Transform Calculus and Its Applications in Differential Equations

Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 19
Introduction to Fourier Series

Welcome all of you. In this particular lecture, we are going to start the Fourier series. In the earlier lectures, we have done the Laplace transform, inverse Laplace transform, convolution theorem and the solution of ordinary differential equations using Laplace transform.

Now, we are going to start a new topic that is Fourier series.

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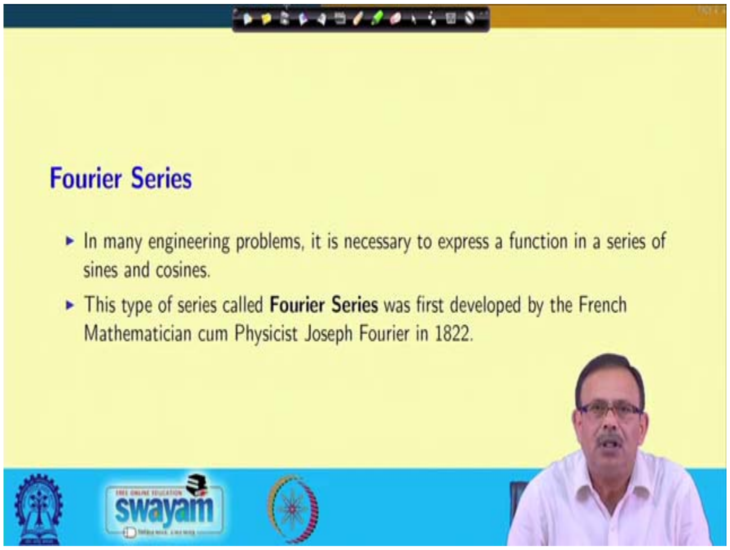

What is a Fourier series? In many engineering problems, it is necessary to express a function as a series of sine and cosine functions. If a function $f(x)$ is given to us, we can express it in terms of sine series and cosine series, that series is called the Fourier series.

Fourier series was first developed by French Mathematician cum Physicist Joseph Fourier in 1822.

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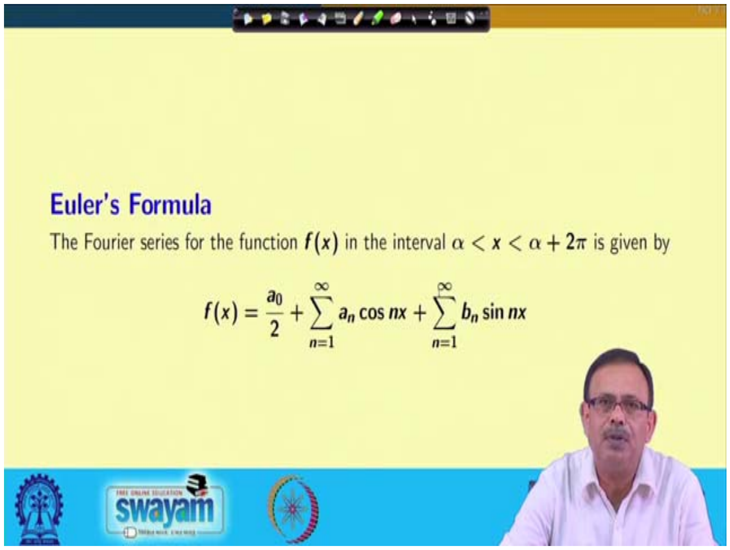

Now for the Fourier series, we study a formula which we call the Euler's formula. The Fourier series of a function $f(x)$ in the interval $(\alpha, \alpha + 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

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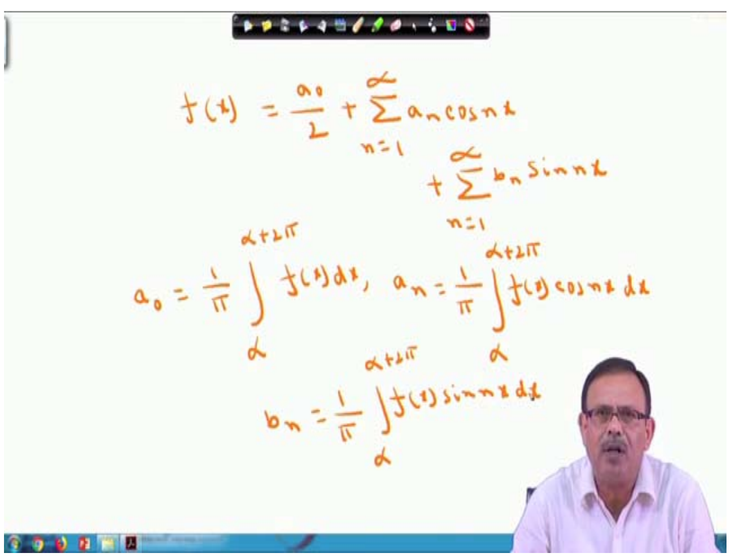

