Transform Calculus and it is Applications in Differential Equations
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Lecture - 20
Fourier Series for Even and Odd Functions

So in the last lecture, we have started the Fourier series. In this lecture, we will start with the convergence of Fourier series of a function, to check whether a function can be expressed in terms of Fourier series or not. We will study and discuss the conditions under which a function can be expanded as a Fourier series.

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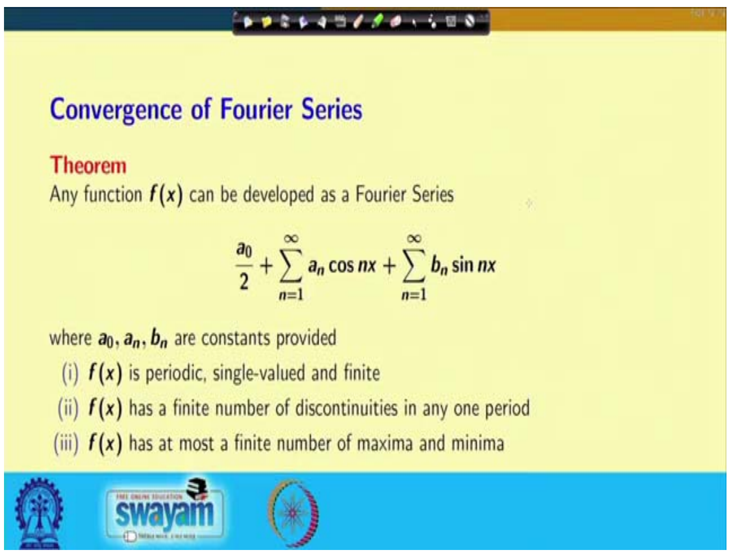

Let us go through a theorem which states the convergence of Fourier series. Any function $f(x)$ can be developed as a Fourier series given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where the coefficients a_0, a_n and b_n are constants (whose values have been derived in the previous lecture), provided a few conditions are satisfied by the function $f(x)$. Firstly, $f(x)$ should be periodic, single-valued function and finite. Secondly, $f(x)$ should have a finite number of discontinuities in any one period. Thirdly, $f(x)$ should have at most a finite number of maxima and minima.

So, if a function $f(x)$ satisfies these 3 conditions, then only we can expand it in terms of a Fourier series i.e., in terms of sine and cosine series.

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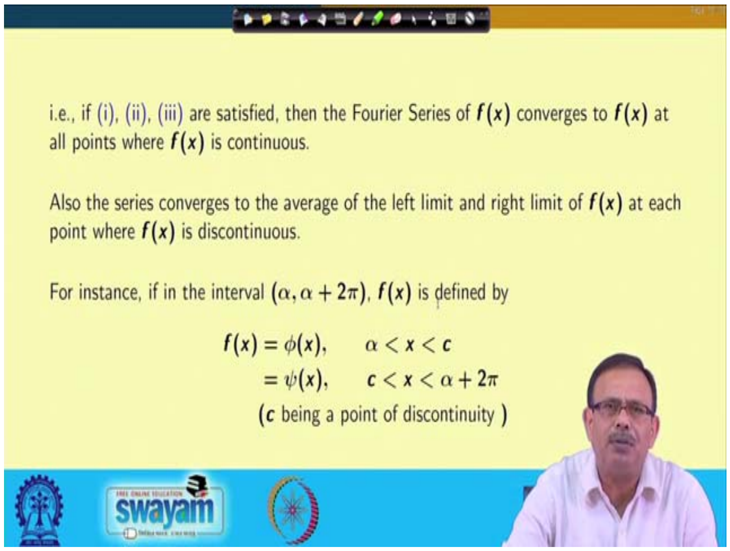

We say, if the given conditions are satisfied, then the Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous. Also the series converges to the average of the left limit and right limit of $f(x)$ at each point where $f(x)$ is discontinuous. So, from the first one, if $f(x)$ is continuous at a point, $x = x_0$, then the series can be convergent to $f(x_0)$. But if there is a point of discontinuity, in that case the value of the series will be equal to average of the left hand limit and the right hand limit of $f(x)$ at each of the points where the function is discontinuous.

