

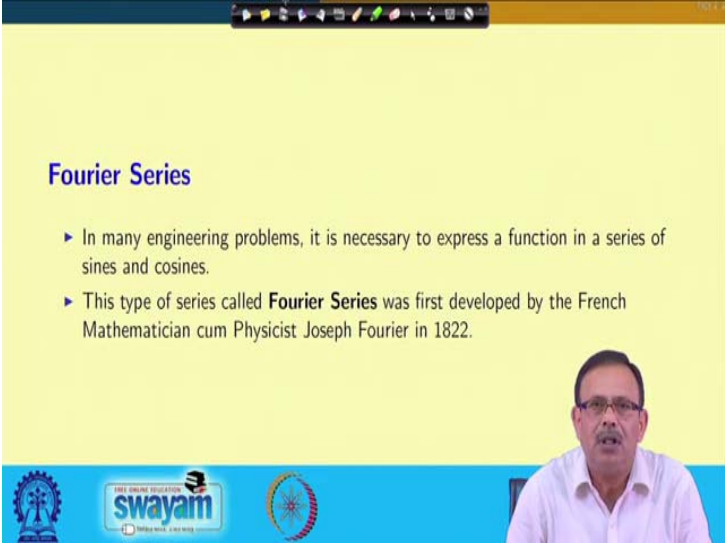
**Transform Calculus and Its Applications in Differential Equations**  
**Prof. Adrijit Goswami**  
**Department of Mathematics**  
**Indian Institute of Technology, Kharagpur**

**Lecture - 19**  
**Introduction to Fourier Series**

Welcome all of you. In this particular lecture, we are going to start the Fourier series. In the earlier lectures, we have done the Laplace transform, inverse Laplace transform, convolution theorem and the solution of ordinary differential equations using Laplace transform.

Now, we are going to start a new topic that is Fourier series.

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The image shows a presentation slide with a yellow background. At the top, there is a dark blue header bar with navigation icons. The slide title is "Fourier Series" in blue text. Below the title, there are two bullet points: "▶ In many engineering problems, it is necessary to express a function in a series of sines and cosines." and "▶ This type of series called **Fourier Series** was first developed by the French Mathematician cum Physicist Joseph Fourier in 1822." In the bottom right corner, there is a small video inset showing a man in a white shirt and glasses. At the bottom of the slide, there is a blue footer bar containing logos for "swayam" and the Indian Institute of Technology, Kharagpur.

What is a Fourier series? In many engineering problems, it is necessary to express a function as a series of sine and cosine functions. If a function  $f(x)$  is given to us, we can express it in terms of sine series and cosine series, that series is called the Fourier series.

Fourier series was first developed by French Mathematician cum Physicist Joseph Fourier in 1822.

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**Euler's Formula**

The Fourier series for the function  $f(x)$  in the interval  $\alpha < x < \alpha + 2\pi$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

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Now for the Fourier series, we study a formula which we call the Euler's formula. The Fourier series of a function  $f(x)$  in the interval  $(\alpha, \alpha + 2\pi)$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

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Handwritten notes on a whiteboard:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$
$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$
$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$$

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where the coefficients  $a_0, a_n$  and  $b_n$  are defined as

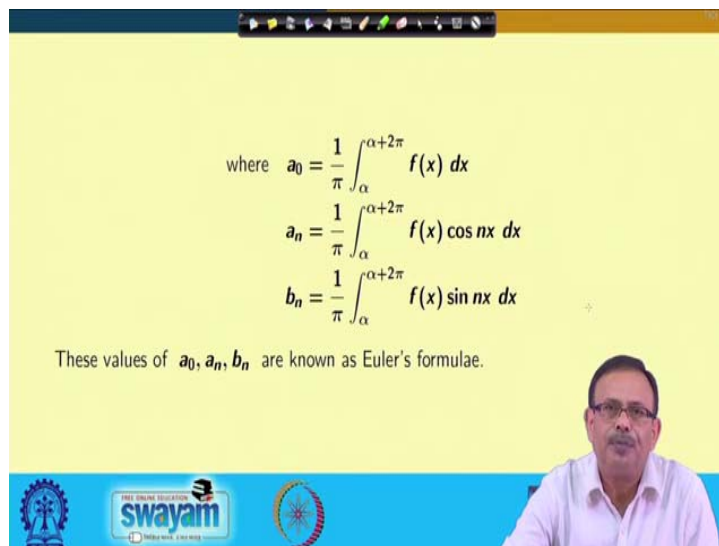
$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx.$$

$a_0$ ,  $a_n$  and  $b_n$  are known as Euler's formulae.

