

Transform Calculus and Its Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 17
Solution of Simultaneous Ordinary Differential Equations using Laplace Transform

So now, in this lecture we will start with the earlier topics that is solution of ordinary differential equation with variable coefficients using the Laplace transform. We have given one example in the last lecture. For easy reference, we should solve one more example of ordinary differential equation with variable coefficients so that the solution process gets even more clear.

(Refer Slide Time: 00:57)

Example
Solve $ty'' + (1 - 2t)y' - 2y = 0$ with $y(0) = 1$, $y'(0) = 2$.

Solution:

$$\therefore L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0 \quad (\text{taking L.T. on both sides})$$

$$\Rightarrow -\frac{d}{ds}(s^2L\{y\} - sy(0) - y'(0)) + (sL\{y\} - y(0)) + 2\frac{d}{ds}(sL\{y\} - y(0)) - 2L\{y\} = 0$$

$$\Rightarrow -\frac{d}{ds}(s^2z - s - 2) + (sz - 1) + 2\frac{d}{ds}(sz - 1) - 2z = 0 \quad (z = L\{y\})$$

$$\Rightarrow -(s^2 - 2s)\frac{dz}{ds} - sz = 0 \Rightarrow \frac{dz}{z} + \frac{ds}{s-2} = 0$$

Let us take this example. Suppose we need to solve

$$t \frac{d^2y}{dt^2} + (1 - 2t) \frac{dy}{dt} - 2y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

So, here ty'' is there which was not present in the earlier example. So, let us see, whenever both y'' and y' consist of coefficients which are functions of t , what happens in such cases.

(Refer Slide Time: 01:28)

The image shows a whiteboard with handwritten mathematical work. At the top, it lists initial conditions: $y(0)=1$ and $y'(0)=2$. The main equation is $L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0$. Below this, it shows the differentiation of the Laplace transform with respect to s : $-\frac{d}{ds}(s^2L\{y\} - sy(0) - y'(0)) + (sL\{y\} - y(0)) - 2\frac{d}{ds}(sL\{y\} - y(0)) - 2L\{y\} = 0$. This is then simplified to $-\frac{d}{ds}(s^2z - s - 2) + (sz - 1) + 2\frac{d}{ds}(sz - 1) - 2z = 0$, where $z = L\{y\}$. The final simplified equation is $2z = 0$.

Initially, we will take Laplace transform of the given ODE as follows:

$$L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0.$$

We can easily simplify this using the well-known properties of Laplace transform as follows:

$$\begin{aligned} &-\frac{d}{ds}L\{y''\} + L\{y'\} + 2\frac{d}{ds}L\{y'\} - 2L\{y\} = 0 \\ \Rightarrow &-\frac{d}{ds}[s^2L\{y\} - sy(0) - y'(0)] + sL\{y\} - y(0) + 2\frac{d}{ds}[sL\{y\} - y(0)] - 2L\{y\} = 0 \\ \Rightarrow &-\frac{d}{ds}[s^2z - s - 2] + sz - 1 + 2\frac{d}{ds}[sz - 1] - 2z = 0, \quad \text{where, } z = L\{y\}. \end{aligned}$$

After differentiation with respect to s and rearranging the terms in a proper way, we obtain

$$\begin{aligned} &\frac{dz}{ds} + \frac{1}{s-2}z = 0 \\ \Rightarrow &\frac{dz}{z} + \frac{ds}{s-2} = 0. \end{aligned}$$

Integrating the above, we get,

$$\begin{aligned} &\log z + \log(s-2) = \log c \\ \Rightarrow &\log[(s-2)z] = \log c \end{aligned}$$

$$\Rightarrow z = L\{y\} = \frac{c}{s-2}$$

Now using the inverse Laplace transform technique, we can evaluate $y(t)$ as

$$\begin{aligned} y(t) &= cL^{-1}\left\{\frac{1}{s-2}\right\} \\ &= ce^{2t}. \end{aligned}$$

