Transform Calculus and Its Application in Differential Equations
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Lecture – 02
Existence of Laplace Transform

In this lecture, we will see how functions of exponential order are related to Laplace transform.

Let us first see the existence of Laplace transform.

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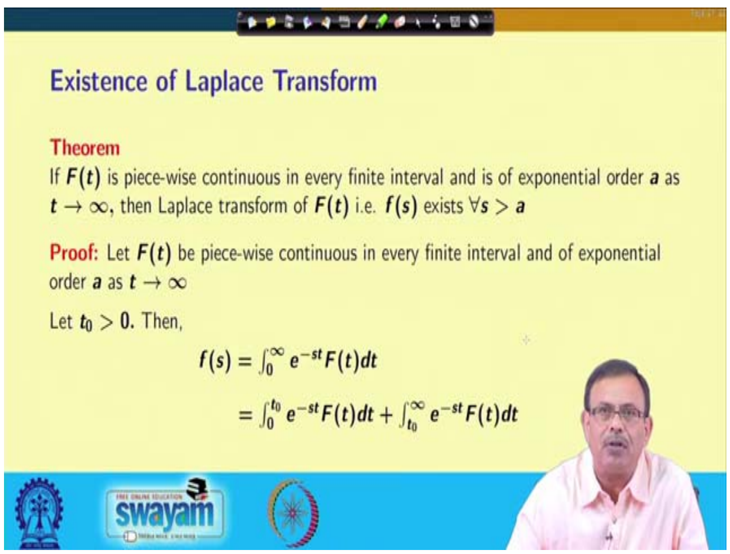

This means, if we consider any function, whether or not the Laplace transform of that function will exist. For that, we have a theorem which says that if $F(t)$ is a piecewise continuous function in every finite interval and is of exponential order a as $t \to \infty$, then Laplace transform of $F(t)$ i.e., $f(s)$ will exist for all $s > a$.

So, we observe two conditions basically: firstly, the function $F(t)$ has to be piecewise continuous in every finite interval and secondly, it should be of exponential order a, then the Laplace transform of $F(t)$ exists and is equal to $f(s)$ $\forall$ $s > a$. Now let us see the proof of this theorem.

It is given that $F(t)$ is a piecewise continuous function in every finite interval and is of exponential order a.

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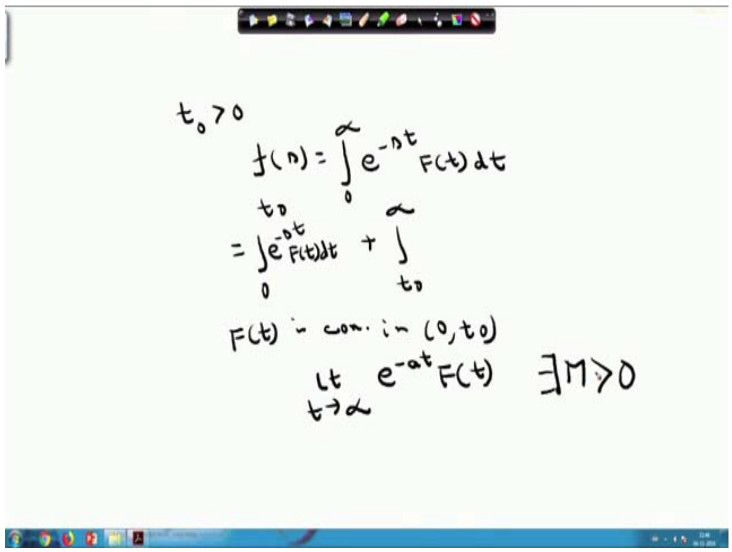

We take $t_0 > 0$, then from the definition of Laplace transform, we have,

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt.$$

We can break $[0,\infty)$ into two intervals namely $[0,t_0]$ and $[t_0,\infty]$ as follows:

$$f(s) = \int_0^{t_0} e^{-st} F(t)\, dt + \int_{t_0}^\infty e^{-st} F(t)\, dt.$$

Now, since $F(t)$ is a piecewise continuous function, so by using continuity of $F(t)$ in $[0,t_0]$, we can say that the first integral will exist. Again $F(t)$ is of exponential order a as $t \to \infty$. This implies

$$\lim_{t\to\infty} e^{-at} F(t)$$

is finite and there will exist some real number $M > 0$ such that the following relation holds:

$$|F(t)| < Me^{at} \quad \forall \quad t > t_0. \qquad (1)$$

Now,

$$\left|\int_{t_0}^{\infty} e^{-st}F(t)\,dt\right| \leq \int_{t_0}^{\infty}|e^{-st}F(t)|\,dt = \int_{t_0}^{\infty} e^{-st}|F(t)|\,dt$$

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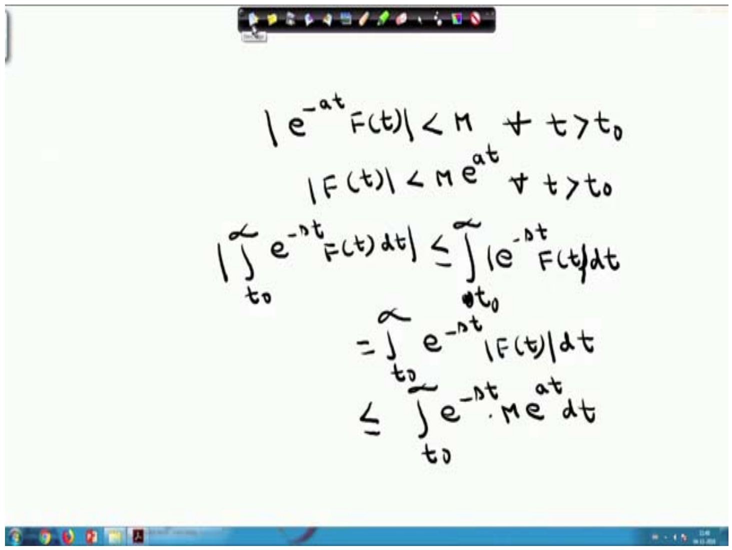

Therefore, using (1), we have,

$$\left|\int_{t_0}^{\infty} e^{-st}F(t)\,dt\right| \leq \int_{t_0}^{\infty} e^{-st}|F(t)|\,dt < \int_{t_0}^{\infty} e^{-st}Me^{at}\,dt$$

