**Transform Calculus and Its Applications in Differential Equations**
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**Lecture - 17**
**Solution of Simultaneous Ordinary Differential Equations using Laplace Transform**

So now, in this lecture we will start with the earlier topics that is solution of ordinary differential equation with variable coefficients using the Laplace transform. We have given one example in the last lecture. For easy reference, we should solve one more example of ordinary differential equation with variable coefficients so that the solution process gets even more clear.

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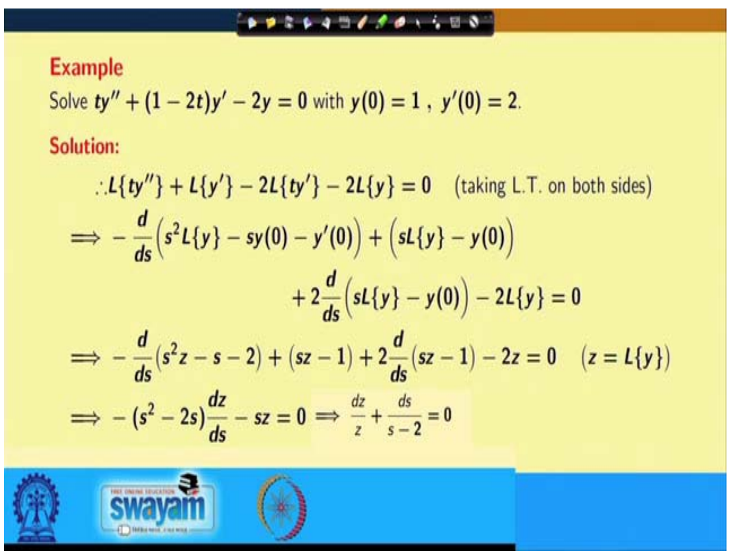


Let us take this example. Suppose we need to solve

$$t\frac{d^2y}{dt^2} + (1-2t)\frac{dy}{dt} - 2y = 0, \qquad y(0) = 1,\ y'(0) = 2.$$

So, here $ty''$ is there which was not present in the earlier example. So, let us see, whenever both $y''$ and $y'$ consist of coefficients which are functions of $t$, what happens in such cases.

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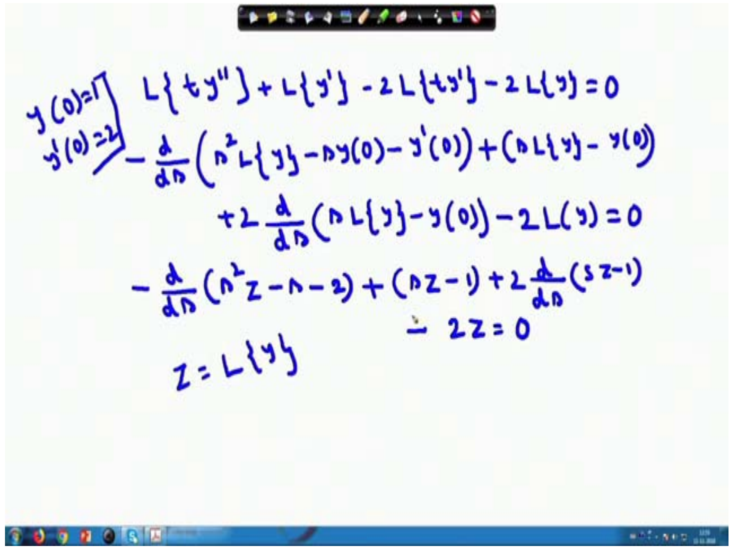


Initially, we will take Laplace transform of the given ODE as follows:

$$L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0.$$

We can easily simplify this using the well-known properties of Laplace transform as follows:

$$-\frac{d}{ds}L\{y''\} + L\{y'\} + 2\frac{d}{ds}L\{y'\} - 2L\{y\} = 0$$

$$\Rightarrow -\frac{d}{ds}[s^2L\{y\} - sy(0) - y'(0)] + sL\{y\} - y(0) + 2\frac{d}{ds}[sL\{y\} - y(0)] - 2L\{y\} = 0$$

$$\Rightarrow -\frac{d}{ds}[s^2z - s - 2] + sz - 1 + 2\frac{d}{ds}[sz - 1] - 2z = 0\,, \quad \text{where}, z = L\{y\}.$$