where the coefficients a_0, a_n and b_n are defined as

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx.$$

a_0, a_n and b_n are known as Euler's formulae.

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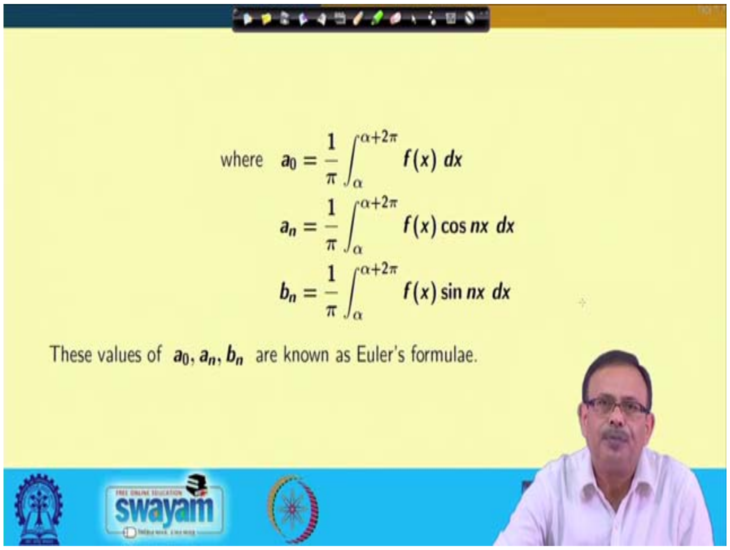

Now, to establish this formula for $f(x)$, we need to know certain standard values of some definite integrals. These are clearly presented in the attached slides.

So, please note these useful values of these definite integrals which we will use frequently afterwards.

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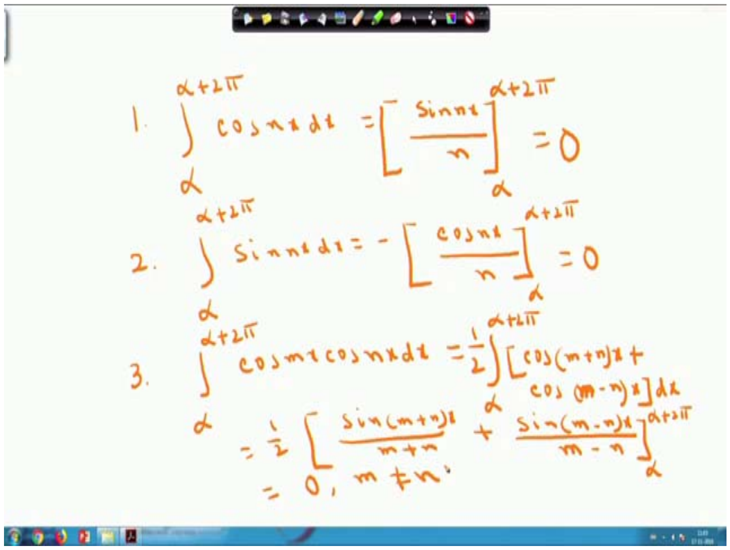