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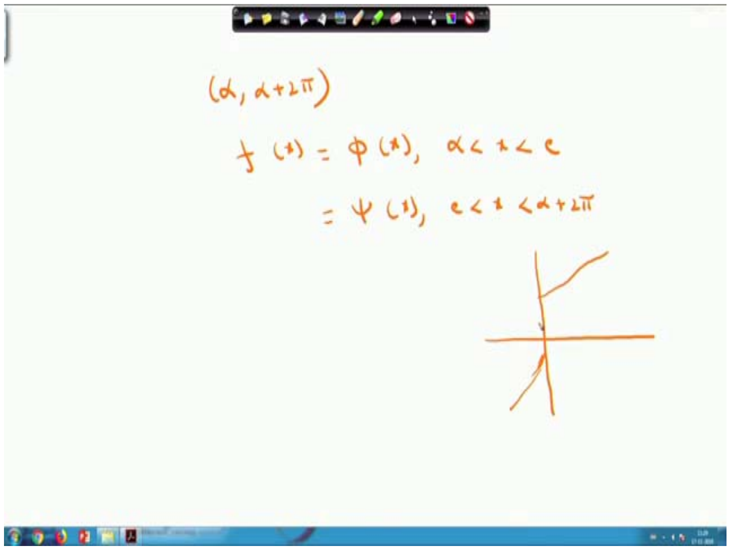

Suppose our function $f(x)$ is defined in the interval α to $\alpha + 2\pi$ as

$$f(x) = \begin{cases} \phi(x), & \alpha < x < c \\ \psi(x), & c < x < \alpha + 2\pi \end{cases}$$

So, we see that c is the point of discontinuity in this case.

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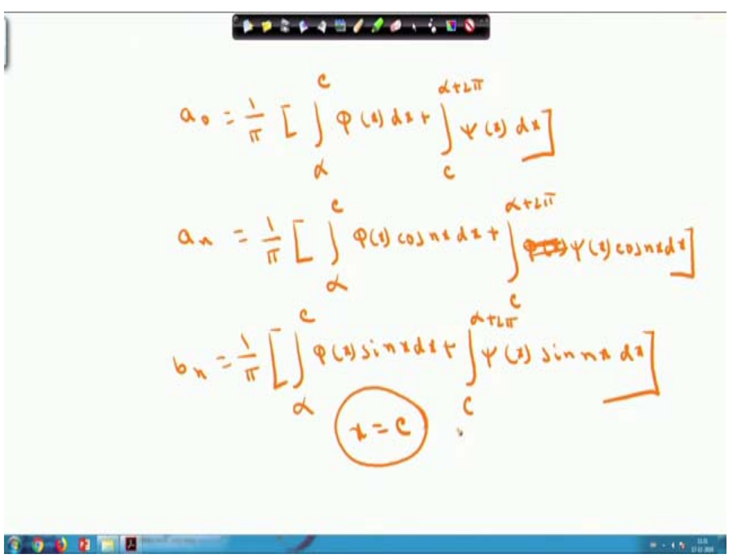

In such a situation,

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c}\phi(x)dx + \int_{c}^{\alpha+2\pi}\psi(x)dx\right]$$
$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c}\phi(x)\cos nx\,dx + \int_{c}^{\alpha+2\pi}\psi(x)\cos nx\,dx\right]$$
$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c}\phi(x)\sin nx\,dx + \int_{c}^{\alpha+2\pi}\psi(x)\sin nx\,dx\right]$$

If there is a point of discontinuity at the point $x = c$, then $a_0,\ a_n,\ b_n$ are evaluated like this.

So, at $x = c$, there is a finite jump in the graph of the function or it is discontinuous. Now, here we see, the left hand limit $[\text{i.e.}, f(c-0)]$ and the right hand limit $[\text{i.e.}, f(c+0)]$ both exist at $x = c$, but they will be different.

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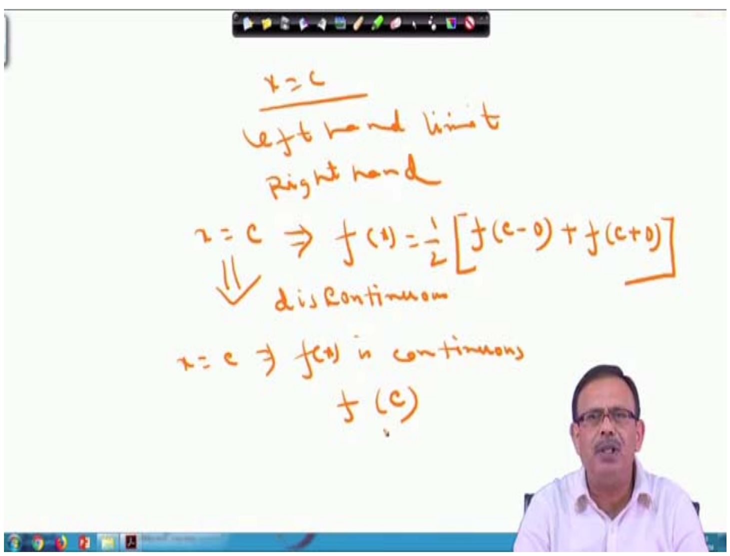

So, at such a point $x = c$, the value of $f(x)$ can be written as

$$f(x) = \frac{1}{2}[f(c-0) + f(c+0)].$$

So, at the point $x = c$, if we have a point of discontinuity, then the value of the Fourier series at the point $x = c$ can be evaluated. Please note that this is true whenever it is discontinuous and if at $x = c$, $f(x)$ is continuous, in that case the value will be $f(c)$.