This integral can be easily evaluated to obtain the following:

$$\left|\int_{t_0}^{\infty} e^{-st}F(t)\,dt\right| < \int_{t_0}^{\infty} e^{-st}Me^{at}\,dt$$

$$= M\int_{t_0}^{\infty} e^{-(s-a)t}\,dt$$

$$= M\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=t_0}^{\infty}$$

$$= M\left[0 + \frac{e^{-(s-a)t_0}}{(s-a)}\right]$$

$$\Rightarrow \left|\int_{t_0}^{\infty} e^{-st}F(t)\,dt\right| < \frac{Me^{-(s-a)t_0}}{s-a}, \qquad s > a.$$

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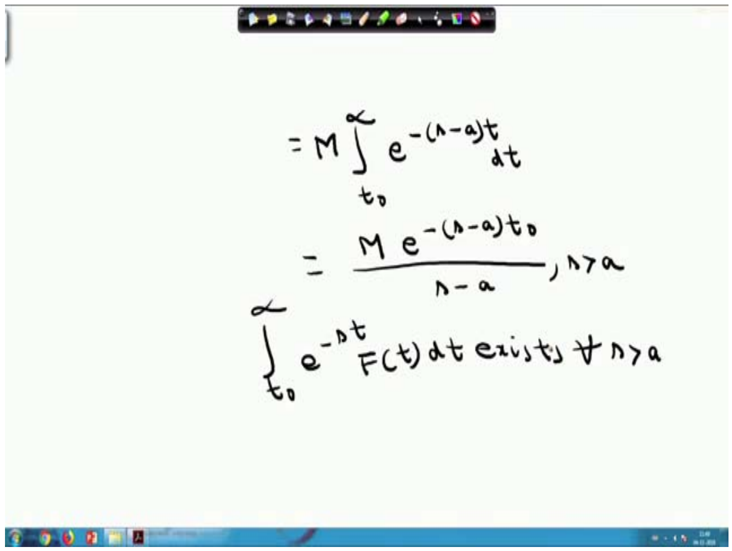

Now, $\frac{Me^{-(s-a)t_0}}{s-a}$ can be made as small as possible by choosing t_0 sufficiently large i.e., if we select a t_0 sufficiently large, then we can make this whole quantity, $\frac{Me^{-(s-a)t_0}}{s-a}$, as small as possible. Thus we can say that the integral $\int_{t_0}^{\infty} e^{-st}F(t)\,dt$ exists for all $s > a$.

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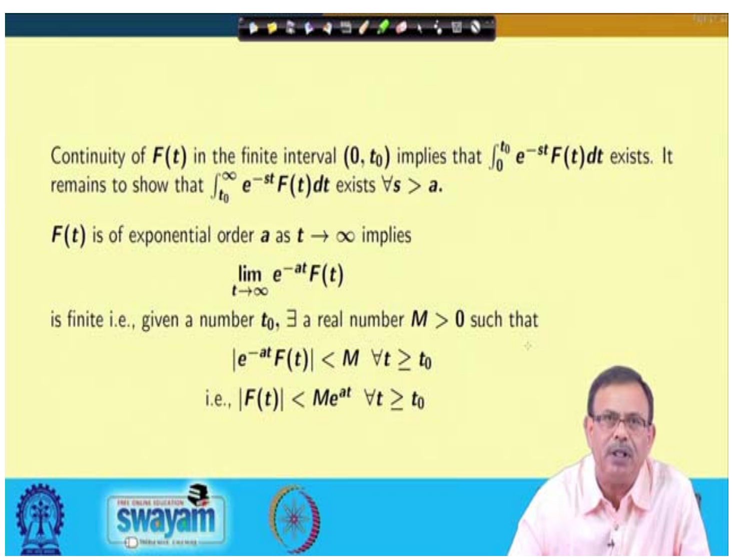

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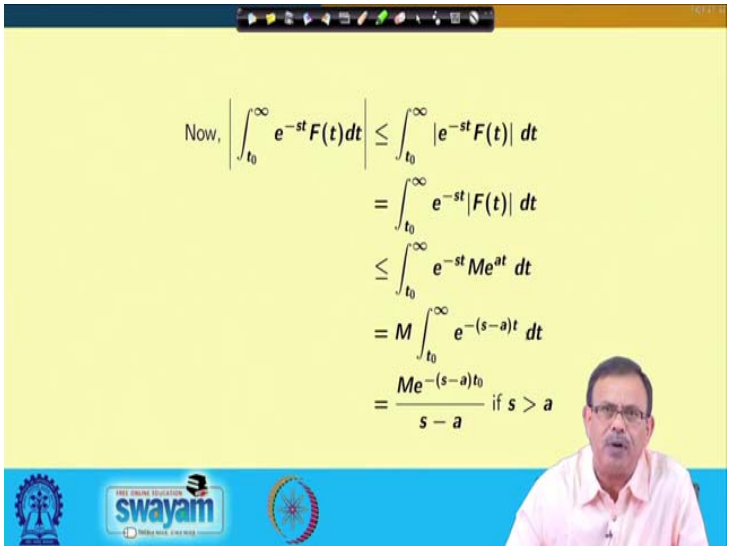

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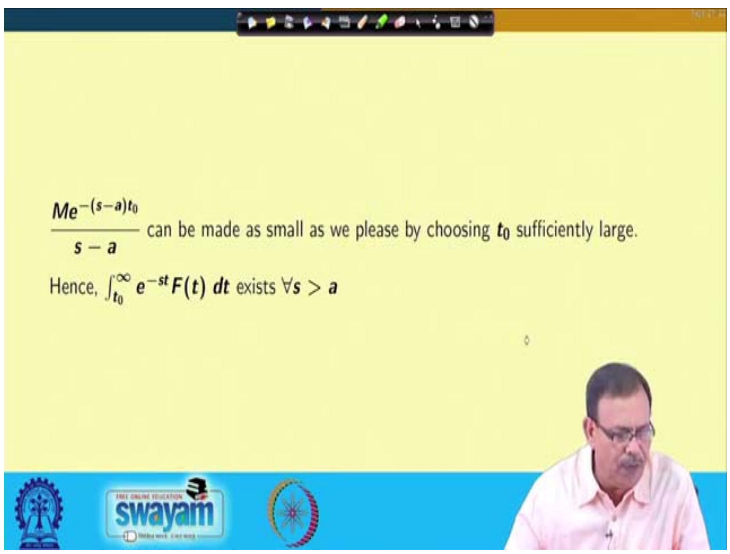

So, this completes the proof that the Laplace transform of a function will exist, if the function is piecewise continuous and is of exponential order.