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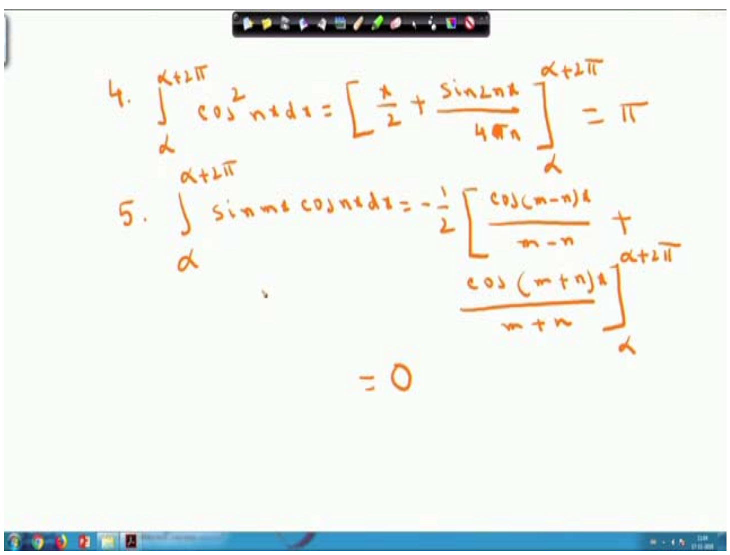

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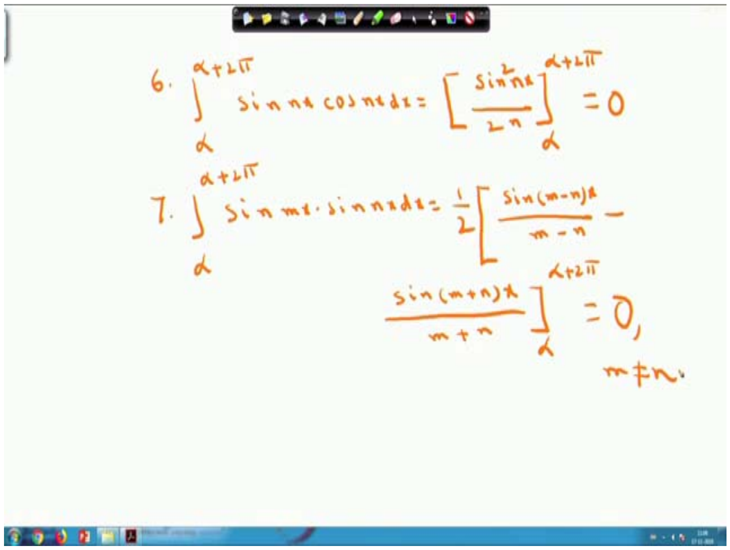

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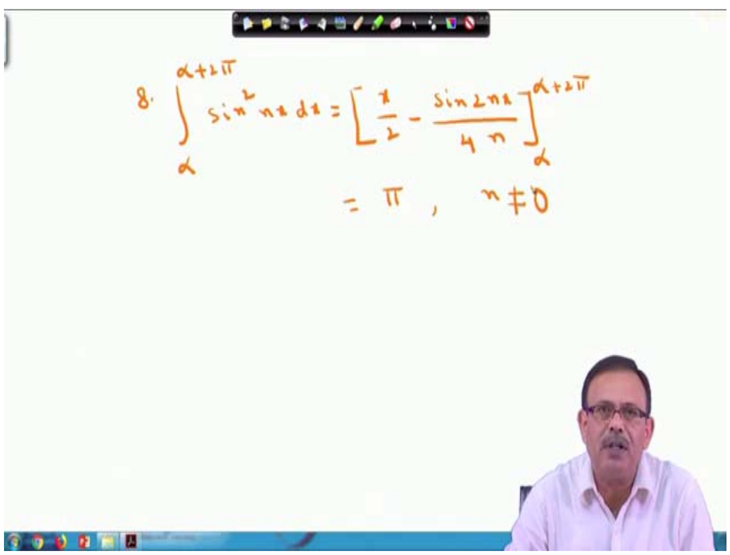

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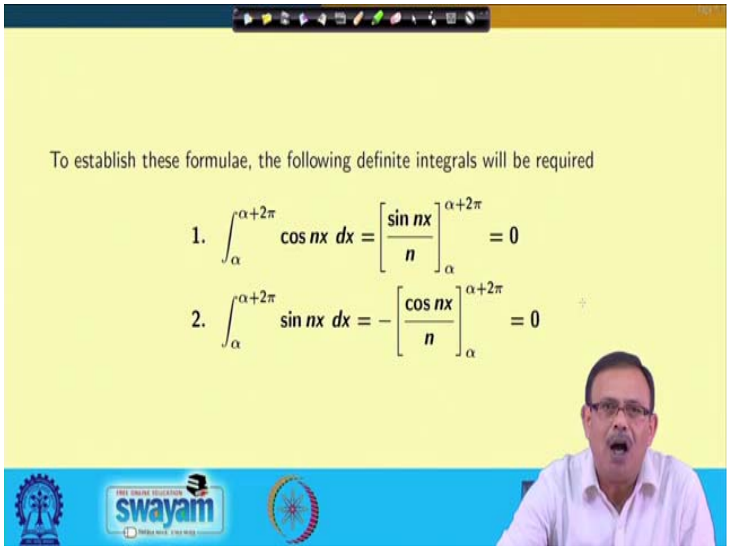