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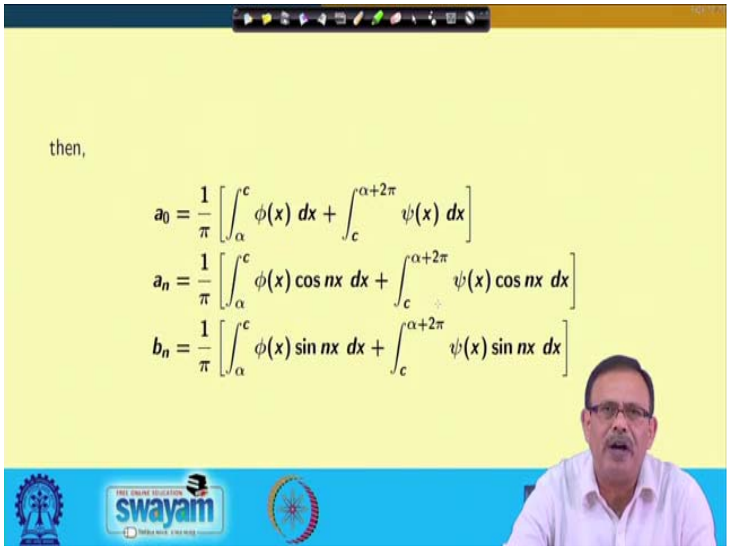

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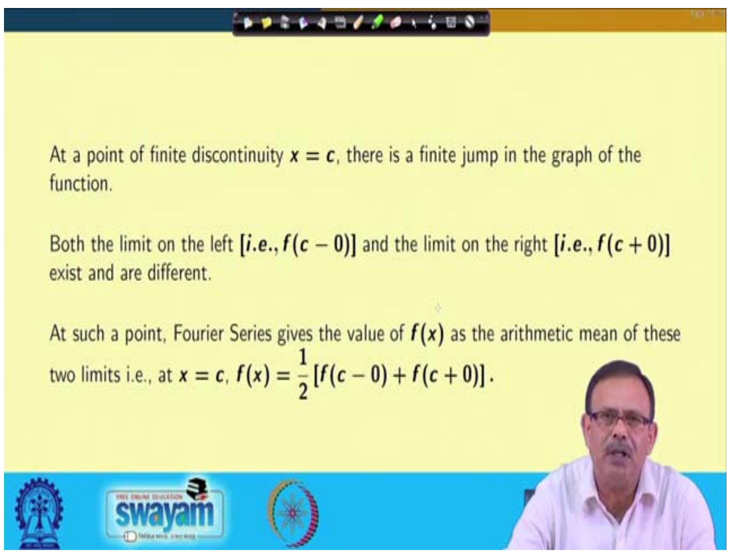

Sometimes we call this one as the Dirichlet's condition also.

Now let us take an example. We want to find the Fourier series of a periodic function $f(x)$ with period 2π which is defined as

$$f(x) = \begin{cases} 0\,, & -\pi \le x < 0 \\ x\,, & 0 \le x \le \pi \end{cases}$$

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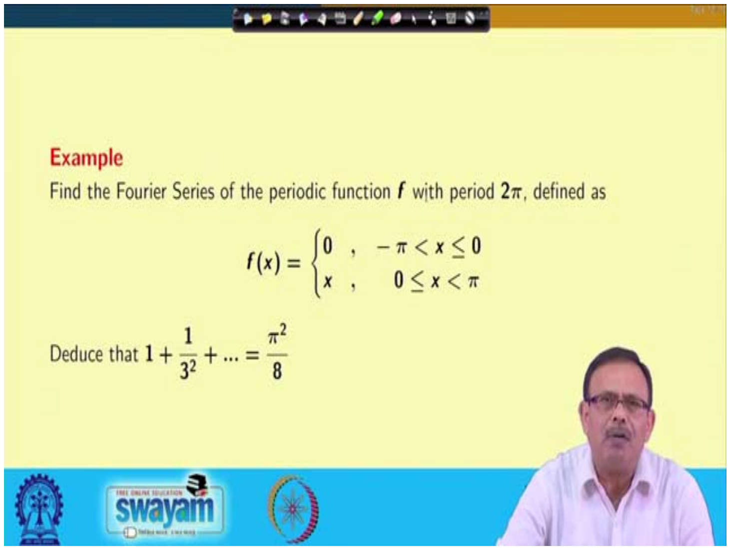