Now our job is to find the value of the constant c with the help of the condition provided to us. We know that $y(0) = 1$, so we use it as

$$\begin{aligned} y(0) &= ce^0 \\ \Rightarrow 1 &= c. \end{aligned}$$

Therefore, $c = 1$ so that the final solution of the given ODE is obtained as

$$y(t) = e^{2t}.$$

(Refer Slide Time: 05:17)

$$-(n^2 - 2n) \frac{dz}{dn} - nZ = 0, Z = L\{y\}$$

$$\frac{dz}{z} + \frac{dn}{n-2} = 0$$

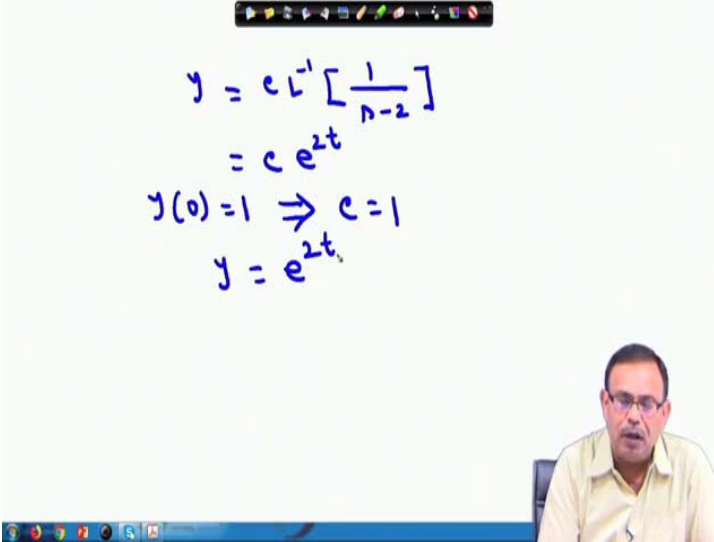
$$\log z + \log(n-2) = \log C$$

$$\log [Z(n-2)] = \log C$$

$$Z = L\{y\} = \frac{e}{n-2}$$

So, whenever we have the variable parameters as coefficients in the ODE, then we have seen that the second order ODE has been converted into a first order ODE. The solution procedure has been well explained through the solved example.

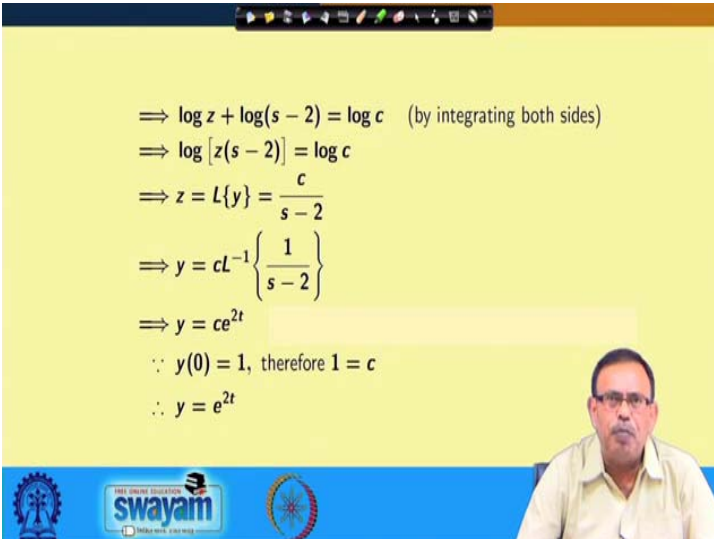
(Refer Slide Time: 07:27)



A whiteboard with handwritten mathematical equations in blue ink. The equations are:
$$y = cL^{-1}\left[\frac{1}{s-2}\right]$$
$$= ce^{2t}$$
$$y(0) = 1 \Rightarrow c = 1$$
$$y = e^{2t}$$

The whiteboard is part of a video recording, with a toolbar at the top and a person's head and shoulders visible in the bottom right corner.