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The screenshot shows a presentation slide with a yellow background. At the top, there is a navigation bar with various icons. The main content of the slide consists of three mathematical formulas for  $a_0$ ,  $a_n$ , and  $b_n$ , each preceded by the word "where". Below the formulas, a line of text states that these values are known as Euler's formulae. In the bottom right corner, there is a small video feed of a man with glasses and a white shirt. At the bottom of the slide, there is a blue banner with logos for "swayam" and other educational institutions.

where  $a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$

$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$

$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$

These values of  $a_0$ ,  $a_n$ ,  $b_n$  are known as Euler's formulae.

Now, to establish this formula for  $f(x)$ , we need to know certain standard values of some definite integrals. These are clearly presented in the attached slides.

So, please note these useful values of these definite integrals which we will use frequently afterwards.

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1. 
$$\int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = \left[ \frac{\sin nx}{n} \right]_{\alpha}^{\alpha+2\pi} = 0$$

2. 
$$\int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = - \left[ \frac{\cos nx}{n} \right]_{\alpha}^{\alpha+2\pi} = 0$$

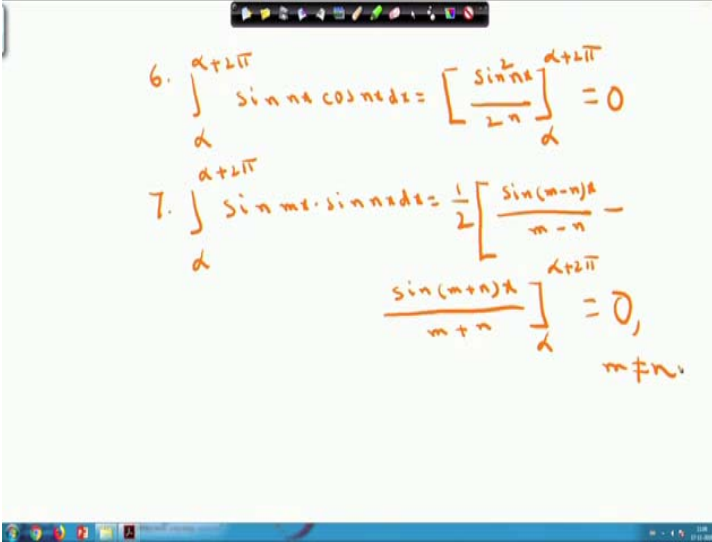
3. 
$$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$$
$$= \frac{1}{2} \left[ \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{\alpha}^{\alpha+2\pi}$$
$$= 0, \quad m \neq n$$

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4. 
$$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \left[ \frac{x}{2} + \frac{\sin 2nx}{4n} \right]_{\alpha}^{\alpha+2\pi} = \pi$$

5. 
$$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx \, dx = -\frac{1}{2} \left[ \frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n} \right]_{\alpha}^{\alpha+2\pi}$$
$$= 0$$

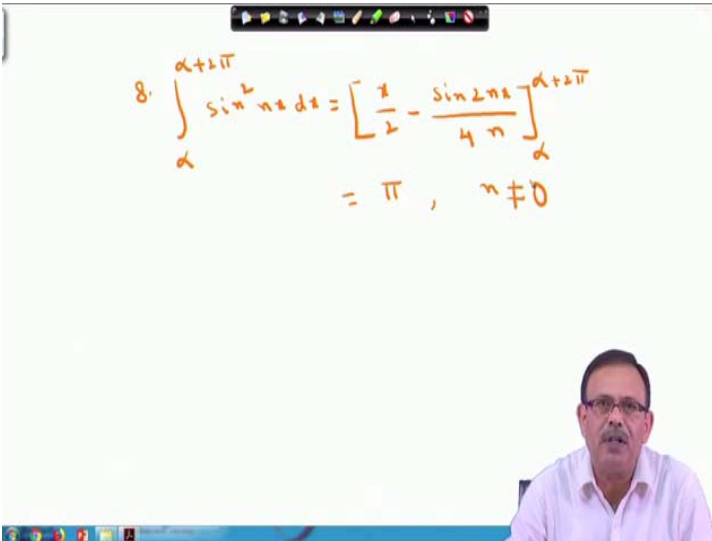
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6.  $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx dx = \left[ \frac{\sin 2nx}{2n} \right]_{\alpha}^{\alpha+2\pi} = 0$