Also we need to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

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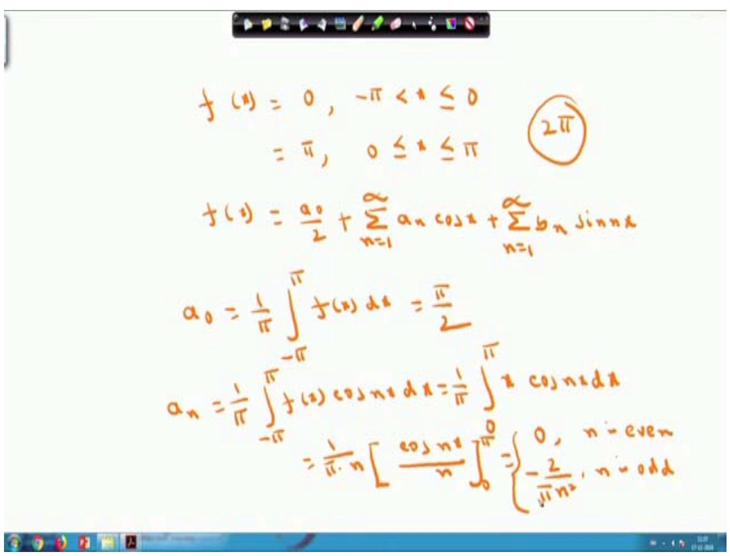

So, from the formula, $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

Our job is to find out the values of the coefficients a_0, a_n and b_n. Using the formula,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{\pi}{2}$$

Here $f(x) = 0$, for $-\pi \leq x < 0$ and $f(x) = x$, for $0 \leq x \leq \pi$. So a_n and b_n can be calculated as,

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx = \frac{1}{\pi n}\left[\frac{\cos n\pi - 1}{n}\right] = \begin{cases} 0 & , \quad \text{when n is even} \\ -\dfrac{2}{\pi n^2} & , \quad \text{when n is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx = -\frac{\cos n\pi}{n} = \begin{cases} -\dfrac{1}{n} & , \quad \text{when n is even} \\ \dfrac{1}{n} & , \quad \text{when n is odd} \end{cases}$$

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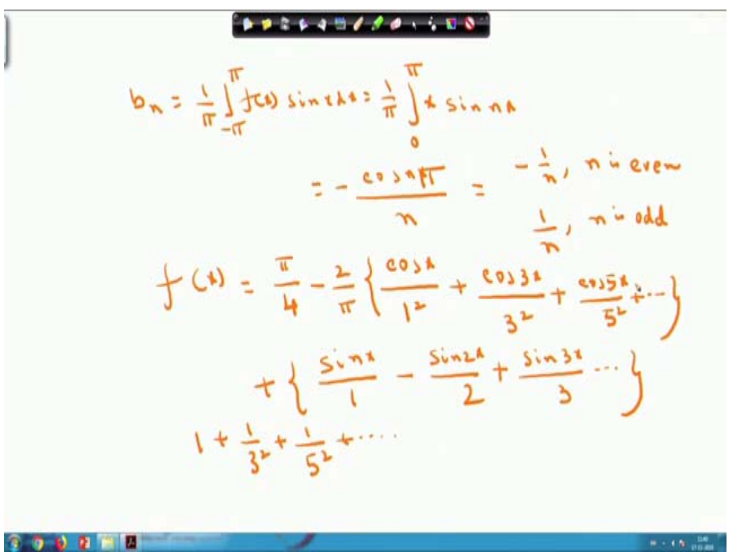

So, depending upon even or odd, the value of $\cos n\pi$ changes. Accordingly, we have to write down what is the values of a_n, b_n whenever n is even or odd. So, we have obtained the values of a_0, a_n and b_n. So, now, we can write down the series, $f(x)$ as,

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right].$$

Please note that whenever n is even, $a_n = 0$. Therefore, even cosine terms will vanish, only the odd terms will be there.

If we put, $x = 0$, in the Fourier series of $f(x)$ then $\cos nx$ will be 1 and $\sin nx$ will be 0.

$$\therefore f(0) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \quad (\because f(0) = 0)$$

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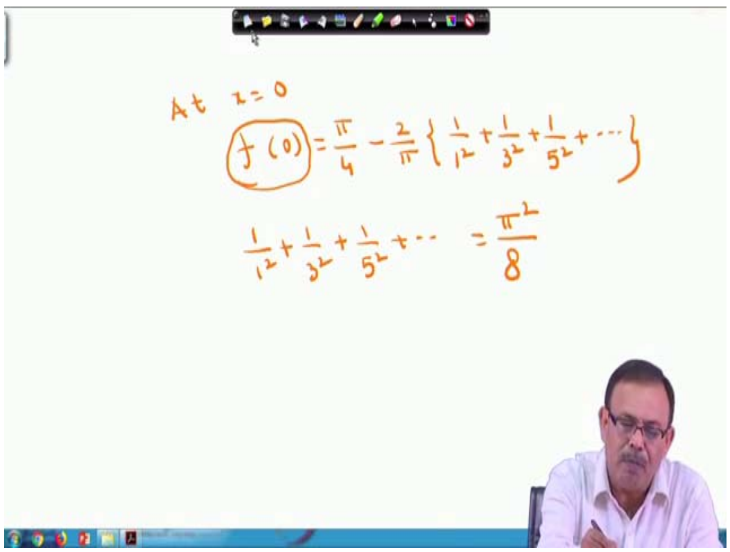

