x) \cos nx dx \right]$$
$$b_n = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) \sin nx dx + \int_c^{\alpha + 2\pi} \psi(x) \sin nx dx \right]$$

The point $x=c$ is circled in the original image.

In such a situation,

$$a_0 = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) dx + \int_c^{\alpha+2\pi} \psi(x) dx \right]$$

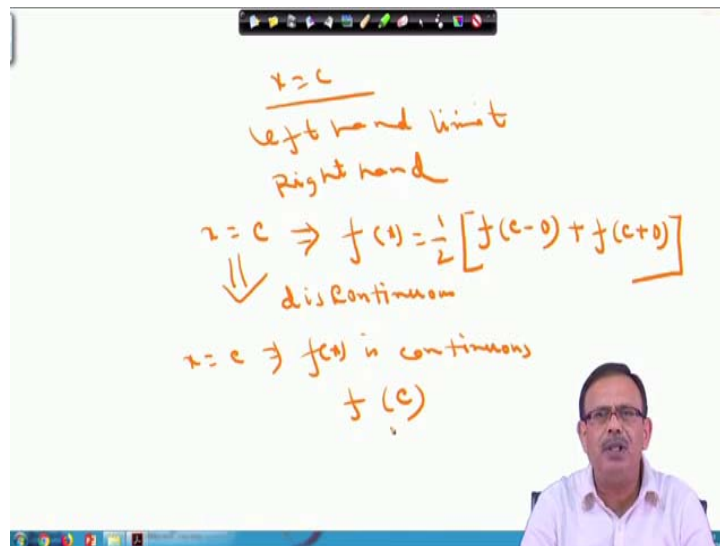
$$a_n = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) \cos nx dx + \int_c^{\alpha+2\pi} \psi(x) \cos nx dx \right]$$

$$b_n = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) \sin nx dx + \int_c^{\alpha+2\pi} \psi(x) \sin nx dx \right]$$

If there is a point of discontinuity at the point $x = c$, then a_0 , a_n , b_n are evaluated like this.

So, at $x = c$, there is a finite jump in the graph of the function or it is discontinuous. Now, here we see, the left hand limit [i. e., $f(c - 0)$] and the right hand limit [i. e., $f(c + 0)$] both exist at $x = c$, but they will be different.

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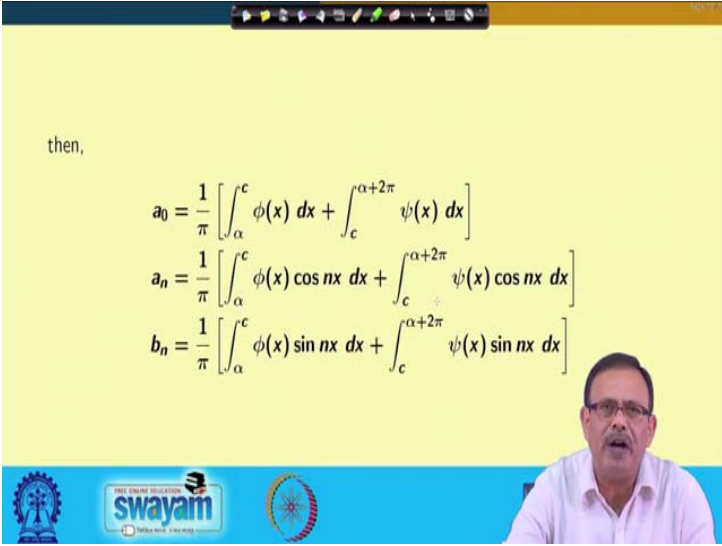
So, at such a point $x = c$, the value of $f(x)$ can be written as

$$f(x) = \frac{1}{2} [f(c - 0) + f(c + 0)].$$

So, at the point $x = c$, if we have a point of discontinuity, then the value of the Fourier series at the point $x = c$ can be evaluated. Please note that this is true whenever it is discontinuous and if at $x = c$, $f(x)$ is continuous, in that case the value will be $f(c)$.

