Transform Calculus and Its Applications in Differential Equations
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Lecture - 18
Introduction to Integral Equation and its Solution Process

In the last few lectures, we have seen how to find out the solution of ordinary differential equations using Laplace transform. We have considered the ordinary differential equations with constant coefficients, with variable coefficients and also the solution of simultaneous linear ordinary differential equations. So, we have discussed these things in the last topics of Laplace transform.

Now we will study a new topic: integral equation and the solution of integral equations. We will discuss the use of Laplace transform in the solution process of the integral equation.

An integral equation is basically an equation, where the function F(t) to be determined, comes under the integral sign.

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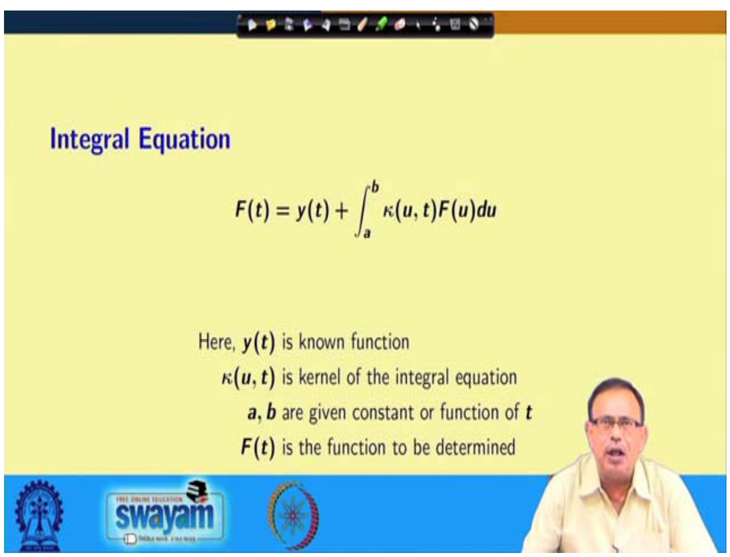

The form of an integral equation is as follows:

$$F(t) = y(t) + \int_a^b \kappa(u,t)F(u)\,du.$$

We have to determine $F(t)$ which comes under the integral sign. This is known as an integral equation, where $y(t)$ is a known function, $\kappa(u,t)$ is basically the kernel of the integral equation, a and b are given constants or they may be functions of t as well.

Various types of integral equations are available; one is Fredholm integral equation. It takes the following form:

$$F(t) = y(t) + \int_a^b \kappa(u,t)F(u)\,du$$

where a and b are constants and not functions of t.

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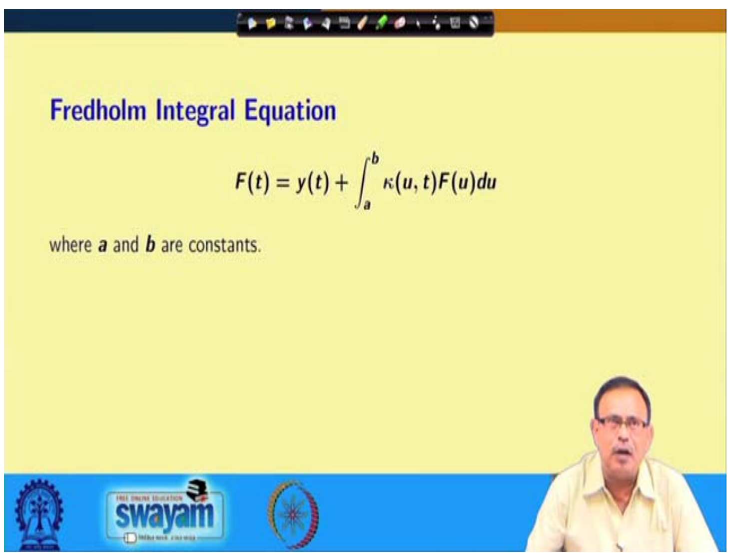

There is another type of integral equation, which we call the Volterra integral equation given by:

$$F(t) = y(t) + \int_a^b \kappa(u,t)F(u)\,du$$

where a is a constant but $b = t$. As we already discussed that a and b may be constants or some functions of t.

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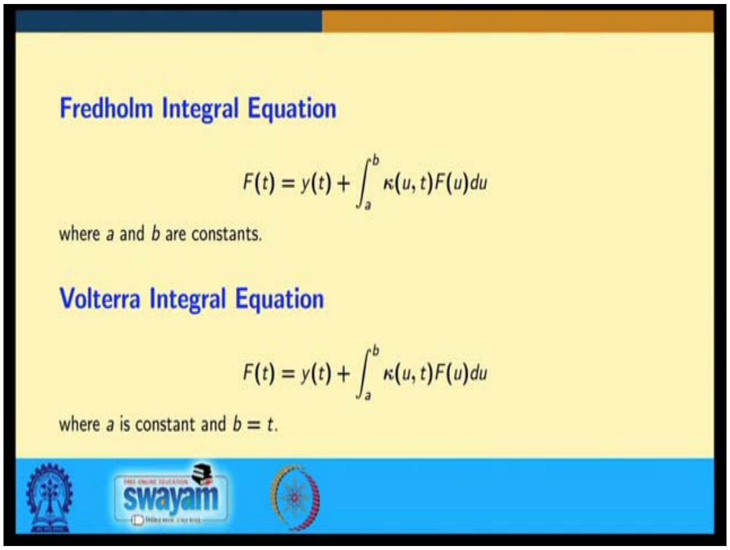