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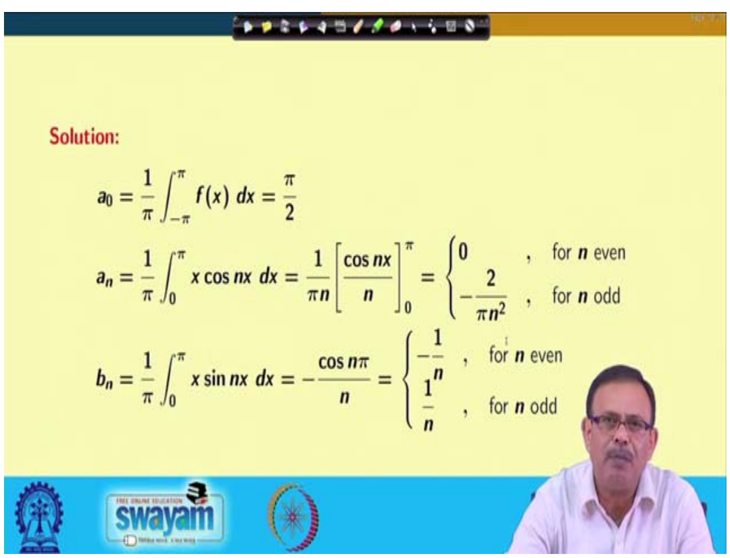

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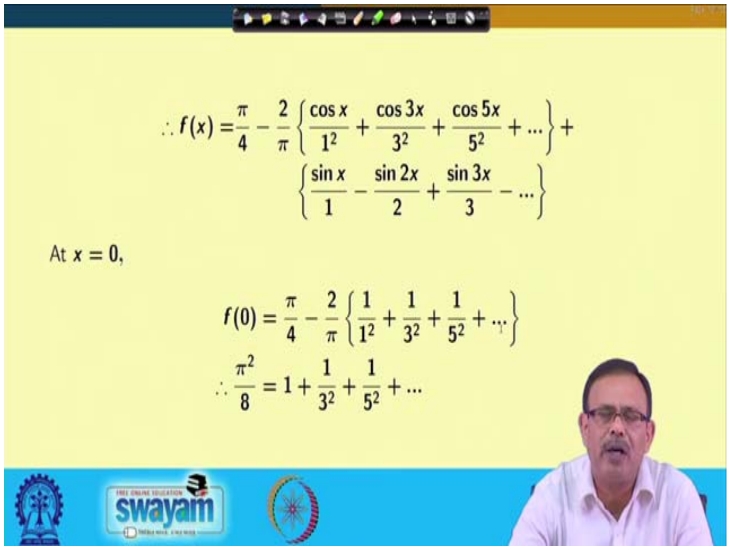

So, whenever we have a function, we can express it in terms of a sine series or cosine series i.e., the Fourier series. Also with the help of this, we are finding the values of some finite series.

Now let us see the next one, that is Fourier series for even and odd functions.

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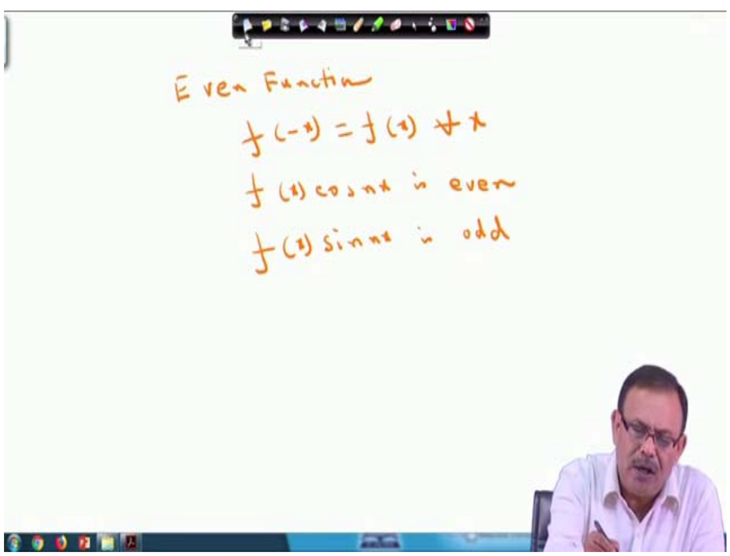

What is even function? As we know, $f(x)$ is called an even function if $f(-x) = f(x)$. If $f(-x) = -f(x)$, then $f(x)$ is called an odd function. Now if $f(x)$ is an even function, then $f(x)\cos nx$ always will be even function and $f(x)\sin nx$ will be odd function.