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then,

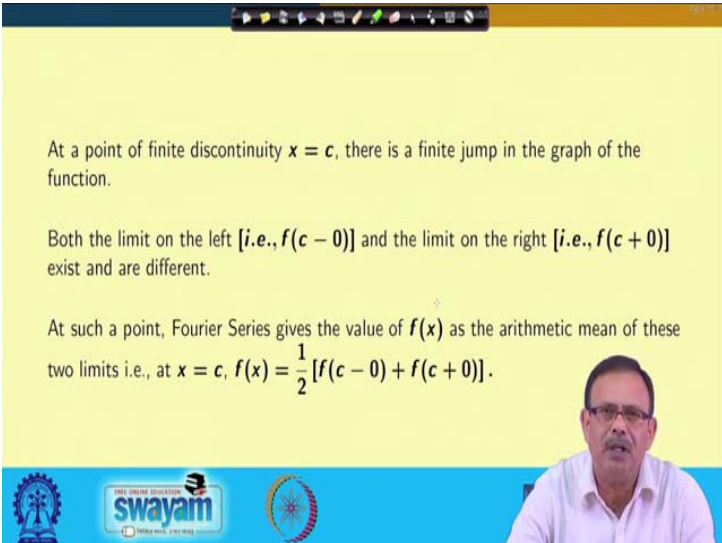
$$a_0 = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) dx + \int_c^{\alpha+2\pi} \psi(x) dx \right]$$
$$a_n = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) \cos nx dx + \int_c^{\alpha+2\pi} \psi(x) \cos nx dx \right]$$
$$b_n = \frac{1}{\pi} \left[\int_{\alpha}^c \phi(x) \sin nx dx + \int_c^{\alpha+2\pi} \psi(x) \sin nx dx \right]$$


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At a point of finite discontinuity $x = c$, there is a finite jump in the graph of the function.

Both the limit on the left [i.e., $f(c - 0)$] and the limit on the right [i.e., $f(c + 0)$] exist and are different.

At such a point, Fourier Series gives the value of $f(x)$ as the arithmetic mean of these two limits i.e., at $x = c$, $f(x) = \frac{1}{2} [f(c - 0) + f(c + 0)]$.



Sometimes we call this one as the Dirichlet's condition also.

Now let us take an example. We want to find the Fourier series of a periodic function $f(x)$ with period 2π which is defined as

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ x, & 0 \leq x \leq \pi \end{cases}$$

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Example
Find the Fourier Series of the periodic function f with period 2π , defined as

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 \leq x < \pi \end{cases}$$

Deduce that $1 + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$

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Also we need to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

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Handwritten notes on a whiteboard background showing the derivation of the Fourier series for the function $f(x)$.

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 \leq x < \pi \end{cases} \quad (2\pi)$$
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{\pi}{2}$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} x \cos nx dx$$
$$= \frac{1}{\pi} \cdot n \left[\frac{\cos nx}{n} \int_0^{\pi} \right] = \begin{cases} 0, & n \text{ - even} \\ -\frac{2}{n^2}, & n \text{ - odd} \end{cases}$$

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So, from the formula, $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

Our job is to find out the values of the coefficients a_0 , a_n and b_n . Using the formula,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{\pi}{2}$$

Here $f(x) = 0$, for $-\pi \leq x < 0$ and $f(x) = x$, for $0 \leq x \leq \pi$. So a_n and b_n can be calculated as,

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \frac{1}{\pi n} [\cos n\pi - 1] = \begin{cases} 0 & , \text{ when } n \text{ is even} \\ -\frac{2}{\pi n^2} & , \text{ when } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx dx = -\frac{\cos n\pi}{n} = \begin{cases} -\frac{1}{n} & , \text{ when } n \text{ is even} \\ \frac{1}{n} & , \text{ when } n \text{ is odd} \end{cases}$$