(Refer Slide Time: 09:13)



A printed slide with a yellow background and black text. The text shows a series of mathematical steps:
$$\Rightarrow \log z + \log(s-2) = \log c \quad (\text{by integrating both sides})$$
$$\Rightarrow \log [z(s-2)] = \log c$$
$$\Rightarrow z = L\{y\} = \frac{c}{s-2}$$
$$\Rightarrow y = cL^{-1}\left\{\frac{1}{s-2}\right\}$$
$$\Rightarrow y = ce^{2t}$$
$$\therefore y(0) = 1, \text{ therefore } 1 = c$$
$$\therefore y = e^{2t}$$

The slide is part of a video recording, with a toolbar at the top and a person's head and shoulders visible in the bottom right corner. At the bottom of the slide, there are logos for 'swayam' and 'Make with e-learning'.

Now we move to another example, linear simultaneous ODE with constant coefficients. Before moving to the example, let us go through the topic once.

(Refer Slide Time: 10:00)

Example
Solve

$$x'' - 3x - 4y = 0$$
$$y'' + x + y = 0$$

when $x(0) = y(0) = y'(0) = 0$, $x'(0) = 2$

Solution: Taking the L.T. on both sides of the two equations

$$L\{x''\} - 3L\{x\} - 4L\{y\} = 0,$$
$$L\{y''\} + L\{x\} + L\{y\} = 0$$

The slide also features logos for Swamyam and other educational institutions at the bottom.

(Refer Slide Time: 10:14)

Solution of Simultaneous ODE

A video frame showing a whiteboard with the handwritten title "Solution of Simultaneous ODE" in blue ink. A man in a light-colored shirt is visible in the bottom right corner of the frame.

We want to study now, solution of simultaneous ordinary differential equations. Till now we have solved only one differential equation. Now the situation is, if we have simultaneous ordinary differential equations of two different dependent variables, then using Laplace transform technique, we are going to see, how to find out the solution for both the dependent variables.

Suppose we want to solve the following problem,

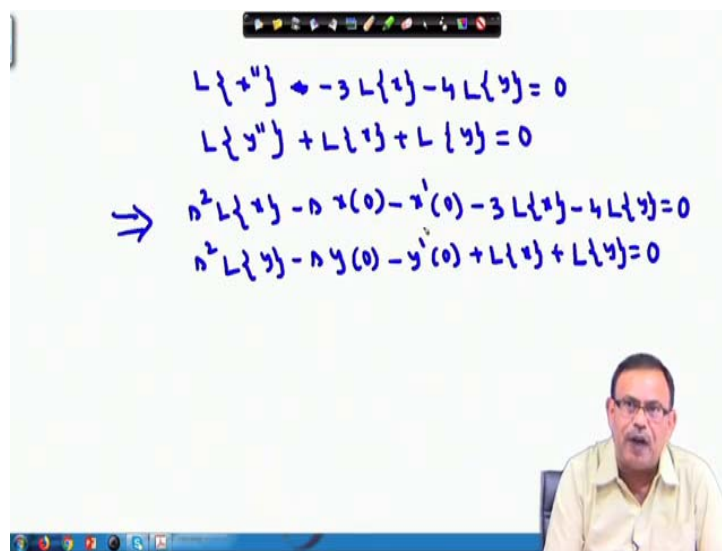
$$x'' - 3x - 4y = 0$$

$$y'' + x + y = 0$$

and the conditions are given as $x(0) = y(0) = y'(0) = 0, x'(0) = 2$.

To find the solution of course, we have to go through the similar process that is, we have to take the Laplace transform on both sides of both the equations.