Next one is integral equation of convolution type which takes the form:

$$F(t) = y(t) + \int_0^t \kappa(t-u)F(u)\,du.$$

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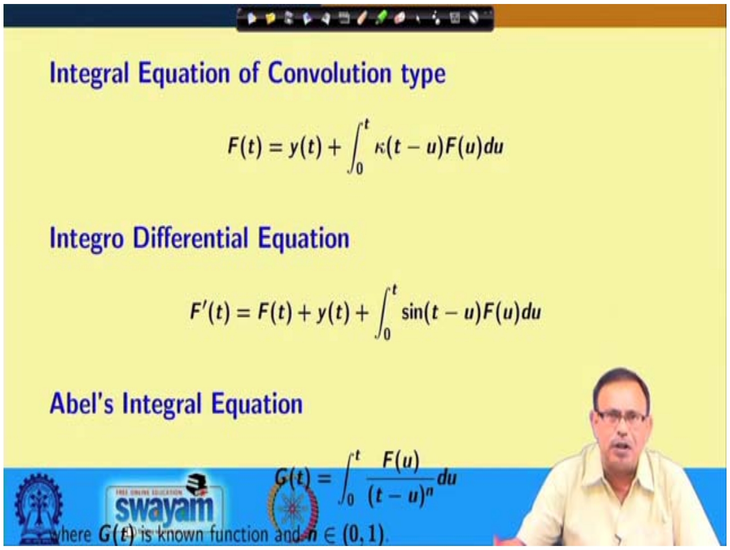

There is yet another type, which we call as Integro differential equation. This Integro differential equation is defined as

$$F'(t) = F(t) + y(t) + \int_0^t \sin(t-u)\, F(u)\, du.$$

It not only involves the function which we have to determine, but also the derivative of the function.

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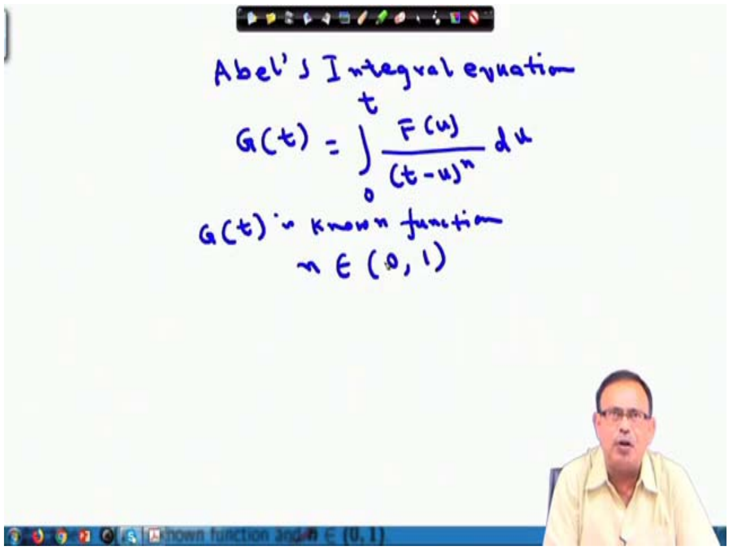

Another type that is Abel's integral equation takes the following form:

$$G(t) = \int_0^t \frac{F(u)}{(t-u)^n} du$$

where, $G(t)$ is a known function and $n \in (0,1)$. We have to find out the value of $F(u)$. So, we have summarized here a few types of well-known integral equations. Now we will see, how to use Laplace transform technique to find out the solution of these integral equations.

First we take this form, suppose we want to solve

$$F(t) = \int_0^t \sin u \cos(t-u)\ du.$$

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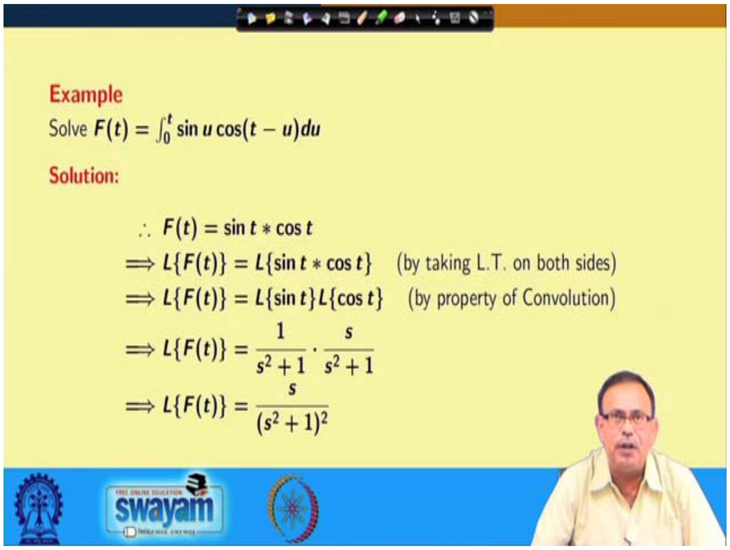