Now we give a general definition here of a function of class A.

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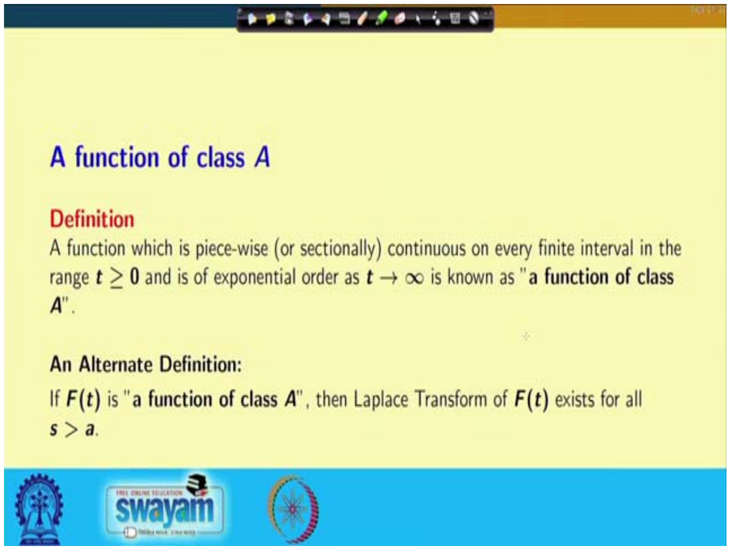

A function $F(t)$, which is piecewise continuous on every finite interval in the range $t \geq 0$ and is of exponential order as $t \to \infty$, then it is known as function of class A. Again, from the given conditions, we can also say that the Laplace transform of $F(t)$ exists.

Now, we give an alternative proof for the existence of Laplace transform.

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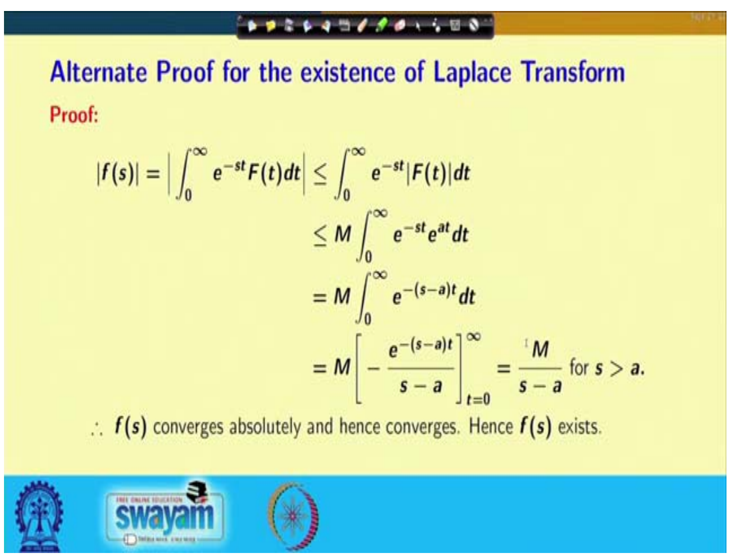

We can start from,

$$|f(s)| = \left|\int_0^\infty e^{-st}F(t)\,dt\right| \le \int_0^\infty |e^{-st}F(t)|\,dt = \int_0^\infty e^{-st}|F(t)|\,dt$$

Now using $|F(t)| \le Me^{at}$, we have

$$|f(s)| \le M\int_0^\infty e^{-st}e^{at}\,dt$$

The RHS can be easily integrated as follows:

$$|f(s)| \le M\int_0^\infty e^{-(s-a)t}\,dt$$

$$= M\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=0}^{\infty}$$

$$= M\left[0 + \frac{1}{s-a}\right]$$

$$= \frac{M}{s-a} \text{ for } s > a. \qquad (2)$$

Therefore, we can say $f(s)$ converges absolutely and hence converges so that $f(s)$ exists.

There is a remark on this.

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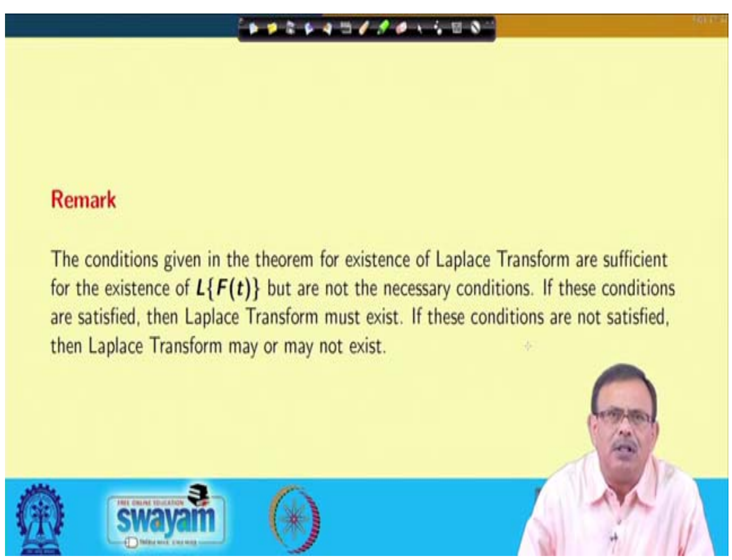

The conditions given in the theorem for existence of Laplace transform are sufficient for the existence but not necessary which means that if these conditions are satisfied by a function $F(t)$, then its Laplace transform will exist, but if these conditions are not satisfied, then $L\{F(t)\}$ may or may not exist.

Let us take one example that will support this stated remark.