After differentiation with respect to $s$ and rearranging the terms in a proper way, we obtain

$$\frac{dz}{ds} + \frac{1}{s-2}z = 0$$

$$\Rightarrow \frac{dz}{z} + \frac{ds}{s-2} = 0.$$

Integrating the above, we get,

$$\log z + \log(s-2) = \log c$$

$$\Rightarrow \log[(s-2)z] = \log c$$

$$\Rightarrow z = L\{y\} = \frac{c}{s-2}$$

Now using the inverse Laplace transform technique, we can evaluate $y(t)$ as

$$y(t) = cL^{-1}\left\{\frac{1}{s-2}\right\}$$
$$= ce^{2t}.$$

Now our job is to find the value of the constant $c$ with the help of the condition provided to us. We know that $y(0) = 1$, so we use it as

$$y(0) = ce^{0}$$
$$\Rightarrow 1 = c.$$

Therefore, $c = 1$ so that the final solution of the given ODE is obtained as

$$y(t) = e^{2t}.$$

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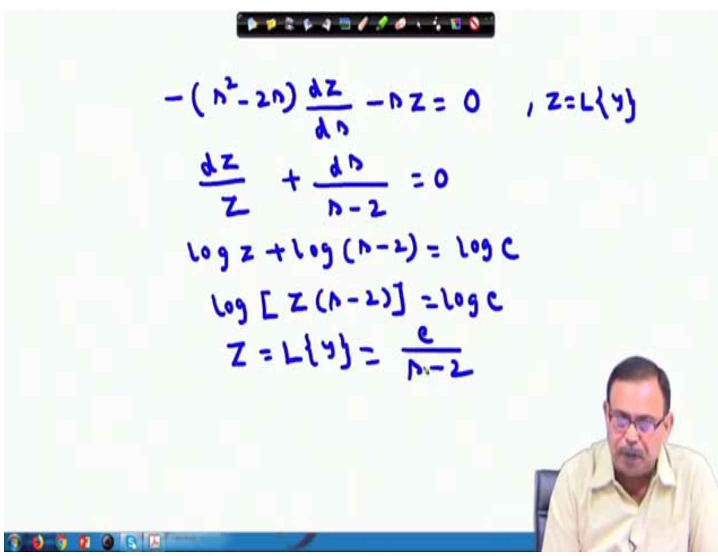


So, whenever we have the variable parameters as coefficients in the ODE, then we have seen that the second order ODE has been converted into a first order ODE. The solution procedure has been well explained through the solved example.

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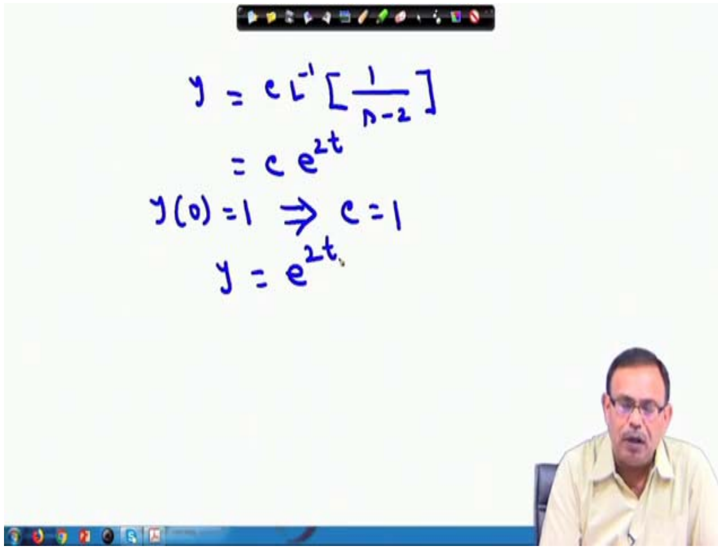


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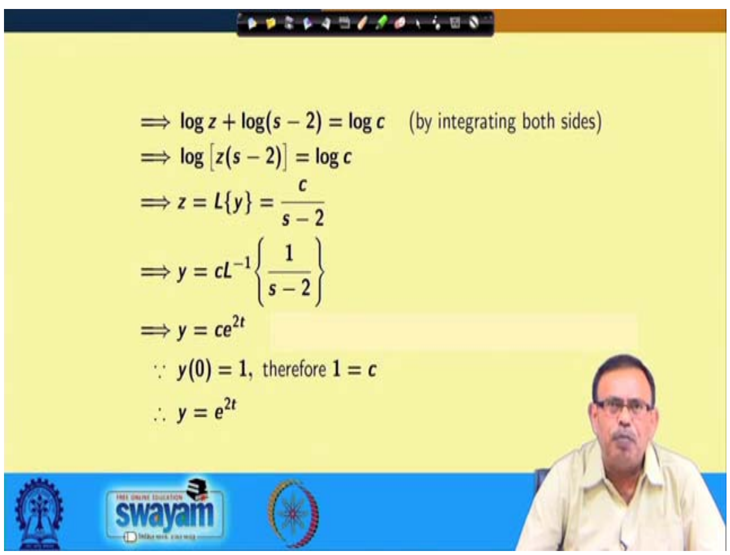


Now we move to another example, linear simultaneous ODE with constant coefficients. Before moving to the example, let us go through the topic once.