(Refer Slide Time: 11:39)



Basically instead of one equation, since now we have two simultaneous equations, we are applying Laplace transform on both sides of those two equations together. So, if we take the Laplace transform on both sides of the given equations, we will obtain,

$$L\{x''\} - 3L\{x\} - 4L\{y\} = 0$$

$$L\{y''\} + L\{x\} + L\{y\} = 0$$

$$\Rightarrow s^2L\{x\} - sx(0) - x'(0) - 3L\{x\} - 4L\{y\} = 0$$

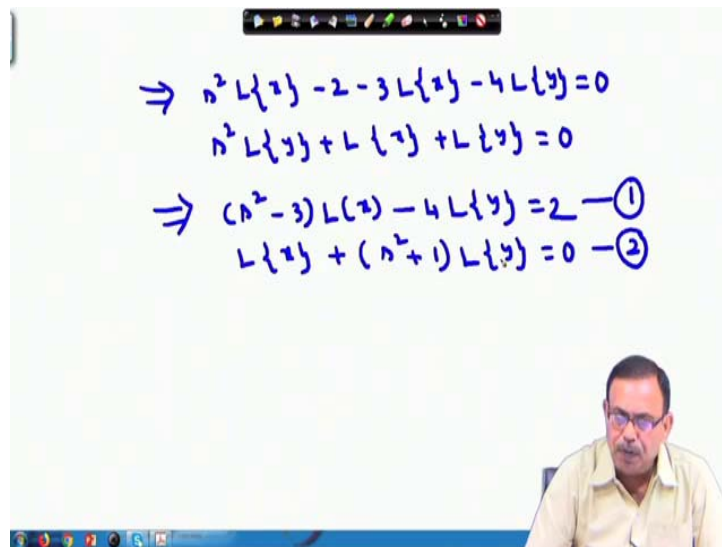
$$s^2L\{y\} - sy(0) - y'(0) + L\{x\} + L\{y\} = 0$$

Please note that, here x and y , both are functions of t . x and y are the dependent variables whose values we need to obtain in terms of the independent variable t .

Now, we have the conditions $x(0) = y(0) = y'(0) = 0, x'(0) = 2$. Since, we already know these values, let us substitute these to obtain the next set of equations that is

$$\begin{aligned} s^2 L\{x\} - 2 - 3L\{x\} - 4L\{y\} &= 0 \\ s^2 L\{y\} + L\{x\} + L\{y\} &= 0 \\ \Rightarrow (s^2 - 3)L\{x\} - 4L\{y\} &= 2 \\ L\{x\} + (s^2 + 1)L\{y\} &= 0 \end{aligned}$$

(Refer Slide Time: 13:53)



Therefore, we have basically two algebraic equations now in two variables namely, $L\{x\}$ and $L\{y\}$ which can be easily solved to obtain $L\{x\}$ and $L\{y\}$ separately as functions of s .

$$\begin{aligned} L\{x\} &= \frac{2(s^2 + 1)}{(s^2 - 1)^2} = \frac{1}{(s - 1)^2} + \frac{1}{(s + 1)^2} \\ L\{y\} &= -\frac{2}{(s + 1)^2(s - 1)^2} = \frac{1}{2} \left[-\frac{1}{s - 1} + \frac{1}{s + 1} - \frac{1}{(s - 1)^2} - \frac{1}{(s + 1)^2} \right]. \end{aligned}$$

We have expressed $L\{x\}$ and $L\{y\}$ in such a way that it becomes easy to evaluate the inverse Laplace transform of these terms. Before taking inverse Laplace transform, we should always express the function of s in some known form. So, now if we take the inverse Laplace transform of the set of equations, then we will get,

$$x(t) = te^t + te^{-t} = t(e^t + e^{-t})$$

$$y(t) = \frac{1}{2}[-e^t + e^{-t} - te^t - te^{-t}].$$

This gives the required result.