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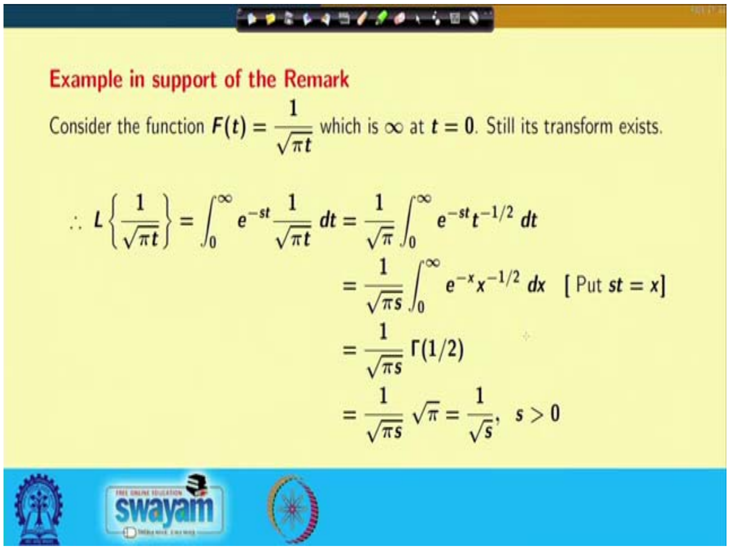

Consider $F(t) = \frac{1}{\sqrt{\pi t}}$, which tends to ∞ at $t = 0$. From the function itself, it is clear that at $t = 0$, the function does not exist. But we will show that its transform always exists. Now, from definition,

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \int_0^\infty e^{-st}\frac{1}{\sqrt{\pi t}}\,dt.$$

which can be written in a simplified way as:

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-st}t^{-1/2}\,dt.$$

In order to integrate the above, we put $st = x$ so that $dt = \frac{1}{s}dx$ and the limits of integration remain unchanged. We have now

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{\pi s}} \int_0^\infty e^{-x} x^{-1/2}\, dx.$$

Now, as we know $\int_0^\infty e^{-x} x^{-1/2}\, dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Therefore,

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}\,, \qquad s > 0$$

So, whenever we have a function $F(t) = \frac{1}{\sqrt{\pi t}}$ which is neither piecewise continuous nor of exponential order (since at $t = 0$, value of the function is ∞), still its Laplace transform exists.

Therefore, the conditions that we have listed for existence of Laplace transform are only sufficient but not necessary.

Now, let us try to find out the Laplace transform of certain well-known functions.

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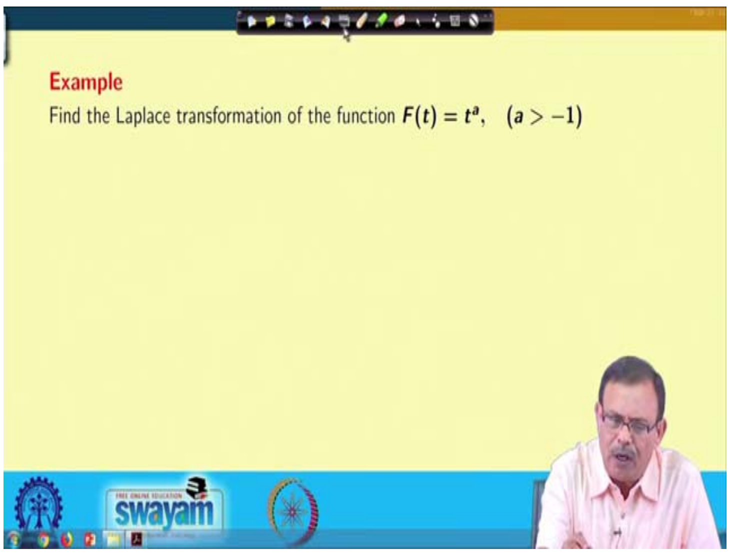

Suppose we need to find out the Laplace Transform of the function $F(t) = t^a$, with $a > -1$.

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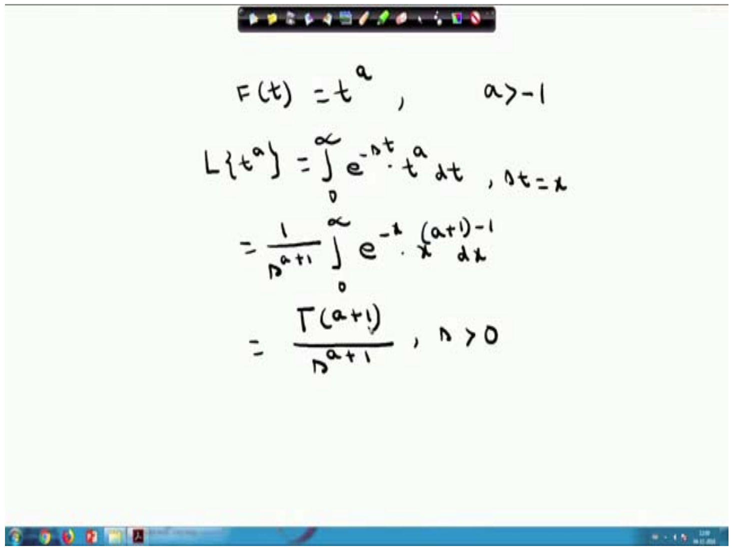

So from definition,

$$L\{F(t)\} = L\{t^a\} = \int_0^\infty e^{-st} t^a \, dt.$$

If we put $st = x$, then we have $dt = \frac{1}{s} dx$ and the limits of the integration will remain unchanged. This reduces the integral to

$$L\{t^a\} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, dx$$

which can be evaluated using Gamma function

$$L\{t^a\} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^{(a+1)-1} \, dx$$

$$= \frac{\Gamma(a+1)}{s^{a+1}}, \qquad s > 0.$$

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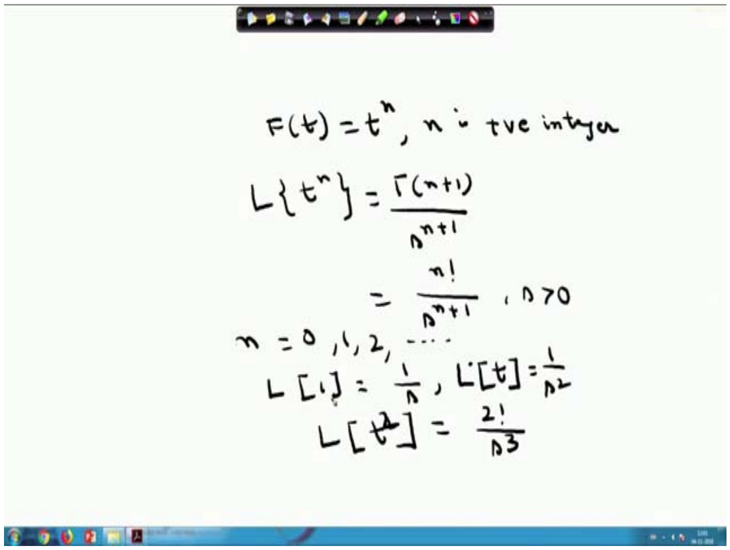

In a similar manner, we can consider $F(t) = t^n$, where n is a positive integer. We can tell that Laplace transform of t^n is equal to $\frac{\Gamma(n+1)}{s^{n+1}}$ and we know, $\Gamma(n+1) = n!$. So, if we have a function $F(t) = t^n$, where n is positive integer, then its Laplace transform is given by