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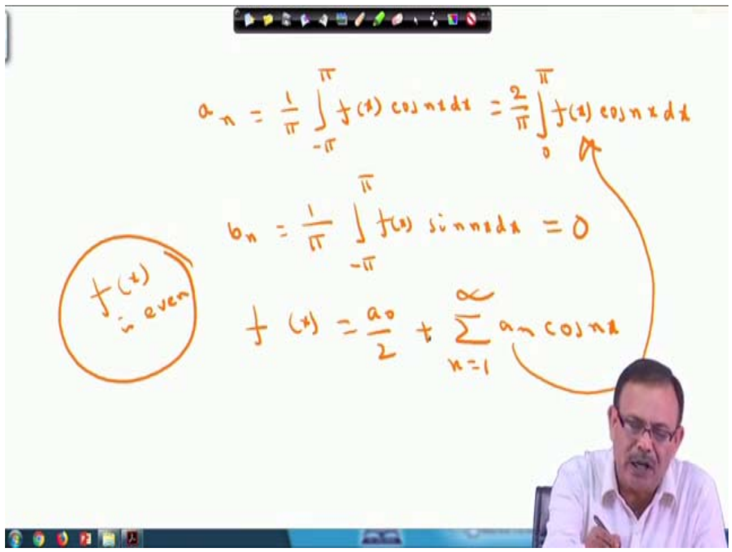

Now we will discuss about the effects on a_0 , a_n and b_n if $f(x)$ is an even function.

We know that, if $f(x)$ is an even function, then $f(x)\cos nx$ always will be even function and $f(x)\sin nx$ will be odd function. Again from the properties of definite integral, we know that,

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd function} \\ 2\int_{0}^{a} f(x)dx & \text{if } f(x) \text{ is even function} \end{cases}$$

Therefore, from the formulae of a_0, a_n and b_n, we have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

So that we can say that the Fourier series of even function consists of cosine terms only, there will be no sine term in this one.

So, if we know the given function is an even function, we do not have to calculate b_n, rather we will evaluate only a_n.

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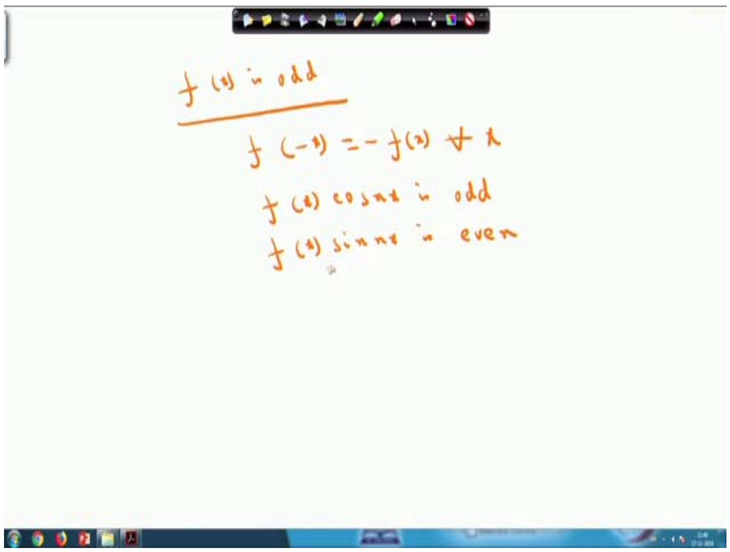

Now, suppose $f(x)$ is odd function. If $f(x)$ is odd function, then $f(x)\cos nx$ always will be odd function and $f(x)\sin nx$ will be even function.

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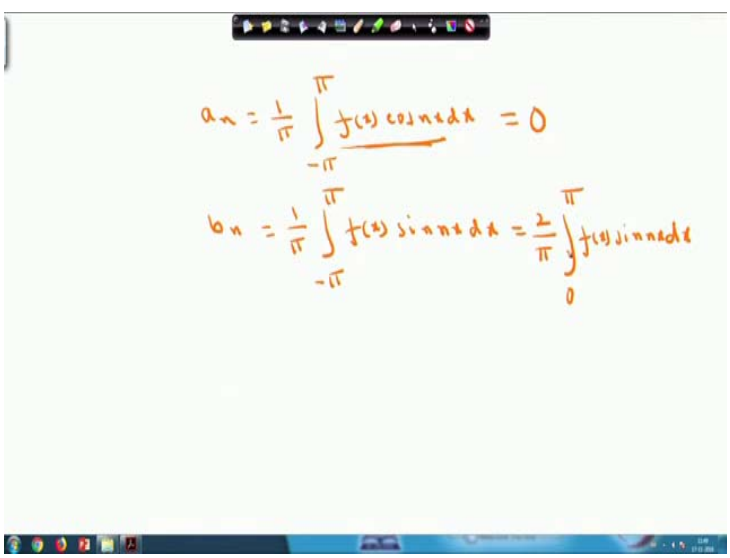

Therefore, from the formulae of a_0, a_n and b_n, we have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

Therefore, Fourier series of an odd function consists of sine terms only. So, in that case, we will calculate only b_n.