(Refer Slide Time: 15:34)

$$L\{x\} = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}$$

$$L\{y\} = -\frac{2}{(s+1)^2(s-1)^2}$$

$$= \frac{1}{2} \left[-\frac{1}{s-1} + \frac{1}{s+1} - \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2} \right]$$

(Refer Slide Time: 17:14)

$$x = L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= te^t + te^{-t}$$


$$y = \frac{1}{2} \left[L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2}\right\} \right]$$

$$= \frac{1}{2} (-e^t + e^{-t} - te^t - te^{-t})$$


So, like this way, whenever we have two simultaneous ordinary differential equations in two dependent variables x and y , applying the Laplace transform on both of them, we will get two simultaneous algebraic equations. Solving these two, we will get the Laplace

transforms of x and y separately. Once we have their Laplace transforms, using inverse Laplace transform, always we can find out the values of x and y . This is basically the solution procedure for simultaneous ODE using Laplace Transform.


(Refer Slide Time: 19:59)

$$\begin{aligned} \Rightarrow s^2 L\{x\} - sx(0) - x'(0) - 3L\{x\} - 4L\{y\} &= 0, \\ s^2 L\{y\} - sy(0) - y'(0) + L\{x\} + L\{y\} &= 0 \\ \Rightarrow s^2 L\{x\} - 2 - 3L\{x\} - 4L\{y\} &= 0, \\ s^2 L\{y\} + L\{x\} + L\{y\} &= 0 \\ \Rightarrow (s^2 - 3)L\{x\} - 4L\{y\} &= 2, \quad (1) \\ L\{x\} + (s^2 + 1)L\{y\} &= 0 \quad (2) \end{aligned}$$


(Refer Slide Time: 20:13)

$$\begin{aligned} \text{Solving (1) \& (2) } L\{x\} &= \frac{2(s^2 + 1)}{(s^2 - 1)^2}, \\ L\{y\} &= -\frac{2}{(s + 1)^2(s - 1)^2} \\ \Rightarrow L\{x\} &= \frac{1}{(s - 1)^2} + \frac{1}{(s + 1)^2}, \\ L\{y\} &= \frac{1}{2} \left(-\frac{1}{s - 1} + \frac{1}{s + 1} - \frac{1}{(s - 1)^2} - \frac{1}{(s + 1)^2} \right) \end{aligned}$$


(Refer Slide Time: 20:27)

$$\begin{aligned}\Rightarrow x &= L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2}\right\}, \\ y &= \frac{1}{2}\left(-L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right) \\ \Rightarrow x &= te^t + te^{-t}, \\ y &= \frac{1}{2}\left(-e^t + e^{-t} - te^t - te^{-t}\right)\end{aligned}$$


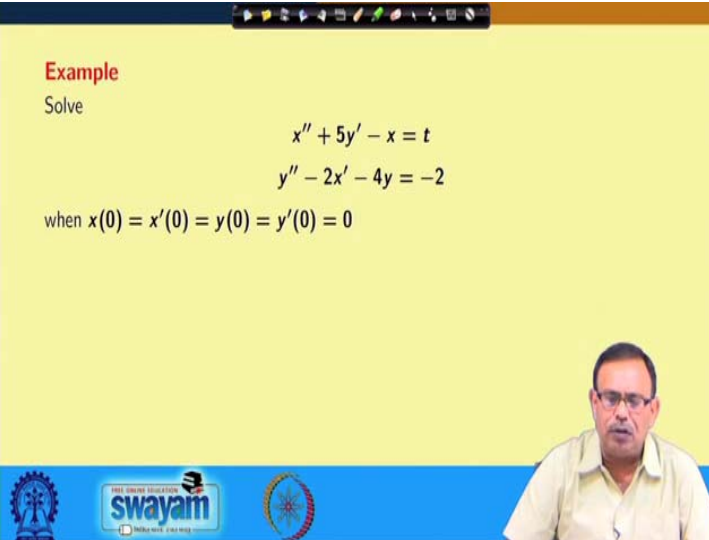
Let us solve one more example so that we can understand the procedure again in much better way.