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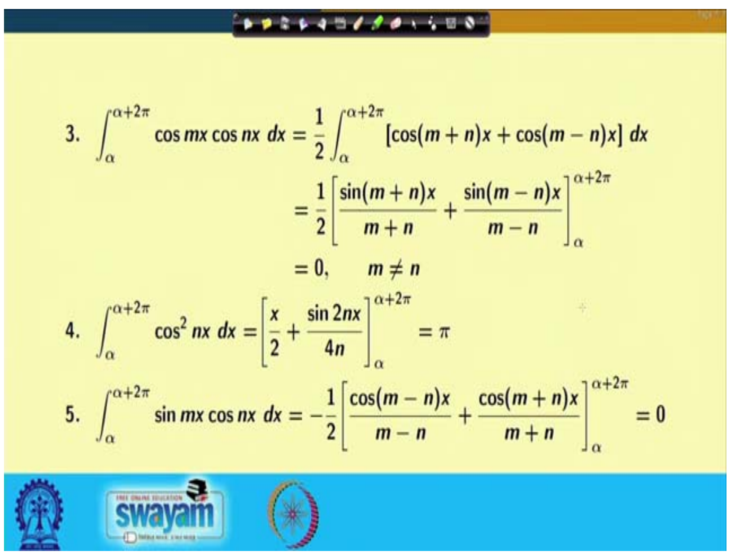

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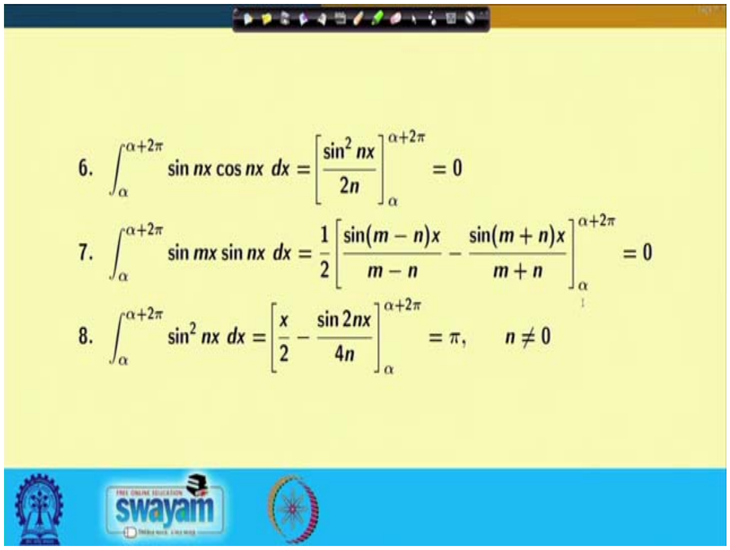

Now, using these integral values, let us establish the formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

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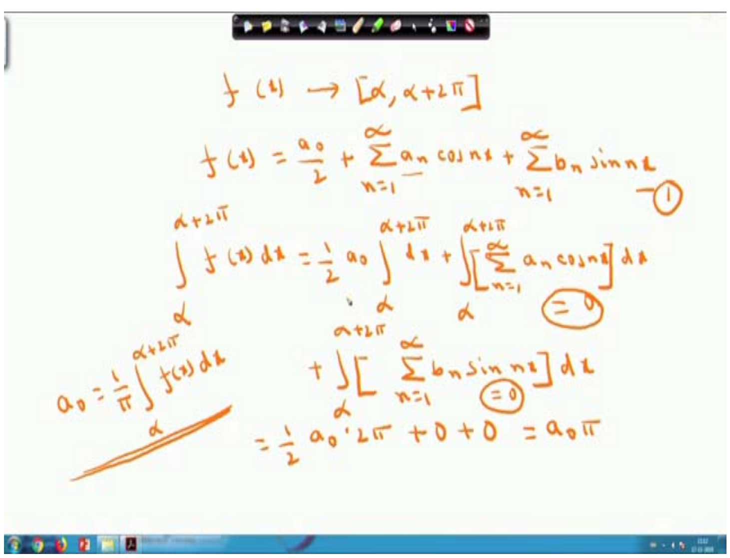

It is given that $f(x)$ is represented as a Fourier series given by (1) in the interval $(\alpha, \alpha + 2\pi)$. So, our aim is to find out the values of the coefficients a_0, a_n and b_n. To find these coefficients from the given series, first we need to find out the value of a_0.