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, \qquad s > 0$$

Now we can take $n = 0,1,2, \ldots$ and we can get the Laplace Transforms of various functions as follows:

$$for\ n = 0, \qquad L\{t^0\} = L\{1\} = \frac{\Gamma(0+1)}{s^{0+1}} = \frac{1}{s}, \qquad s > 0$$

$$for\ n = 1, \qquad L\{t^1\} = L\{t\} = \frac{\Gamma(1+1)}{s^{1+1}} = \frac{1}{s^2}, \qquad s > 0$$

$$for\ n = 2, \qquad L\{t^2\} = \frac{\Gamma(2+1)}{s^{2+1}} = \frac{2}{s^3}, \qquad s > 0.$$

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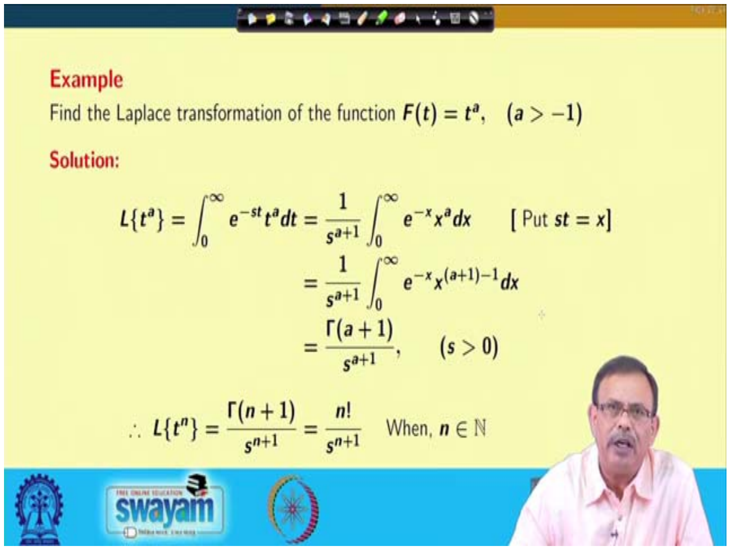

The next example is to find the Laplace transform of the function $F(t) = e^{at}$.

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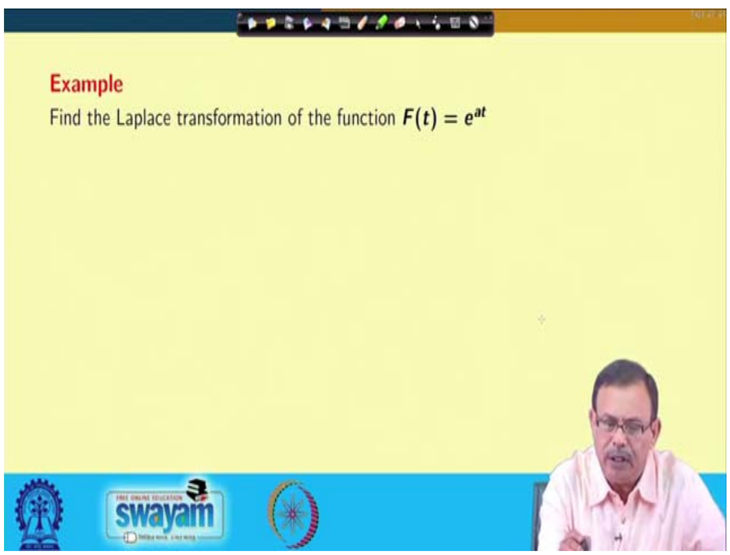

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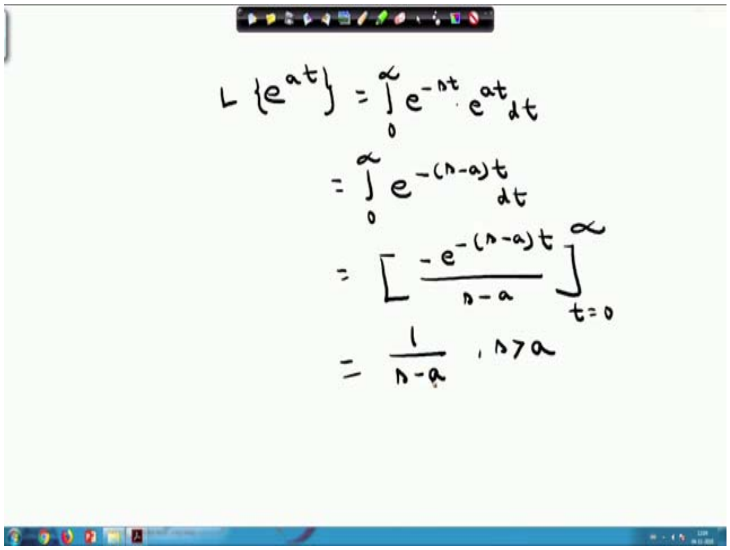

So, Laplace transform of e^{at} equals $\int_0^\infty e^{-st} e^{at}\, dt$ (by definition). This can be written as:

$$L\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\, dt.$$

And if we calculate the value of this integral, the result is obtained as

$$L\{e^{at}\} = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=0}^{\infty}$$

$$= \frac{1}{s-a}, \qquad s > a.$$

Therefore, we can find out the Laplace transform of e^{at} for any given value of a from the above formula.

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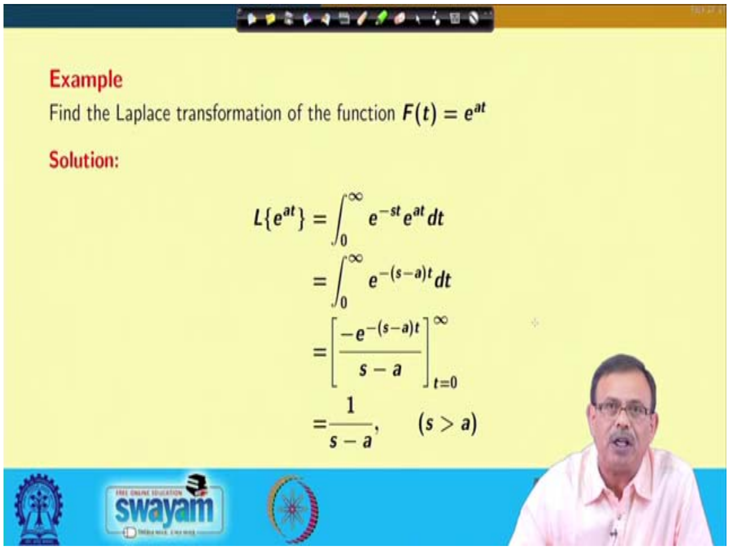

Now, let us move to the next example.