Our aim is to find out the value of the function $F(t)$.

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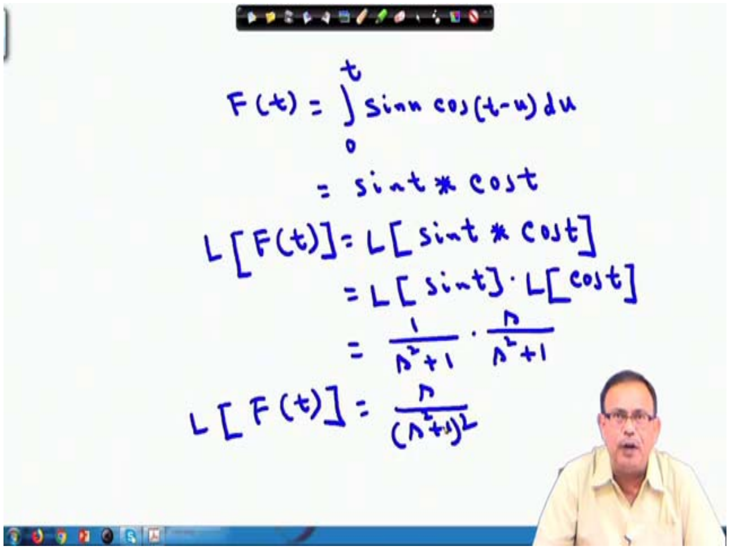

Clearly, from the definition of convolution, we can write $F(t)$ as

$$F(t) = \sin t * \cos t.$$

Now, we can take Laplace transform on both sides of the above equation. If we take Laplace transform on both sides and use the property of the convolution theorem, we can write it as

$$
\begin{aligned}
L\{F(t)\} &= \mathrm{L}\{\sin t * \cos t\} \\
&= L\{\sin t\}\, L\{\cos t\} \\
&= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} \\
&= \frac{s}{(s^2+1)^2}.
\end{aligned}
$$

So, once we are getting Laplace transform of $F(t)$, we can find out the value of $F(t)$ using inverse Laplace transform.

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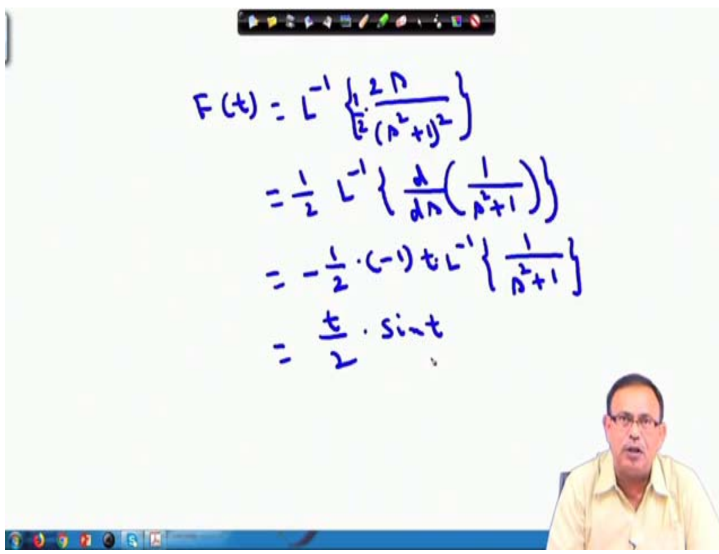

$$
\begin{aligned}
\therefore F(t) &= L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \\
&= -\frac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}
\end{aligned}
$$

Using the property of inverse Laplace transform, we get the solution as

$$
F(t) = -\frac{1}{2}\cdot(-1)^1\cdot t^1 L^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{t}{2}\sin t.
$$

So, even if we have the integral equation, using the convolution property and other properties of Laplace and inverse Laplace transforms, we can find out the solution easily.