So, we will integrate both sides of (1) from $x = \alpha$ to $x = \alpha + 2\pi$.

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{1}{2}a_0 \int_{\alpha}^{\alpha+2\pi} dx$$
$$+ \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} a_n \cos nx\right] dx + \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} b_n \sin nx\right] dx.$$

Value of the second integral is 0 using the formula 1. of definite integrals as already discussed in this lecture. For similar reason, by formula 2., we have the value of the third integral as 0. Therefore, we are left with

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{1}{2}a_0 \int_{\alpha}^{\alpha+2\pi} dx = a_0\pi$$
$$\Rightarrow a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx.$$

This gives us the value of the coefficient a_0.

Now, to obtain a_n, first we multiply both sides of equation (1) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. So, we obtain

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\,dx$$
$$= \frac{1}{2}a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx$$
$$+ \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} a_n \cos nx\right] \cos nx\,dx + \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} b_n \sin nx\right] \cos nx\,dx.$$

If we clearly observe each of these integrals, then we can see that different deduced formulae on definite integrals will be applicable in this case. So, by using formulae 1., 3., 4., 5. and 6., we obtain,

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\,dx = a_n\pi$$
$$\therefore a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\,dx.$$

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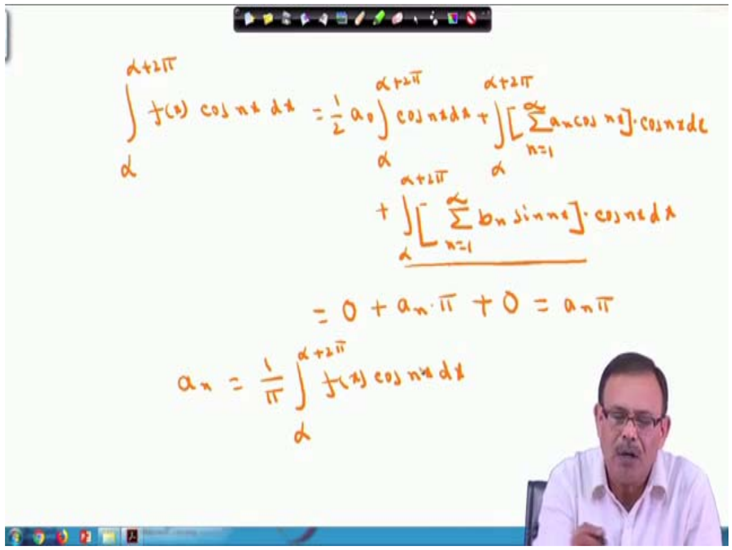

In the same way, if we want to find out the value of the coefficient b_n, we will multiply the equation (1) on both sides with $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$ to obtain

$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} a_n \cos nx \right] \sin nx \, dx + \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} b_n \sin nx \right] \sin nx \, dx.$$

If we clearly observe each of these integrals, then we can see that different deduced formulae on definite integrals will be applicable in this case. So, by using formulae 2., 5., 6., 7. and 8., we obtain,

$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx = b_n \pi$$

$$\therefore b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx.$$

Thus we have derived the values of all the coefficients involved in the Fourier series expansion of $f(x)$ namely a_0, a_n and b_n.