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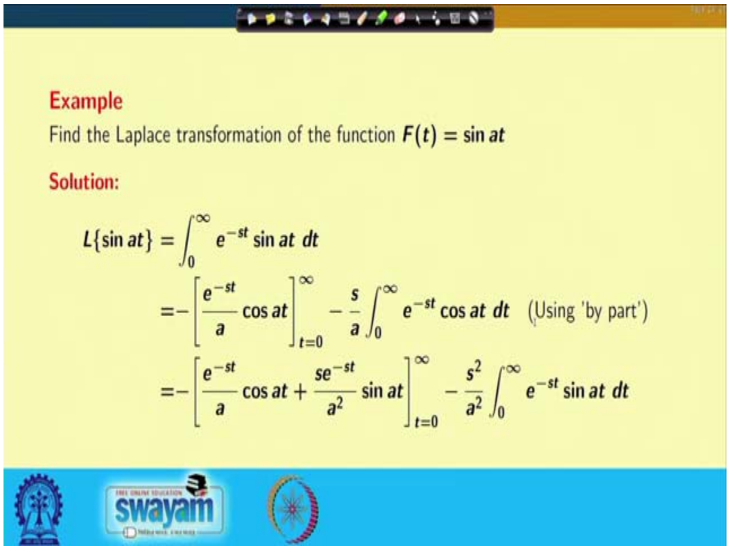

We have $F(t) = \sin at$. Again, by using the definition of Laplace Transform, we can obtain the desired result as follows:

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \; dt.$$

We can use integration by parts twice to evaluate the above integral. Therefore,

$$\begin{aligned}
L\{\sin at\} &= \int_0^\infty e^{-st} \sin at \; dt \\
&= -\left[\frac{e^{-st}}{a} \cos at\right]_{t=0}^{\infty} - \frac{s}{a}\int_0^\infty e^{-st} \cos at \; dt \\
&= -\left[0 - \frac{1}{a}\right] - \frac{s}{a}\left[\left[\frac{e^{-st}}{a} \sin at\right]_{t=0}^{\infty} + \frac{s}{a}\int_0^\infty e^{-st} \sin at \; dt\right] \\
&= \frac{1}{a} - \frac{s}{a}\left[0 - 0 + \frac{s}{a} L\{\sin at\}\right] \\
&= \frac{1}{a} - \frac{s^2}{a^2} L\{\sin at\}
\end{aligned}$$

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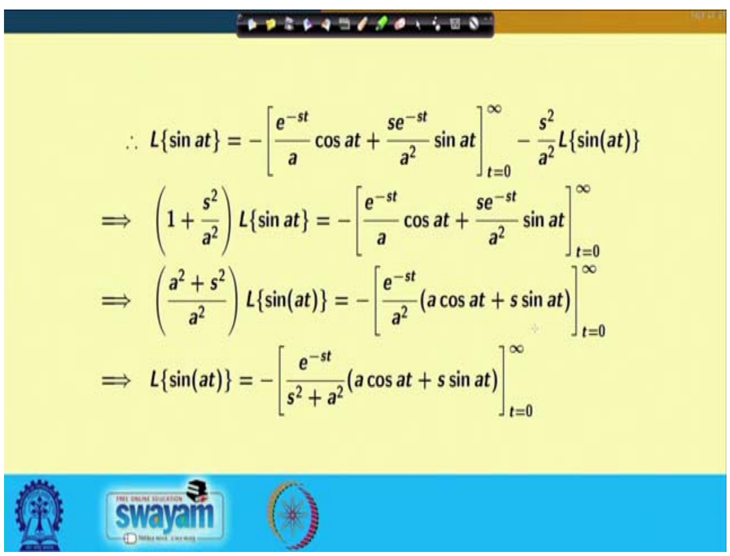

Clearly, we observe that after applying integration by parts twice, $L\{\sin at\}$ is obtained on the right hand side as well.

$$L\{\sin at\} = \frac{1}{a} - \frac{s^2}{a^2} L\{\sin at\}$$

$$\Rightarrow \left(1+\frac{s^2}{a^2}\right) L\{\sin at\} = \frac{1}{a}$$

$$\Rightarrow L\{\sin at\} = \frac{1}{a}\left(\frac{a^2}{s^2+a^2}\right)$$

$$\Rightarrow L\{\sin at\} = \frac{a}{s^2+a^2}, \quad s > 0.$$

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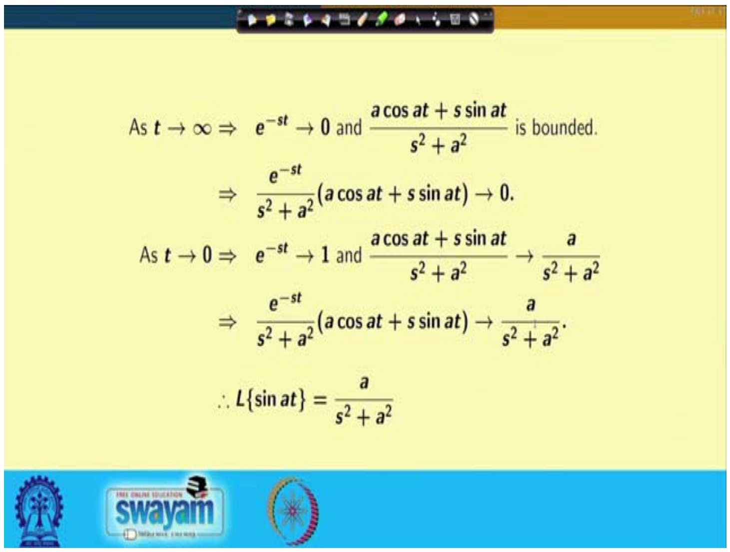

So, using the obtained result, we can calculate the Laplace transform of $\sin at$ for any given value of a.

Next we come to the Laplace transform of the function $\cos at$.