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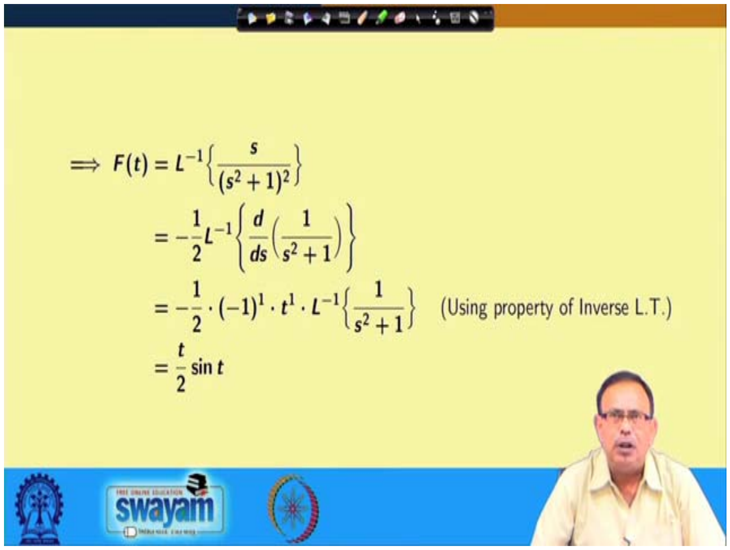

Now, let us take another problem. We want to solve the equation

$$F(t) = 1 + \int_0^t F(u)\sin(t-u)\ du.$$

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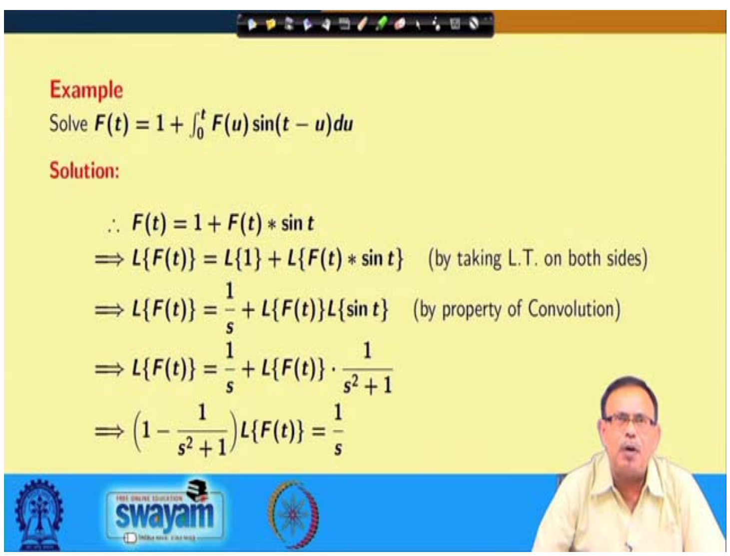

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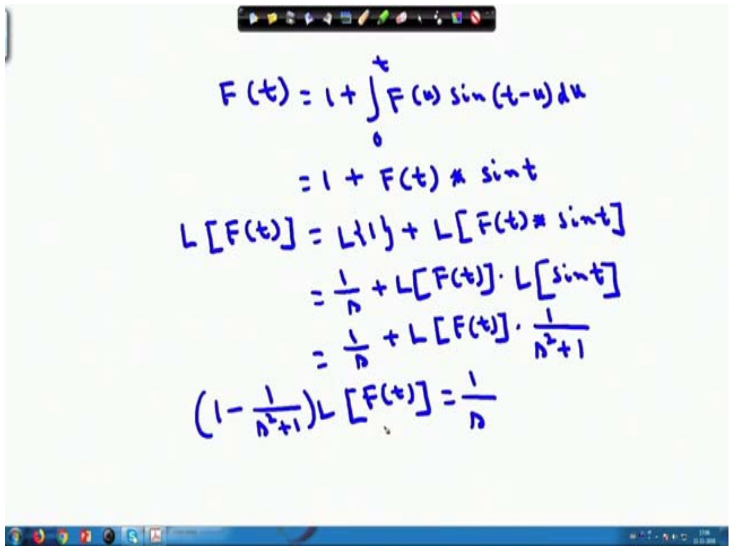

Let us see the solution process. Again from the definition of convolution, we can write it as

$$F(t) = 1 + F(t) * \sin t.$$

Now, we take Laplace transform on both sides of the above equation so that it equals

$$L\{F(t)\} = L\{1\} + L\{F(t) * \sin t\}$$

$$\Rightarrow L\{F(t)\} = L\{1\} + L\{F(t)\}L\{\sin t\} \quad \text{(by Convolution Theorem)}$$

$$\Rightarrow L\{F(t)\} = \frac{1}{s} + L\{F(t)\} \cdot \frac{1}{s^2 + 1}$$

After simplification, we can write it as,

$$L\{F(t)\} = \frac{1}{s} + \frac{1}{s^3}$$

Now we take the inverse Laplace transform so that the function $F(t)$ will be

$$F(t) = L^{-1}\left\{\frac{1}{s} + \frac{1}{s^3}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 + \frac{t^2}{2}.$$