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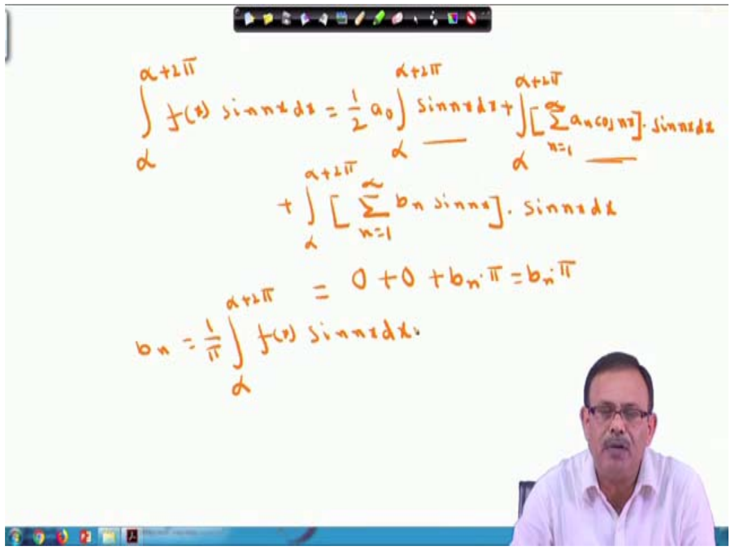

We will now take a look at some special cases that may arise:

Suppose $\alpha = 0$. Then the interval of series expansion reduces to $(0,2\pi)$ and the Euler's formulae take the following form

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

So that whenever we are taking a special case, $\alpha = 0$, then x varies within the range from 0 to 2π.

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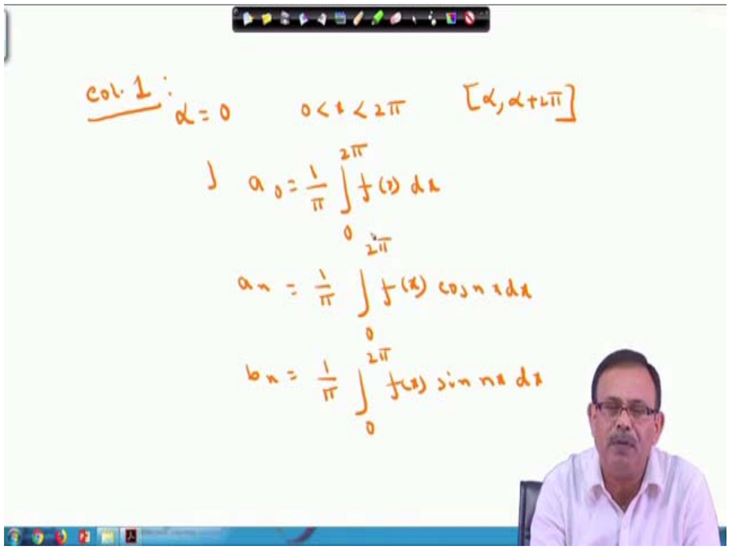

Suppose $\alpha = -\pi$. Then the interval of series expansion reduces to $(-\pi, \pi)$ and the Euler's formulae take the following form

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

So that whenever we are taking a special case, $\alpha = -\pi$, then x varies within the range from $-\pi$ to π.

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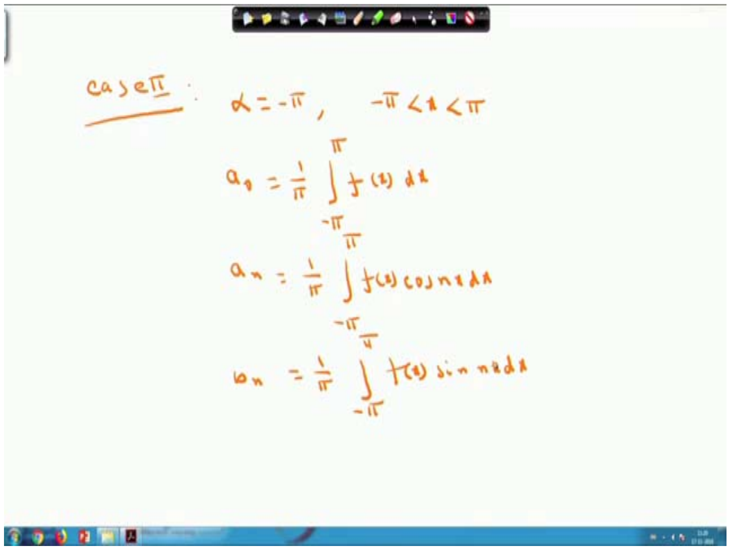

Therefore, we have discussed the generalized case taking α and in this case, the variable x varies within the limits α to $\alpha + 2\pi$. We can also assume the case for $\alpha = 0$, when the range of x will be from 0 to 2π. Similarly, when $\alpha = -\pi$, in that case the range will be $-\pi$ to π. In all these cases, only change will be in the limit of the integration.