(Refer Slide Time: 20:42)

Example
Solve

$$\begin{aligned}x'' + 5y' - x &= t \\ y'' - 2x' - 4y &= -2\end{aligned}$$

when $x(0) = x'(0) = y(0) = y'(0) = 0$



Suppose we need to solve the following problem,

$$\begin{aligned}x'' + 5y' - x &= t \\ y'' - 2x' - 4y &= -2\end{aligned}$$

and the conditions are given as $x(0) = y(0) = x'(0) = y'(0) = 0$.

(Refer Slide Time: 21:35)

$$L\{x''\} + 5L\{y'\} - L\{x\} = L\{t\}$$

$$L\{y''\} - 2L\{x'\} - 4L\{y\} = L\{-2\}$$

$$\Rightarrow (s^2 L\{x\} - sx(0) - x'(0)) + 5(sL\{y\} - y(0)) - L\{x\} = \frac{1}{s^2}$$

$$(s^2 L\{y\} - sy(0) - y'(0)) - 2(sL\{x\} - x(0)) - 4L\{y\} = -\frac{2}{s}$$

As in the previous case, here also, we take the Laplace transform on both sides of the given equations to obtain,

$$L\{x''\} + 5L\{y'\} - L\{x\} = L\{t\}$$

$$L\{y''\} - 2L\{x'\} - 4L\{y\} = -L\{2\}$$

$$\Rightarrow s^2 L\{x\} - sx(0) - x'(0) + 5[sL\{y\} - y(0)] - L\{x\} = \frac{1}{s^2}$$

$$s^2 L\{y\} - sy(0) - y'(0) - 2[sL\{x\} - x(0)] - 4L\{y\} = -\frac{2}{s}$$

Now, we have the conditions $x(0) = y(0) = x'(0) = y'(0) = 0$. Since, we already know these values, let us substitute these to obtain the next set of equations that is

$$s^2 L\{x\} + 5sL\{y\} - L\{x\} = \frac{1}{s^2}$$

$$s^2 L\{y\} - 2sL\{x\} - 4L\{y\} = -\frac{2}{s}$$

$$\Rightarrow (s^2 - 1)L\{x\} + 5sL\{y\} = \frac{1}{s^2}$$

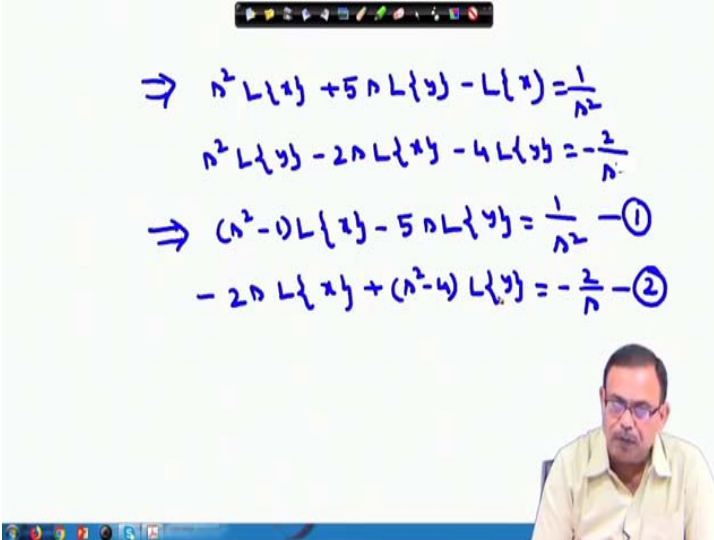
$$-2sL\{x\} + (s^2 - 4)L\{y\} = -\frac{2}{s}$$

Therefore, we have basically two algebraic equations now in two variables namely, $L\{x\}$ and $L\{y\}$ which can be easily solved to obtain $L\{x\}$ and $L\{y\}$ separately as functions of s .