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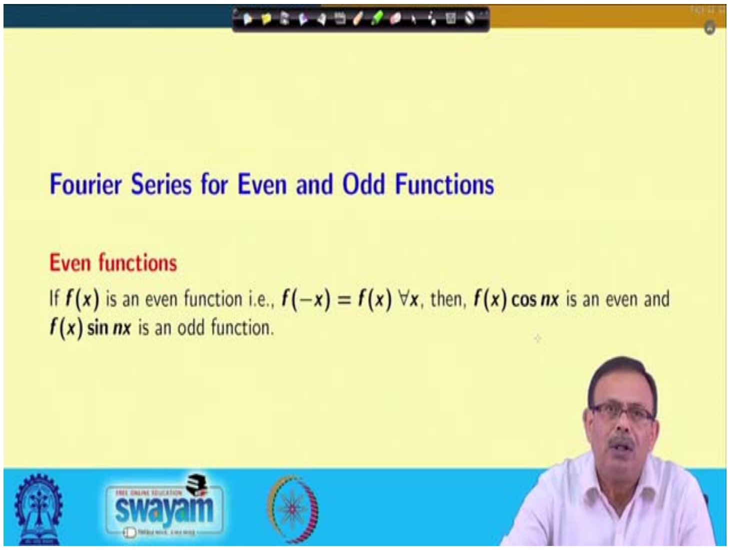

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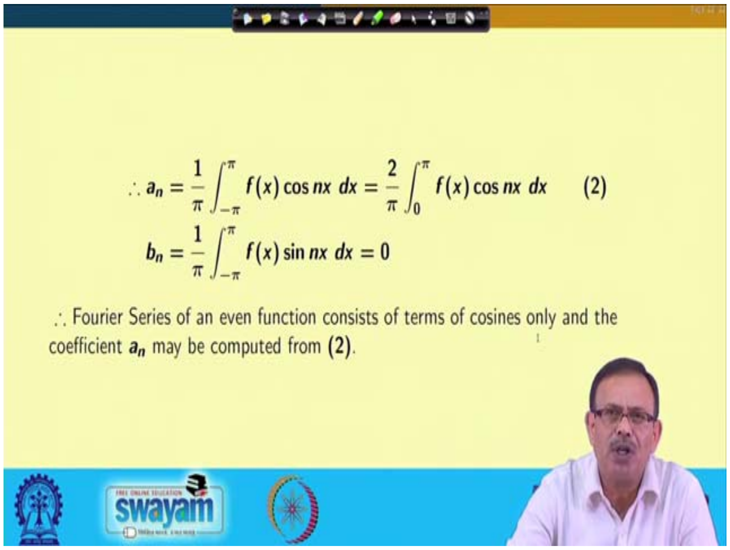

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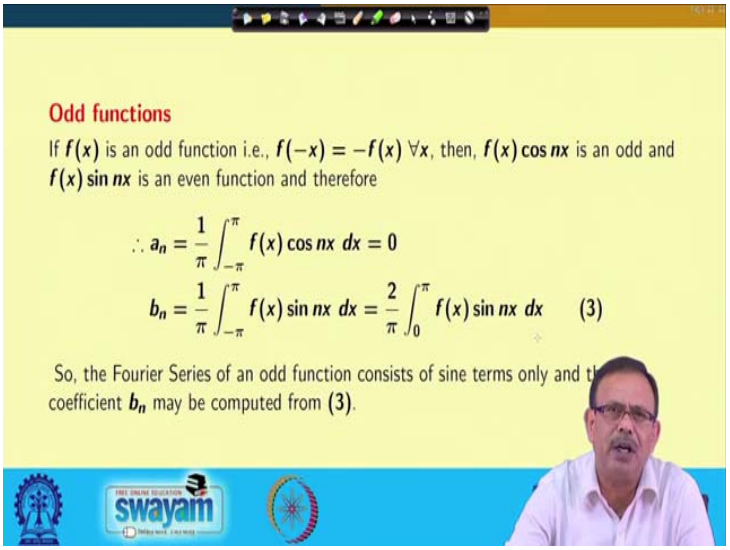

So, for even and odd functions, we can calculate it like this.

Let us consider one example. We want to find out the Fourier series for a periodic function $|x|$ of period 2π. Afterwards, we also need to compute the values of the series at $0, 2\pi$.