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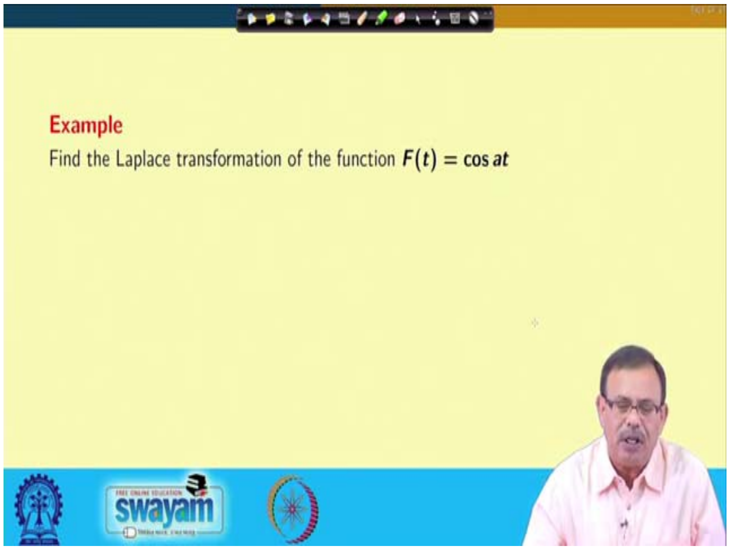

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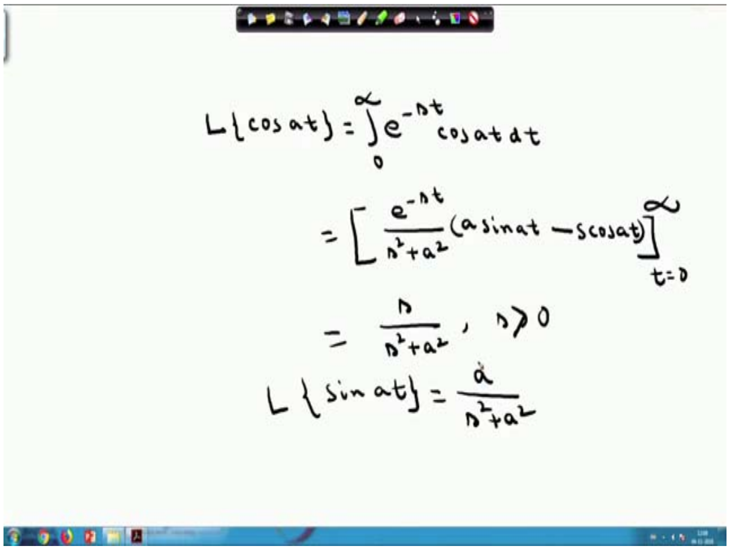

Similar to the previous problem, we use the definition of Laplace Transform to evaluate $L\{\cos at\}$. Therefore,

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at \ dt.$$

In order to evaluate this, we can also directly apply the integration formula for $\int e^{ax} \cos bx \ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$. So, we obtain the following:

$$L\{\cos at\} = \left[\frac{e^{-st}}{s^2 + a^2}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= 0 - \frac{1}{s^2+a^2}(-s)$$

$$= \frac{s}{s^2 + a^2}, \qquad s > 0.$$

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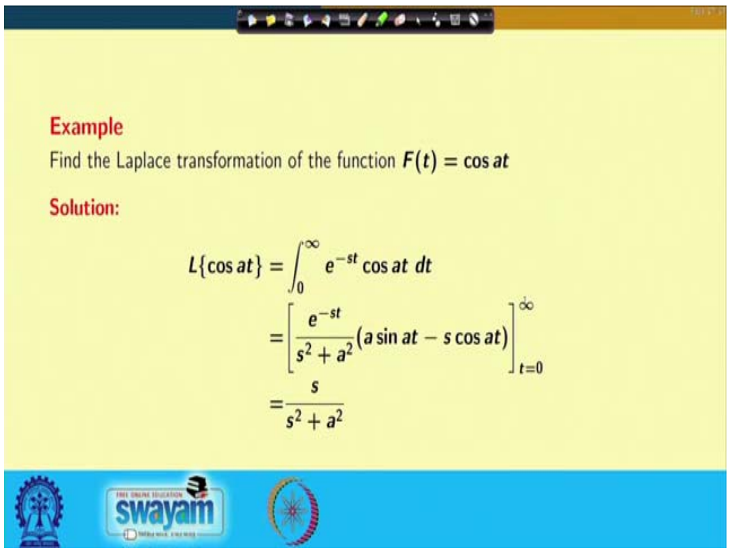

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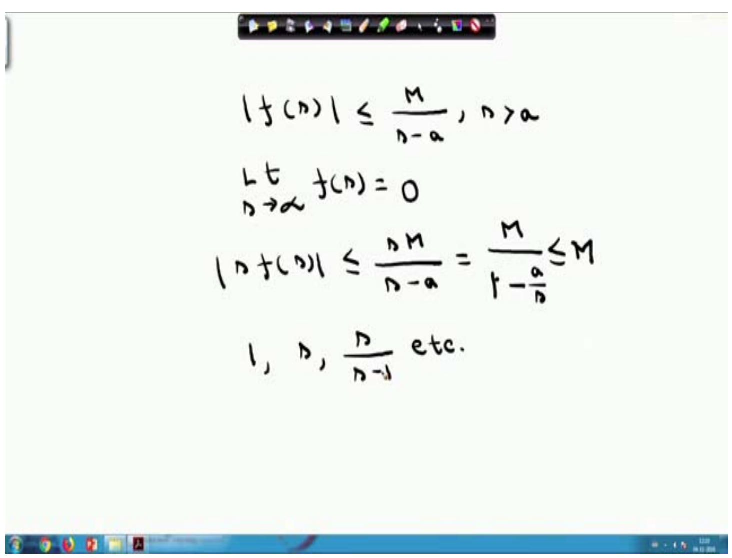

As already proved, from (2), we have $\frac{M}{s-a}, s > a$. Therefore, whenever we take $\lim_{s\to\infty} f(s)$, this value should always be equal to 0 since M is finite. So, we obtain

$$\lim_{s\to\infty} f(s) = \lim_{s\to\infty} L\{F(t)\} = 0. \qquad (3)$$

Now, from (2), we have,

$$|f(s)| \le \frac{M}{s-a}, \qquad s > a$$

$$\Rightarrow |sf(s)| \le \frac{sM}{s-a}$$

$$\le \frac{M}{1-\frac{a}{s}}$$

$$\le M \qquad \text{for sufficiently large } M.$$

From here, we can conclude that there are functions like $1,\ s,\ \frac{s}{s-1}$ which can never be the Laplace transform of any function. This is because, as $s \to \infty$, these functions will never approach 0 as is derived in (3). In short, if a function $f(s)$ is the Laplace Transform of any function $F(t)$, then it should tend to 0 as $s \to \infty$ i.e., (3) should be satisfied.