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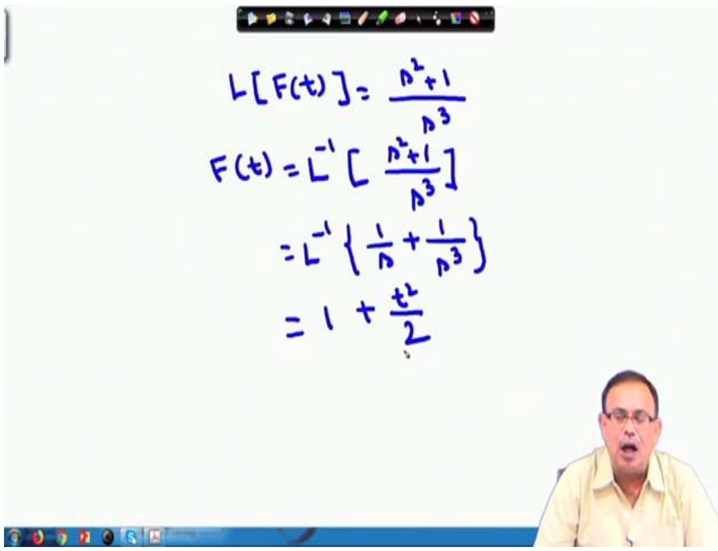

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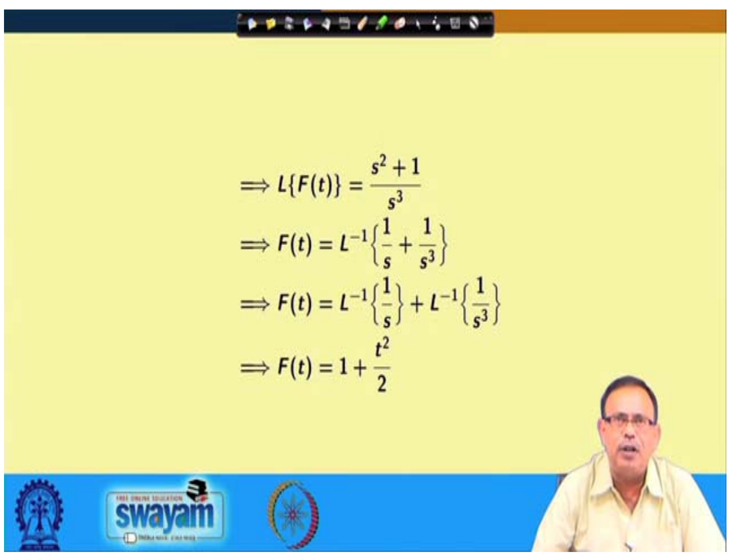

Now, let us see another problem involving the Bessel function. Suppose we want to find the value of

$$\int_0^t J_0(u)\,J_0(t-u)du.$$

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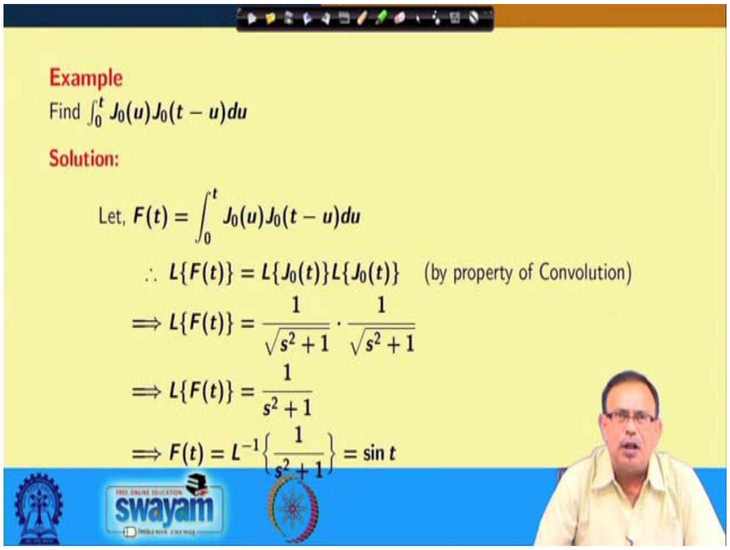

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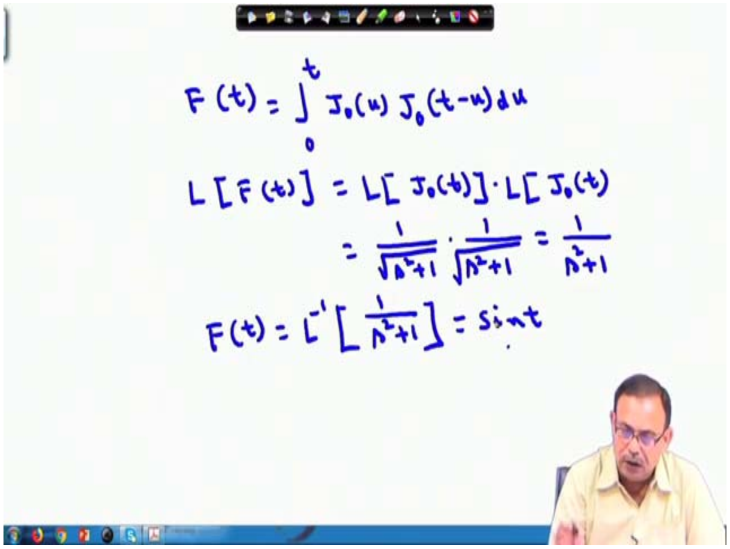