7.  $\int_{\alpha}^{\alpha+2\pi} \sin mx \cdot \sin nx dx = \frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{\alpha}^{\alpha+2\pi} = 0, m \neq n$

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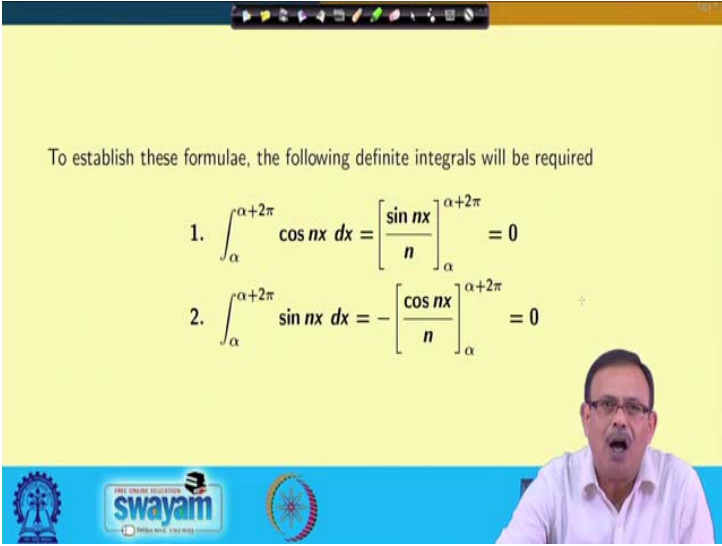
8.  $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx dx = \left[ \frac{x}{2} - \frac{\sin 2nx}{4n} \right]_{\alpha}^{\alpha+2\pi} = \pi, n \neq 0$

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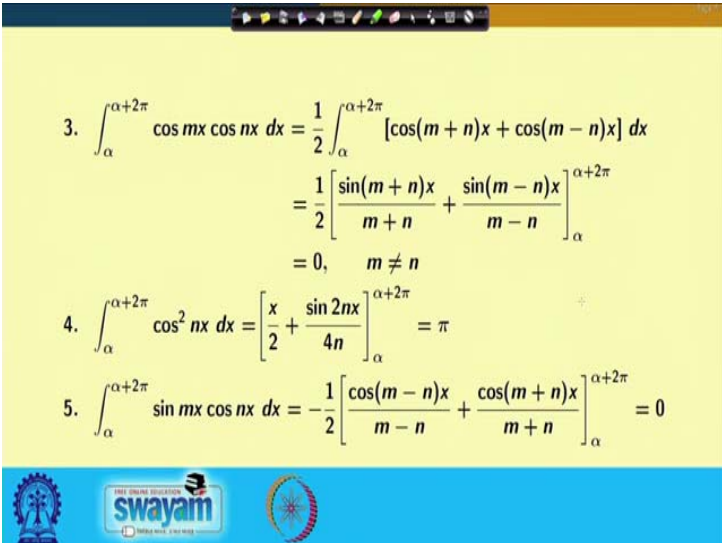
To establish these formulae, the following definite integrals will be required

- $$1. \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = \left[ \frac{\sin nx}{n} \right]_{\alpha}^{\alpha+2\pi} = 0$$
- $$2. \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = - \left[ \frac{\cos nx}{n} \right]_{\alpha}^{\alpha+2\pi} = 0$$



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- $$3. \int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$$
$$= \frac{1}{2} \left[ \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{\alpha}^{\alpha+2\pi}$$
$$= 0, \quad m \neq n$$
- $$4. \int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \left[ \frac{x}{2} + \frac{\sin 2nx}{4n} \right]_{\alpha}^{\alpha+2\pi} = \pi$$
- $$5. \int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx \, dx = -\frac{1}{2} \left[ \frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n} \right]_{\alpha}^{\alpha+2\pi} = 0$$



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6.  $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx \, dx = \left[ \frac{\sin^2 nx}{2n} \right]_{\alpha}^{\alpha+2\pi} = 0$

7.  $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{\alpha}^{\alpha+2\pi} = 0$

8.  $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx \, dx = \left[ \frac{x}{2} - \frac{\sin 2nx}{4n} \right]_{\alpha}^{\alpha+2\pi} = \pi, \quad n \neq 0$

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Now, using these integral values, let us establish the formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

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$f(x) \rightarrow [\alpha, \alpha + 2\pi]$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$

$\int_{\alpha}^{\alpha+2\pi} f(x) \, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] dx$

$\int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] dx = 0$

$\int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] dx = 0$

$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \, dx$

$= \frac{1}{2} a_0 \cdot 2\pi + 0 + 0 = a_0 \pi$

The handwritten derivation shows the integration of the Fourier series over one full period, demonstrating that the integral of the cosine and sine terms is zero, leaving only the constant term.