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The image shows handwritten mathematical work on a whiteboard. It starts with the formula for b_n :
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} x \sin nx dx$$

$$= -\frac{\cos n\pi}{n} = \begin{cases} -\frac{1}{n}, & n \text{ even} \\ \frac{1}{n}, & n \text{ odd} \end{cases}$$
Then it shows the function $f(x)$ as a sum of two series:
$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left\{ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right\}$$

$$+ \left\{ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right\}$$
Below this, there is a note:
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

So, depending upon even or odd, the value of $\cos n\pi$ changes. Accordingly, we have to write down what is the values of a_n , b_n whenever n is even or odd. So, we have obtained the values of a_0 , a_n and b_n . So, now, we can write down the series, $f(x)$ as,

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right].$$

Please note that whenever n is even, $a_n = 0$. Therefore, even cosine terms will vanish, only the odd terms will be there.

If we put, $x = 0$, in the Fourier series of $f(x)$ then $\cos nx$ will be 1 and $\sin nx$ will be 0.

$$\begin{aligned} \therefore f(0) &= \frac{\pi}{4} - \frac{2}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \\ \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots &= \frac{\pi^2}{8} \quad (\because f(0) = 0) \end{aligned}$$

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$$\therefore f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left\{ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right\} + \left\{ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right\}$$

At $x = 0$,

$$f(0) = \frac{\pi}{4} - \frac{2}{\pi} \left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right\}$$
$$\therefore \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

So, whenever we have a function, we can express it in terms of a sine series or cosine series i.e., the Fourier series. Also with the help of this, we are finding the values of some finite series.

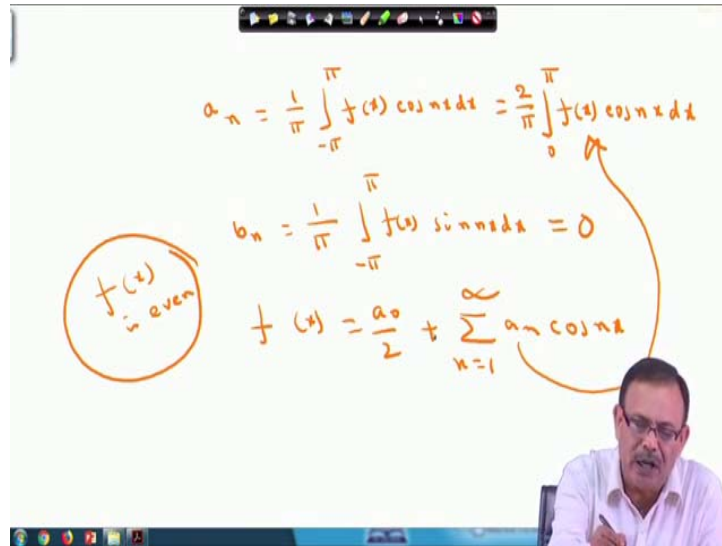
Now let us see the next one, that is Fourier series for even and odd functions.

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Even Function
 $f(-x) = f(x) \forall x$
 $f(x) \cos nx$ is even
 $f(x) \sin nx$ is odd

What is even function? As we know, $f(x)$ is called an even function if $f(-x) = f(x)$. If $f(-x) = -f(x)$, then $f(x)$ is called an odd function. Now if $f(x)$ is an even function, then $f(x) \cos nx$ always will be even function and $f(x) \sin nx$ will be odd function.

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Now we will discuss about the effects on a_0 , a_n and b_n if $f(x)$ is an even function.