Let us now consider the unit step function.

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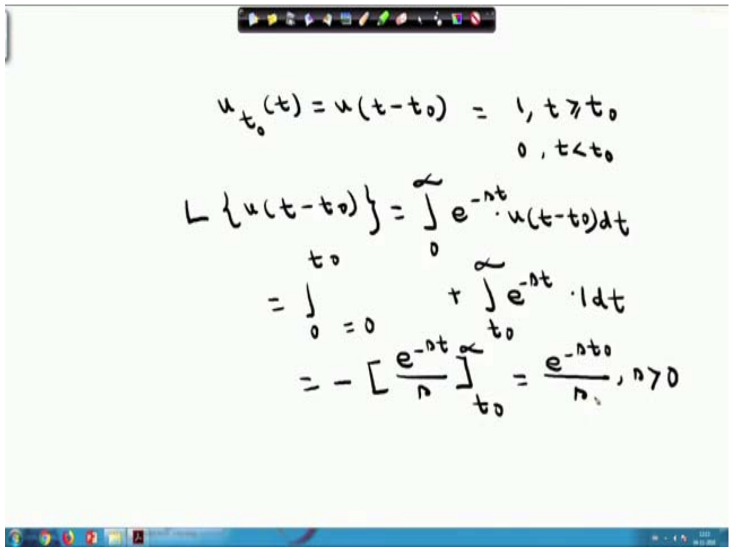

Unit step function is denoted in various ways as $u_{t_0}(t)$ or sometimes as $u(t-t_0)$ and is defined as

$$u_{t_0}(t) = u(t-t_0) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < t_0 \end{cases}$$

This is not a continuous function, instead it is piecewise continuous.

If we try to find out the Laplace transform of $u(t-t_0)$, we will use the definition again.

$$L\{u(t-t_0)\} = \int_0^\infty e^{-st}\, u(t-t_0)\, dt$$

According to the definition of the unit step function, we can break this integral into two parts, one from 0 to t_0 and the other from t_0 to ∞ as follows:

$$L\{u(t-t_0)\} = \int_0^{t_0} e^{-st}.0\, dt + \int_{t_0}^\infty e^{-st}.1\, dt$$

$$= \int_{t_0}^\infty e^{-st}\, dt.$$

This integral can be easily solved to obtain the following result:

$$L\{u(t-t_0)\} = \left[\frac{e^{-st}}{-s}\right]_{t=t_0}^{\infty} = \frac{e^{-st_0}}{s}, \qquad s > 0.$$

Therefore, the Laplace transform of unit step function is $\frac{e^{-st_0}}{s}$.

Next, we have to evaluate the Laplace Transform of cosh at and sinh at.

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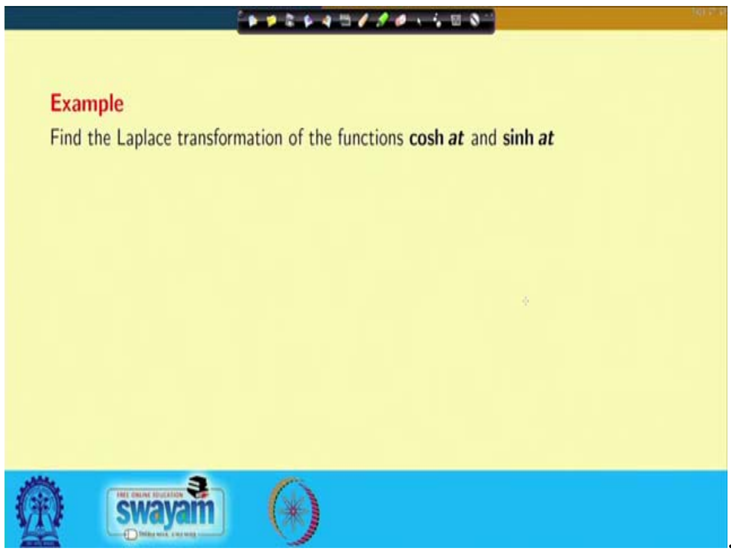

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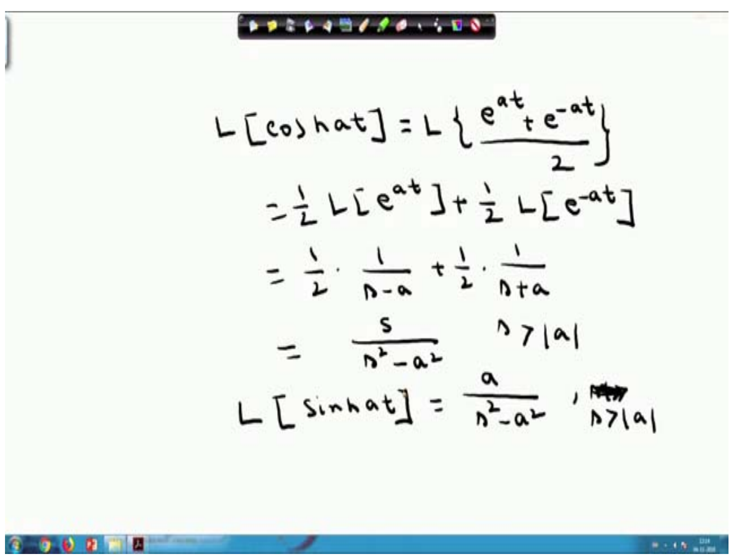

As we are familiar with the formulas of $\cosh at$ and $\sinh at$, therefore, we proceed as below:

$$L\{cosh\ at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \tfrac{1}{2}L\{e^{at}\} + \tfrac{1}{2}L\{e^{-at}\} \qquad \text{(using linearity property)}$$

$$= \frac{1}{2(s-a)} + \frac{1}{2(s+a)}$$

$$= \frac{s}{s^2 - a^2}, \qquad s > |a|.$$

Therefore, Laplace transform of $cosh\ at$ equals $\frac{s}{s^2-a^2}$ and in a similar fashion, we can evaluate the Laplace transform of $\sinh at$ as well.

$$L\{sinh\ at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)}$$

$$= \frac{a}{s^2 - a^2}, \qquad s > |a|.$$

Therefore, Laplace transform of $\sinh at$ equals $\frac{a}{s^2-a^2}$.

Thank you.