Suppose we take

$$F(t) = \int_0^t J_0(u)\, J_0(t-u)du.$$

Then directly we can write using the definition of convolution of two functions that

$$F(t) = J_0(t) * J_0(t).$$

Now, if we take Laplace transform on both sides, we get,

$$\begin{aligned} L\{F(t)\} &= L\{J_0(t) * J_0(t)\} \\ &= L\{J_0(t)\}L\{J_0(t)\} \quad \text{(using convolution theorem)} \\ &= \frac{1}{\sqrt{s^2+1}} \cdot \frac{1}{\sqrt{s^2+1}} \\ &= \frac{1}{s^2+1}. \end{aligned}$$

Therefore, in order to obtain $F(t)$, we take the Laplace inverse as follows:

$$\begin{aligned} F(t) &= L^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \sin t. \end{aligned}$$

Therefore, value of the integral $\int_0^t J_0(u)\, J_0(t-u)du$ is obtained as $\sin t$.

Let us see another example which involves not only the desired function $F(t)$ but also its derivative $F'(t)$.

We need to solve

$$F'(t) = \sin t + \int_0^t F(t-u)\cos u\, du \quad \text{if} \quad F(0) = 0.$$

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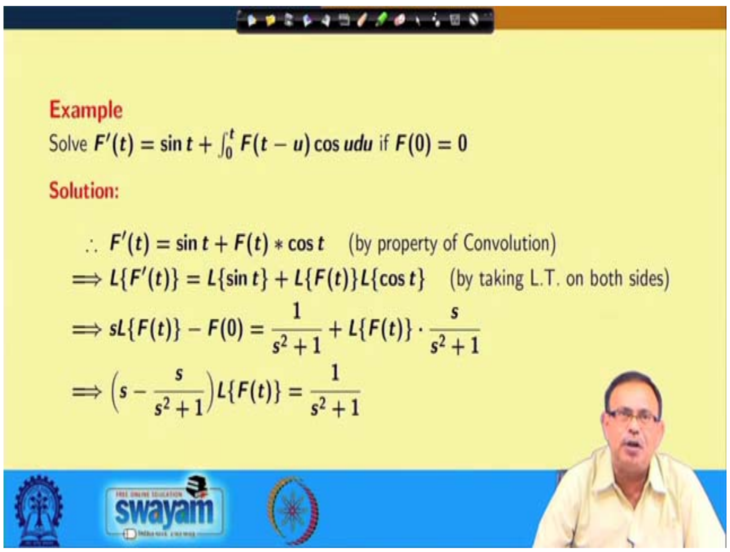

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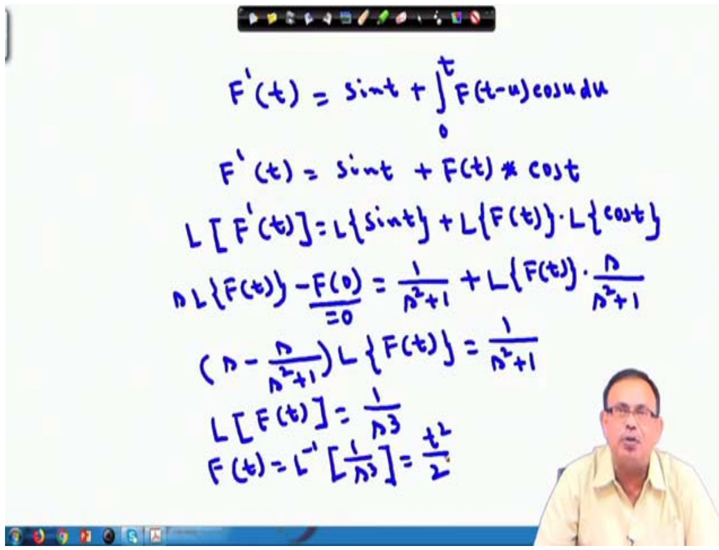