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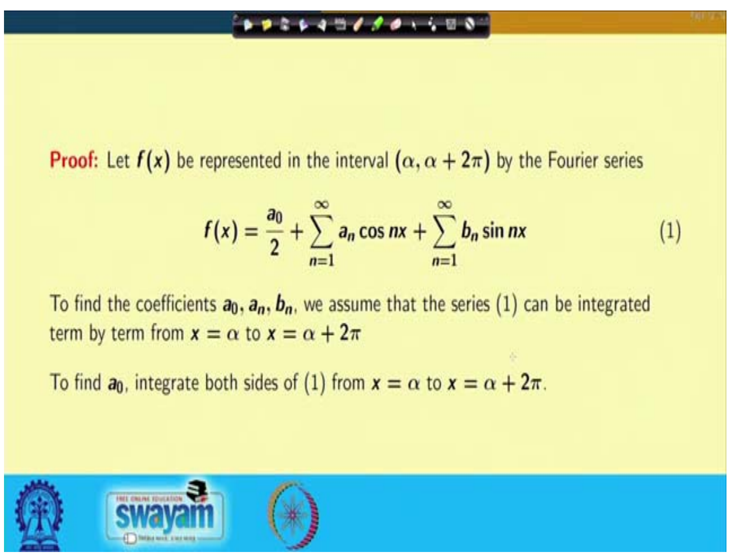

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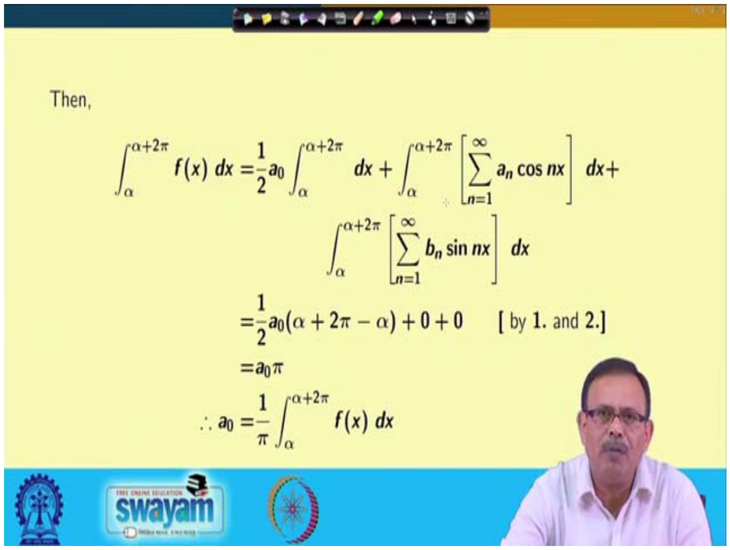

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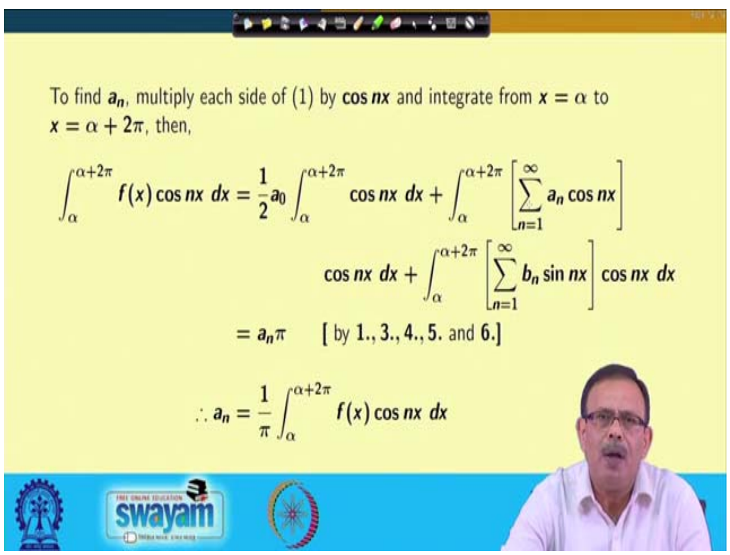