It is given that  $f(x)$  is represented as a Fourier series given by (1) in the interval  $(\alpha, \alpha + 2\pi)$ . So, our aim is to find out the values of the coefficients  $a_0$ ,  $a_n$  and  $b_n$ . To find these coefficients from the given series, first we need to find out the value of  $a_0$ .

So, we will integrate both sides of (1) from  $x = \alpha$  to  $x = \alpha + 2\pi$ .

$$\int_{\alpha}^{\alpha+2\pi} f(x) dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] dx.$$

Value of the second integral is 0 using the formula 1. of definite integrals as already discussed in this lecture. For similar reason, by formula 2., we have the value of the third integral as 0. Therefore, we are left with

$$\int_{\alpha}^{\alpha+2\pi} f(x) dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx = a_0 \pi$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx.$$

This gives us the value of the coefficient  $a_0$ .

Now, to obtain  $a_n$ , first we multiply both sides of equation (1) by  $\cos nx$  and integrate from  $x = \alpha$  to  $x = \alpha + 2\pi$ . So, we obtain

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

$$= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \cos nx dx.$$

If we clearly observe each of these integrals, then we can see that different deduced formulae on definite integrals will be applicable in this case. So, by using formulae 1., 3., 4., 5. and 6., we obtain,

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx = a_n \pi$$

$$\therefore a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx.$$



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$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \cdot \cos nx \, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \cdot \cos nx \, dx$$

$$= 0 + a_n \cdot \pi + 0 = a_n \pi$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx$$

In the same way, if we want to find out the value of the coefficient  $b_n$ , we will multiply the equation (1) on both sides with  $\sin nx$  and integrate from  $x = \alpha$  to  $x = \alpha + 2\pi$  to obtain

$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \sin nx \, dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \sin nx \, dx.$$

If we clearly observe each of these integrals, then we can see that different deduced formulae on definite integrals will be applicable in this case. So, by using formulae 2., 5., 6., 7. and 8., we obtain,

$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx = b_n \pi$$

$$\therefore b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx.$$

Thus we have derived the values of all the coefficients involved in the Fourier series expansion of  $f(x)$  namely  $a_0$ ,  $a_n$  and  $b_n$ .

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$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \sin nx dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \sin nx dx$$
$$= 0 + 0 + b_n \pi = b_n \pi$$
$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$$

We will now take a look at some special cases that may arise:

Suppose  $\alpha = 0$ . Then the interval of series expansion reduces to  $(0, 2\pi)$  and the Euler's formulae take the following form

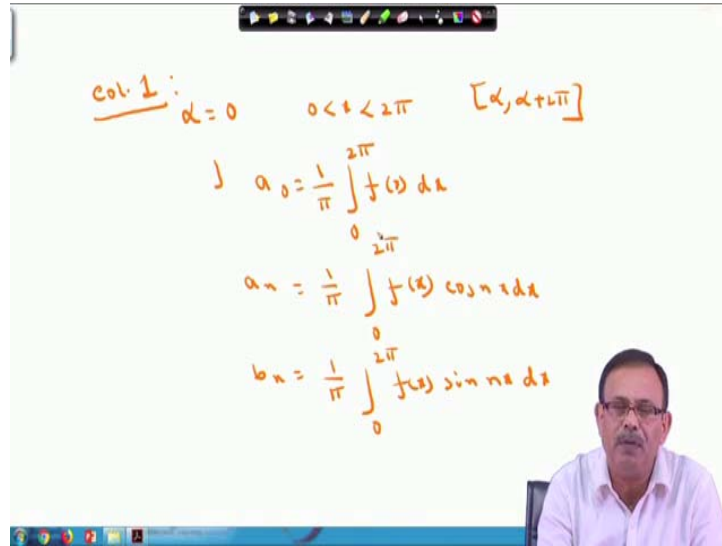
$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx.$$

So that whenever we are taking a special case,  $\alpha = 0$ , then  $x$  varies within the range from 0 to  $2\pi$ .