First of all, clearly, by the definition of convolution of two functions $F(t)$ and $\cos t$, we can write

$$F'(t) = \sin t + [F(t) * \cos t].$$

Now, we take Laplace transform on both sides of the above equation to get

$$L\{F'(t)\} = L\{\sin t\} + L\{F(t) * \cos t\}$$

$$\Rightarrow sL\{F(t)\} - F(0) = \frac{1}{s^2+1} + L\{F(t)\}L\{\cos t\}$$

$$\Rightarrow sL\{F(t)\} = \frac{1}{s^2+1} + L\{F(t)\}L\{\cos t\} \qquad [\because F(0) = 0]$$

$$\Rightarrow sL\{F(t)\} = \frac{1}{s^2+1} + \frac{s}{s^2+1}L\{F(t)\}$$

$$\Rightarrow L\{F(t)\} = \frac{1}{s^3}.$$

Now clearly, if we take the inverse Laplace transform, we will easily obtain the value of $F(t)$ as

$$F(t) = L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}.$$

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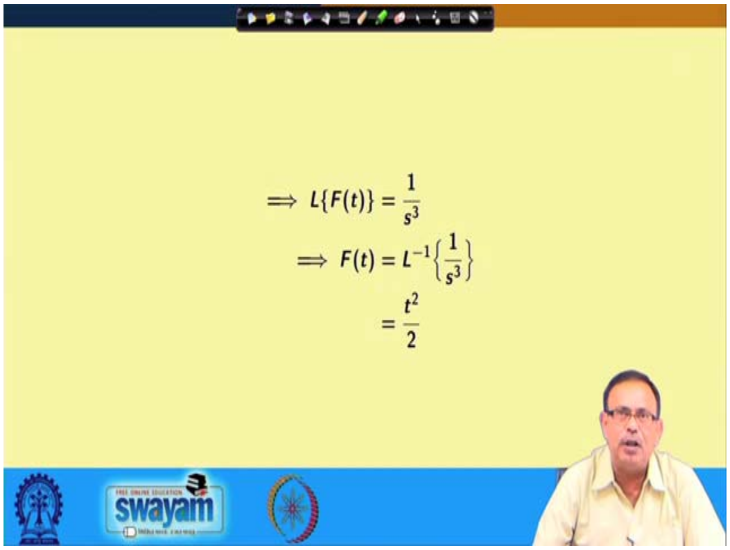

Let us take the next example. We want to solve

$$\int_0^t \frac{F(u)}{(t-u)^{1/3}}\,du = t(t+1).$$

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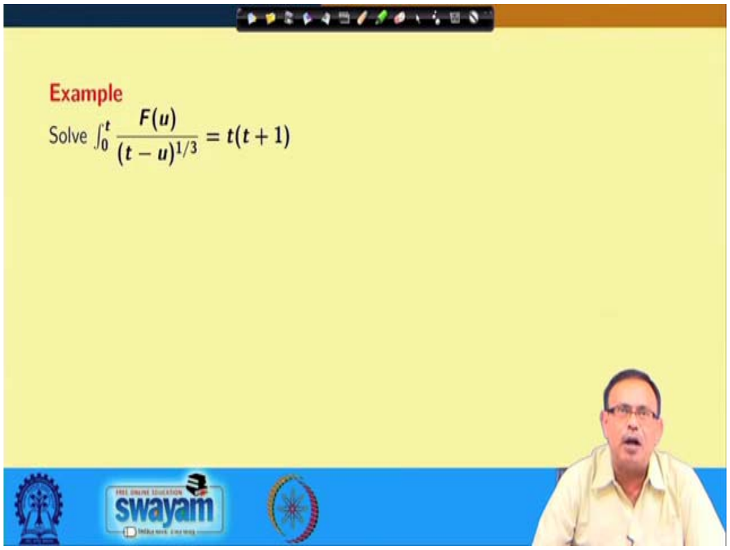

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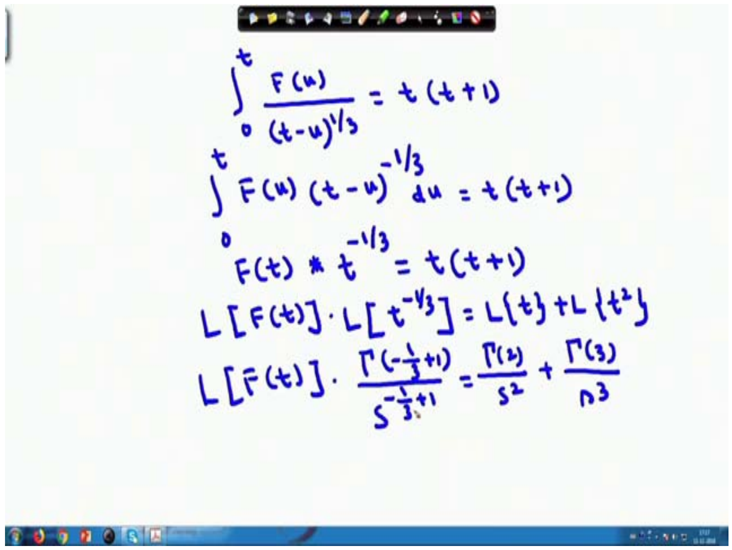