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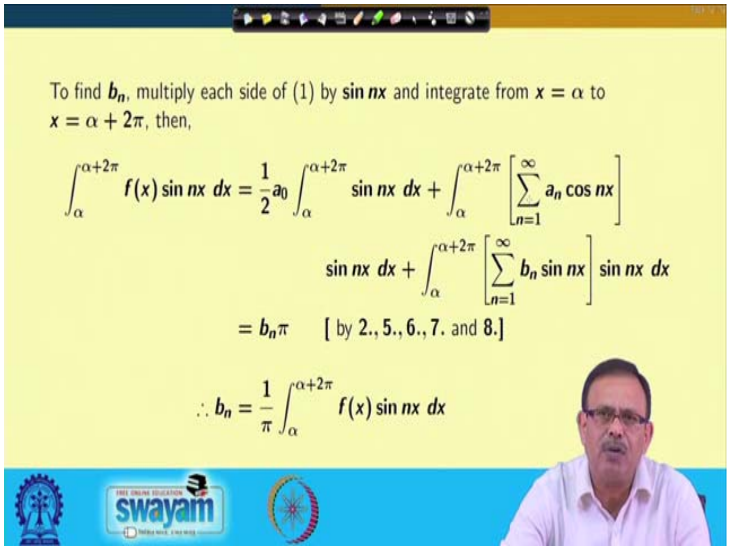

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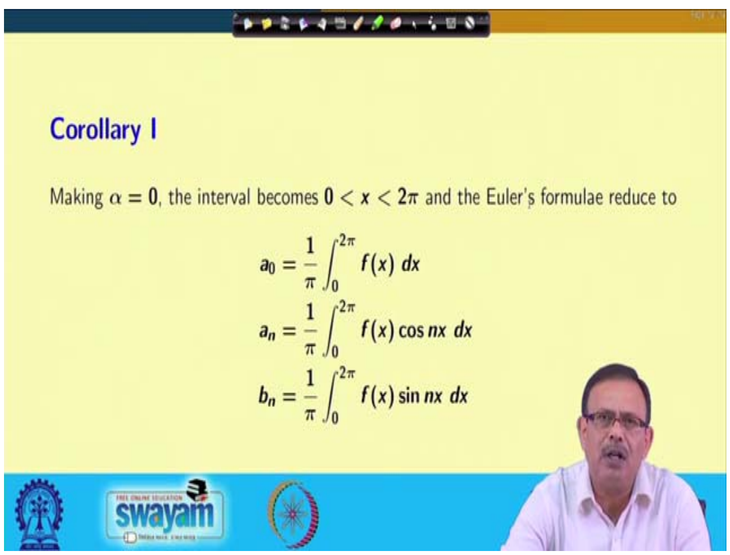

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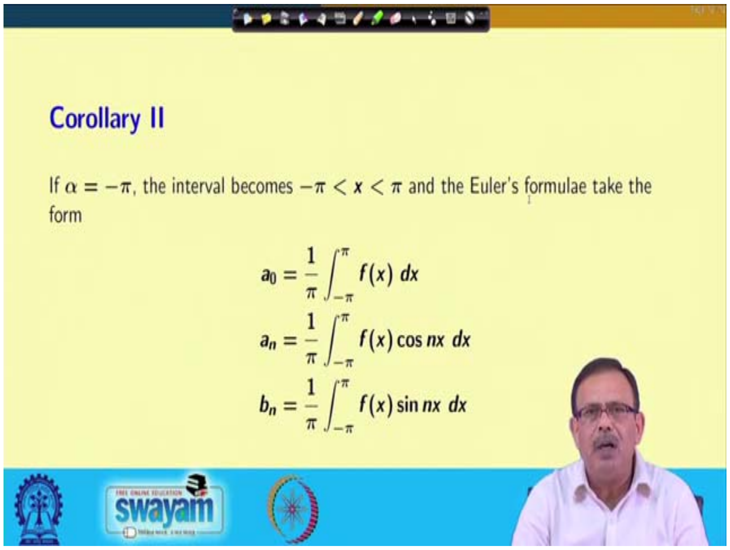

In the next lecture, we will see the conditions for which the Fourier series of a function exists i.e., whether the Fourier series exists for all functions or there are certain criteria which should be satisfied for the existence of the Fourier series. Thank you.