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Suppose  $\alpha = -\pi$ . Then the interval of series expansion reduces to  $(-\pi, \pi)$  and the Euler's formulae take the following form

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

So that whenever we are taking a special case,  $\alpha = -\pi$ , then  $x$  varies within the range from  $-\pi$  to  $\pi$ .

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Handwritten mathematical formulas on a whiteboard:

Case II:  $\alpha = -\pi, -\pi < x < \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Therefore, we have discussed the generalized case taking  $\alpha$  and in this case, the variable  $x$  varies within the limits  $\alpha$  to  $\alpha + 2\pi$ . We can also assume the case for  $\alpha = 0$ , when the range of  $x$  will be from 0 to  $2\pi$ . Similarly, when  $\alpha = -\pi$ , in that case the range will be  $-\pi$  to  $\pi$ . In all these cases, only change will be in the limit of the integration.

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**Proof:** Let  $f(x)$  be represented in the interval  $(\alpha, \alpha + 2\pi)$  by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$


To find the coefficients  $a_0, a_n, b_n$ , we assume that the series (1) can be integrated term by term from  $x = \alpha$  to  $x = \alpha + 2\pi$

To find  $a_0$ , integrate both sides of (1) from  $x = \alpha$  to  $x = \alpha + 2\pi$ .

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Then,

$$\begin{aligned}\int_{\alpha}^{\alpha+2\pi} f(x) dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] dx + \\ &\quad \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] dx \\ &= \frac{1}{2} a_0 (\alpha + 2\pi - \alpha) + 0 + 0 \quad [\text{by 1. and 2.}] \\ &= a_0 \pi \\ \therefore a_0 &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx\end{aligned}$$



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To find  $a_n$ , multiply each side of (1) by  $\cos nx$  and integrate from  $x = \alpha$  to  $x = \alpha + 2\pi$ , then,

$$\begin{aligned}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \\ &\quad \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \cos nx dx \\ &= a_n \pi \quad [\text{by 1., 3., 4., 5. and 6.}] \\ \therefore a_n &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx\end{aligned}$$


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To find  $b_n$ , multiply each side of (1) by  $\sin nx$  and integrate from  $x = \alpha$  to  $x = \alpha + 2\pi$ , then,

$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} a_n \cos nx \right] \sin nx \, dx + \int_{\alpha}^{\alpha+2\pi} \left[ \sum_{n=1}^{\infty} b_n \sin nx \right] \sin nx \, dx$$
$$= b_n \pi \quad [\text{by 2., 5., 6., 7. and 8.}]$$
$$\therefore b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx$$


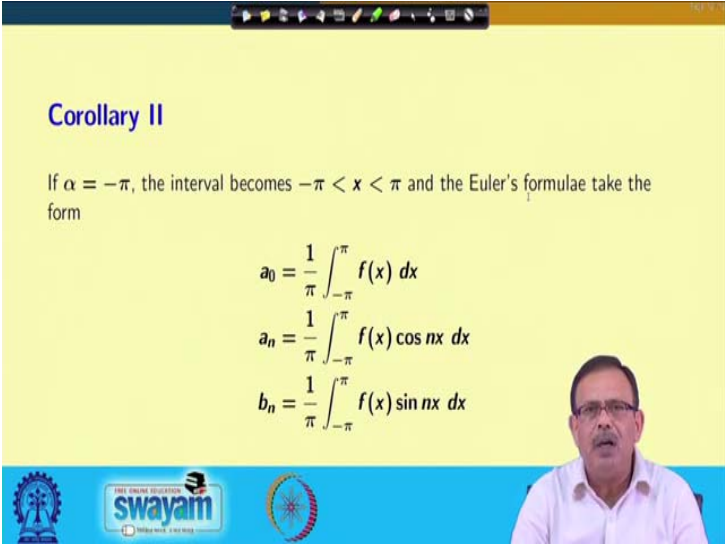
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### Corollary I

Making  $\alpha = 0$ , the interval becomes  $0 < x < 2\pi$  and the Euler's formulae reduce to

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx$$
$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$


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**Corollary II**

If  $\alpha = -\pi$ , the interval becomes  $-\pi < x < \pi$  and the Euler's formulae take the form

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

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In the next lecture, we will see the conditions for which the Fourier series of a function exists i.e., whether the Fourier series exists for all functions or there are certain criteria which should be satisfied for the existence of the Fourier series. Thank you.