The given integral can be re-written as

$$\int_0^t F(u)(t-u)^{-\frac{1}{3}}du = t(t+1)$$

$$\Rightarrow F(t) * t^{-\frac{1}{3}} = t(t+1).$$

Now we take Laplace transform on both sides to obtain

$$L\left\{F(t) * t^{-\frac{1}{3}}\right\} = L\{t^2\} + L\{t\}$$

$$\Rightarrow L\{F(t)\}L\left\{t^{-\frac{1}{3}}\right\} = \frac{2}{s^3} + \frac{1}{s^2}$$

$$\Rightarrow L\{F(t)\}\frac{\Gamma\left(\frac{2}{3}\right)}{s^{\frac{2}{3}}} = \frac{2}{s^3} + \frac{1}{s^2}$$

$$\Rightarrow L\{F(t)\} = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{2}{s^{7/3}} + \frac{1}{s^{4/3}}\right].$$

Now we take the inverse Laplace transform to obtain

$$F(t) = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[2L^{-1}\left\{\frac{1}{s^{7/3}}\right\} + L^{-1}\left\{\frac{1}{s^{4/3}}\right\}\right]$$

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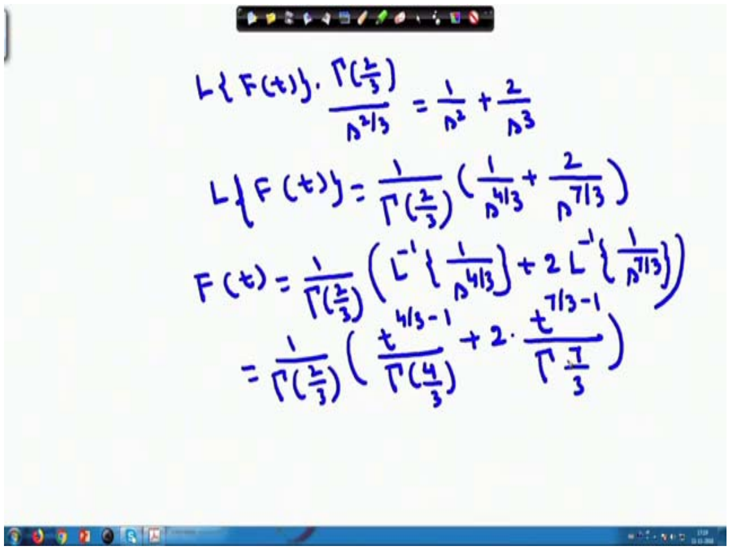

Solving this and after some simplifications, we obtain $F(t)$ as

$$F(t) = \frac{3\sqrt{3}}{4\pi} t^{\frac{1}{3}}(2 + 3t).$$

For details regarding the steps involved to reach this solution, refer to the lecture slides attached.

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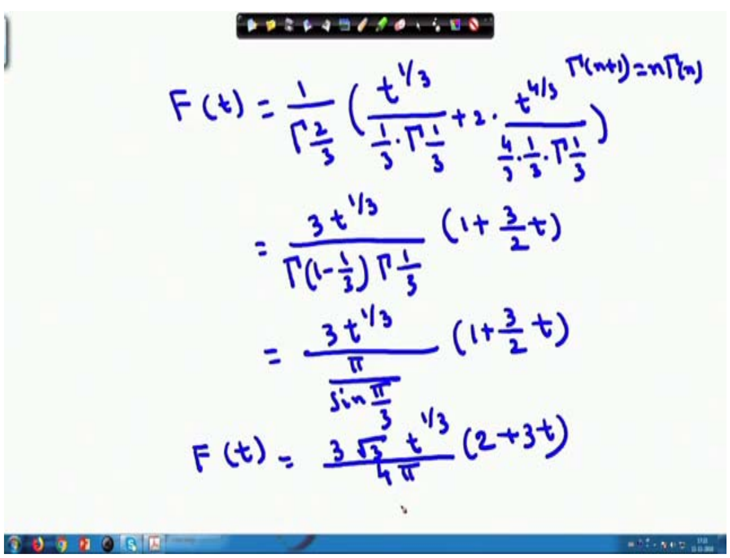

So, we observe that if we have the integral equations, then always we can find out the solution of these using Laplace transform. This completes the topics on Laplace transform. In Laplace transform, we have studied the basic properties of Laplace transform, Laplace transform of derivatives, integrations, multiplication by powers of t, division by t, convolution theorem etc. Then we have done the inverse Laplace transform. We have discussed the Laplace transform of certain special function like Dirac delta function, error function, Bessel function etc.

We have elaborately studied convolution and using convolution, how to find out the solution of various equations. Towards the end of this topic, we have seen the use of Laplace transform in solution of ordinary differential equations with constant coefficients, with variable coefficients, solution of simultaneous ordinary differential equations. Besides these, in the present lecture, we have covered solution of integral equations using Laplace transform.

In the next lecture, we will start with the Fourier series and then, we will go to some other transform that is Fourier transform.

Thank you.