$$L\{x\} = \frac{11s^2 - 4}{s^2(s^2 + 1)(s^2 + 4)} = -\frac{1}{s^2} + \frac{5}{s^2 + 1} - \frac{4}{s^2 + 4}$$

$$L\{y\} = \frac{-2s^2 + 4}{s(s^2 + 1)(s^2 + 4)} = \frac{1}{s} - \frac{2s}{s^2 + 1} + \frac{s}{s^2 + 4}$$

(Refer Slide Time: 24:19)



Handwritten equations on a whiteboard:

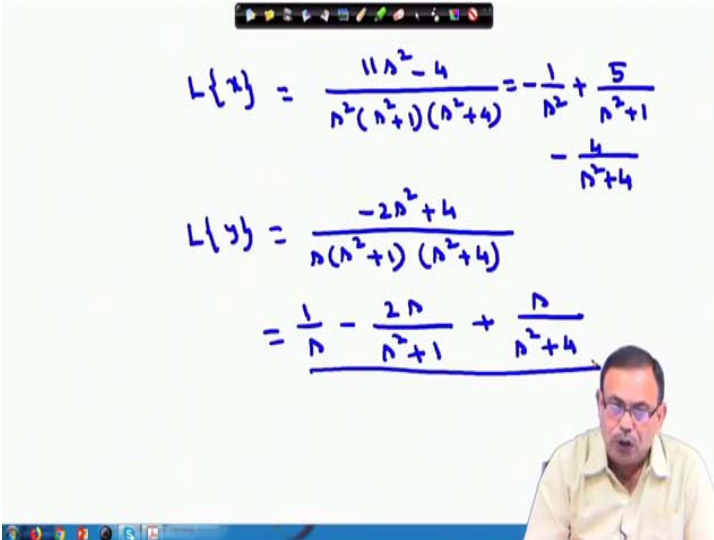
$$\Rightarrow n^2 L\{x\} + 5n L\{y\} - L\{x\} = \frac{1}{n^2}$$

$$n^2 L\{y\} - 2n L\{x\} - 4L\{y\} = -\frac{2}{n}$$

$$\Rightarrow (n^2 - 1)L\{x\} - 5n L\{y\} = \frac{1}{n^2} \quad \text{--- ①}$$

$$-2n L\{x\} + (n^2 - 4)L\{y\} = -\frac{2}{n} \quad \text{--- ②}$$

(Refer Slide Time: 25:47)



Handwritten equations on a whiteboard:

$$L\{x\} = \frac{11s^2 - 4}{s^2(s^2 + 1)(s^2 + 4)} = -\frac{1}{s^2} + \frac{5}{s^2 + 1} - \frac{4}{s^2 + 4}$$

$$L\{y\} = \frac{-2s^2 + 4}{s(s^2 + 1)(s^2 + 4)}$$

$$= \frac{1}{s} - \frac{2s}{s^2 + 1} + \frac{s}{s^2 + 4}$$

So, now if we take the inverse Laplace transform of the set of equations, then we will get,

$$x(t) = -t + 5 \sin t - 2 \sin 2t$$

$$y(t) = 1 - 2 \cos t + \cos 2t.$$

This gives the required result.

(Refer Slide Time: 27:34)

$$\begin{aligned}x &= L^{-1}\left\{\frac{1}{s^2}\right\} + 5L^{-1}\left\{\frac{1}{s^2+1}\right\} - 2L^{-1}\left\{\frac{4}{s^2+4}\right\} \\&= -t + 5 \sin t - 2 \sin 2t \\y &= L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{2s}{s^2+1}\right\} + L^{-1}\left\{\frac{s}{s^2+4}\right\} \\&= 1 - 2 \cos t + \cos 2t\end{aligned}$$

So, we have covered almost all the things of ODE, how to solve one ODE using Laplace transform with constant coefficients, with variable coefficients. If the initial condition at $t = 0$ is not provided, then also how to handle it, that also we have done. We have also gone through simultaneous ODE. So, in the next lecture, we will see the next part that is solution of integral equations using Laplace transform. Thank you.