We know that, if $f(x)$ is an even function, then $f(x) \cos nx$ always will be even function and $f(x) \sin nx$ will be odd function. Again from the properties of definite integral, we know that,

$$\int_{-a}^a f(x) dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd function} \\ 2 \int_0^a f(x) dx & \text{if } f(x) \text{ is even function} \end{cases}$$

Therefore, from the formulae of a_0 , a_n and b_n , we have,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

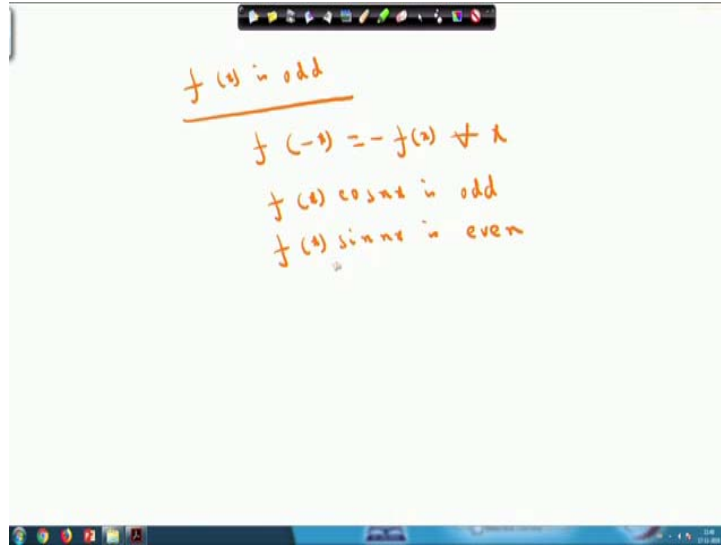
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0$$

So that we can say that the Fourier series of even function consists of cosine terms only, there will be no sine term in this one.

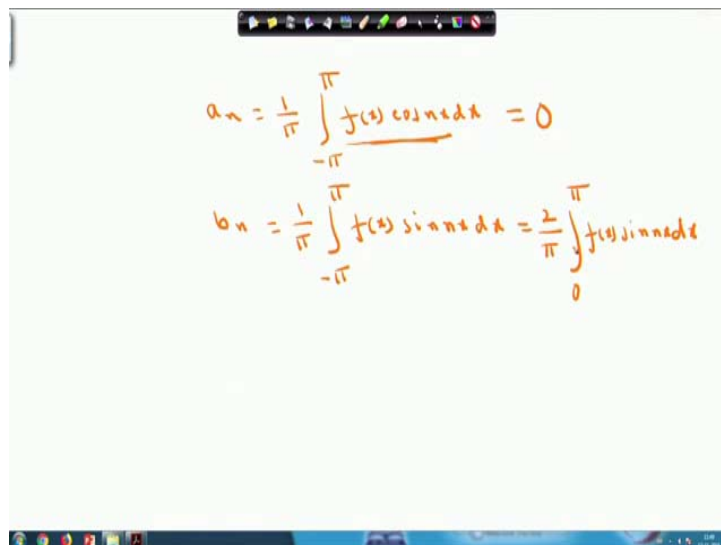
So, if we know the given function is an even function, we do not have to calculate b_n , rather we will evaluate only a_n .

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Now, suppose $f(x)$ is odd function. If $f(x)$ is odd function, then $f(x) \cos nx$ always will be odd function and $f(x) \sin nx$ will be even function.

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Therefore, from the formulae of a_0 , a_n and b_n , we have,

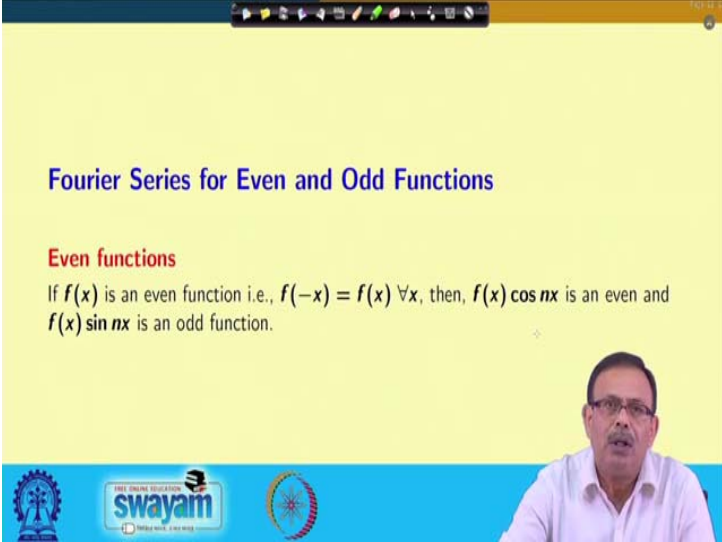
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

Therefore, Fourier series of an odd function consists of sine terms only. So, in that case, we will calculate only b_n .