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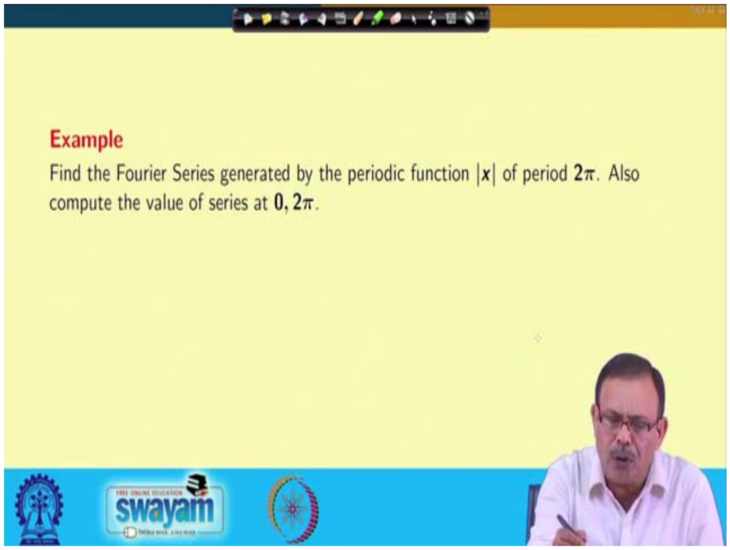

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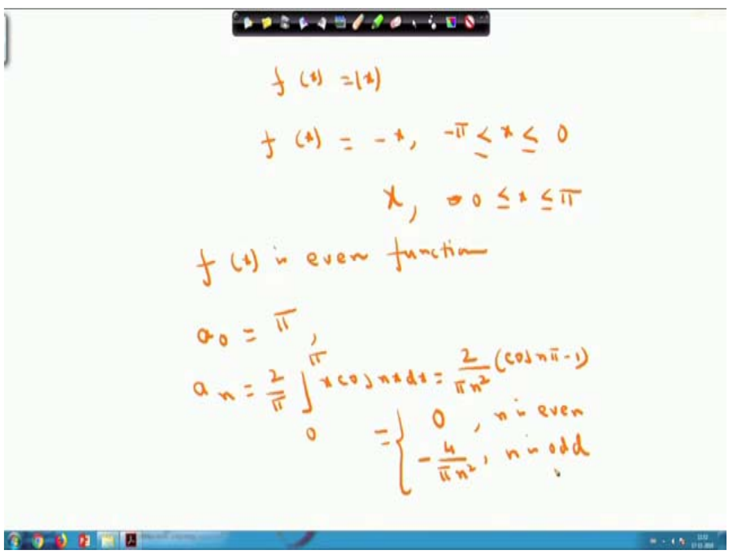

Given function can be represented as

$$f(x) = \begin{cases} -x\ , & -\pi \le x \le 0 \\ \ \ x\ , & \ \ 0 \le x \le \pi \end{cases}$$

So, $f(x)$ is an even function. Therefore, once we know that $f(x)$ is an even function, then we know even function is expressed in terms of cosine series only. So, we have to calculate only a_0 and a_n as follows:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} 0 & , \quad \text{when n is even} \\ -\dfrac{4}{\pi n^2} & , \quad \text{when n is odd} \end{cases}$$

So, this value will vary depending upon the value of n.

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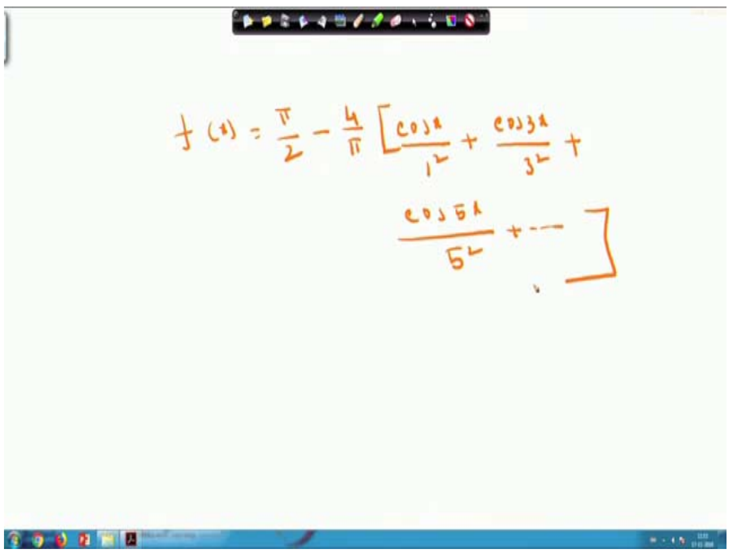

So, we can write $f(x)$ as

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

At $x = 0$, we have,

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Similarly, at $x = 2\pi$, we have,

$$f(2\pi) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right].$$

So, please note that, at the beginning itself, before finding the Fourier series of a function $f(x)$, if we check whether $f(x)$ is an even function or an odd function, in that case, our calculation burden will be reduced a lot. This is because if it is even function, it will have only the cosine terms in the Fourier series or in other sense, we will have to evaluate only

a_0 and a_n. Whereas, if the function is odd function, in that case, it will have the sine terms only and we will need to evaluate only b_n, that we will see in the next lecture. Thank you.