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The image shows a screenshot of a video lecture slide. The slide has a yellow background and a blue header bar at the top. The title "Fourier Series for Even and Odd Functions" is written in blue. Below the title, the text "Even functions" is written in red. The main content of the slide states: "If $f(x)$ is an even function i.e., $f(-x) = f(x) \forall x$, then, $f(x) \cos nx$ is an even and $f(x) \sin nx$ is an odd function." In the bottom right corner, there is a small video inset showing a man with glasses speaking. At the bottom of the slide, there are logos for "swayam" and other educational institutions.

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The slide displays the following mathematical derivations:

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \quad (2)$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

∴ Fourier Series of an even function consists of terms of cosines only and the coefficient a_n may be computed from (2).

The slide also features a video feed of a presenter in the bottom right corner and logos for 'swayam' and 'All India Institute of Space Technology' at the bottom.

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The slide is titled "Odd functions" and contains the following text:

If $f(x)$ is an odd function i.e., $f(-x) = -f(x) \forall x$, then, $f(x) \cos nx$ is an odd and $f(x) \sin nx$ is an even function and therefore

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad (3)$$

So, the Fourier Series of an odd function consists of sine terms only and the coefficient b_n may be computed from (3).

The slide also features a video feed of a presenter in the bottom right corner and logos for 'swayam' and 'All India Institute of Space Technology' at the bottom.

So, for even and odd functions, we can calculate it like this.

Let us consider one example. We want to find out the Fourier series for a periodic function $|x|$ of period 2π . Afterwards, we also need to compute the values of the series at $0, 2\pi$.

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Example
Find the Fourier Series generated by the periodic function $|x|$ of period 2π . Also compute the value of series at $0, 2\pi$.

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$f(x) = |x|$
 $f(x) = -x, -\pi \leq x \leq 0$
 $f(x) = x, 0 \leq x \leq \pi$
 $f(x)$ is even function
 $a_0 = \pi$
 $a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx = \frac{2}{\pi^2} (\cos n\pi - 1)$
 $= \begin{cases} 0, & n \text{ is even} \\ -\frac{4}{\pi^2}, & n \text{ is odd} \end{cases}$

Given function can be represented as

$$f(x) = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \end{cases}$$

So, $f(x)$ is an even function. Therefore, once we know that $f(x)$ is an even function, then we know even function is expressed in terms of cosine series only. So, we have to calculate only a_0 and a_n as follows:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi n^2} (\cos n\pi - 1) = \begin{cases} 0 & , \text{ when } n \text{ is even} \\ -\frac{4}{\pi n^2} & , \text{ when } n \text{ is odd} \end{cases}$$

So, this value will vary depending upon the value of n .

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The image shows a whiteboard with handwritten mathematical work. The formula written is $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$. The handwriting is in orange ink on a light green background. There are some small marks and a bracket at the end of the series.

So, we can write $f(x)$ as

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

At $x = 0$, we have,

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

Similarly, at $x = 2\pi$, we have,

$$f(2\pi) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right].$$

So, please note that, at the beginning itself, before finding the Fourier series of a function $f(x)$, if we check whether $f(x)$ is an even function or an odd function, in that case, our calculation burden will be reduced a lot. This is because if it is even function, it will have only the cosine terms in the Fourier series or in other sense, we will have to evaluate only

a_0 and a_n . Whereas, if the function is odd function, in that case, it will have the sine terms only and we will need to evaluate only b_n , that we will see in the next lecture. Thank you.