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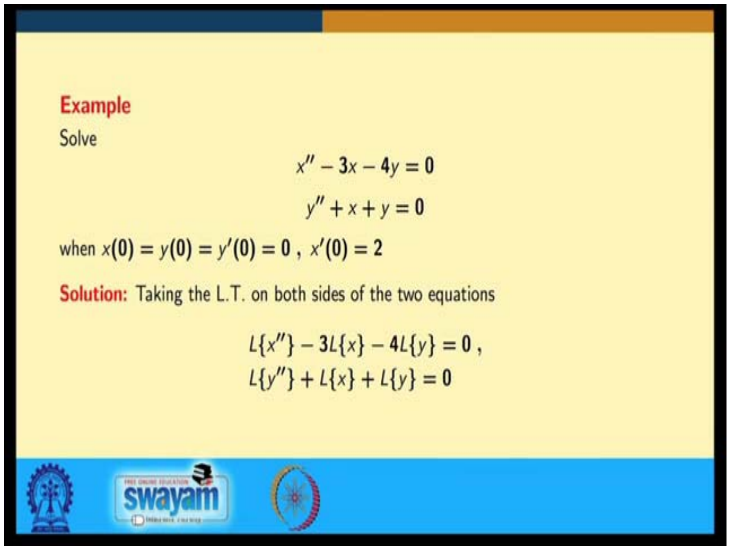


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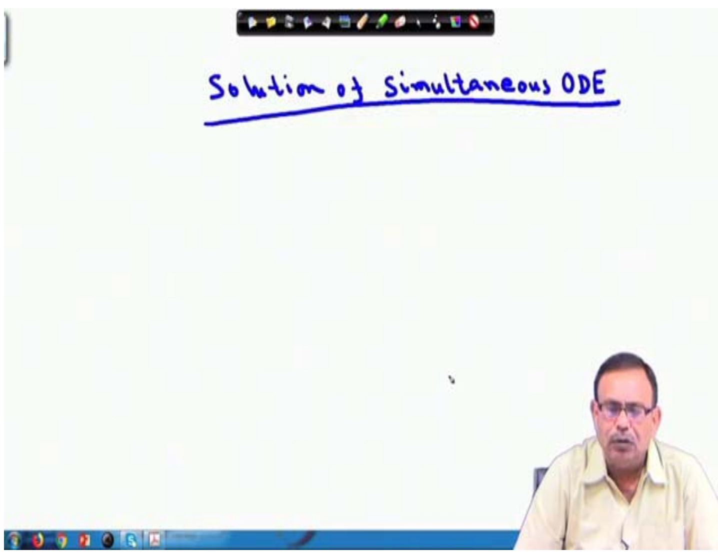


We want to study now, solution of simultaneous ordinary differential equations. Till now we have solved only one differential equation. Now the situation is, if we have simultaneous ordinary differential equations of two different dependent variables, then using Laplace transform technique, we are going to see, how to find out the solution for both the dependent variables.

Suppose we want to solve the following problem,

$$x'' - 3x - 4y = 0$$
$$y'' + x + y = 0$$

and the conditions are given as $x(0) = y(0) = y'(0) = 0, x'(0) = 2$.

To find the solution of course, we have to go through the similar process that is, we have to take the Laplace transform on both sides of both the equations.

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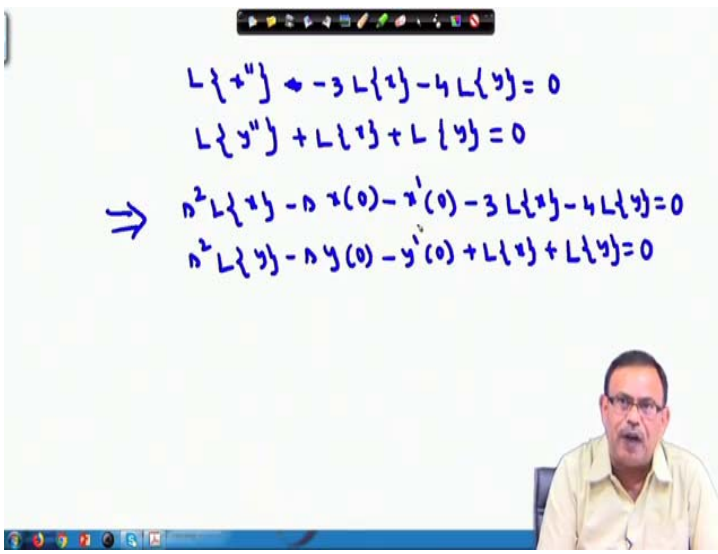

Basically instead of one equation, since now we have two simultaneous equations, we are applying Laplace transform on both sides of those two equations together. So, if we take the Laplace transform on both sides of the given equations, we will obtain,

$$L\{x''\} - 3L\{x\} - 4L\{y\} = 0$$
$$L\{y''\} + L\{x\} + L\{y\} = 0$$
$$\Rightarrow s^2L\{x\} - sx(0) - x'(0) - 3L\{x\} - 4L\{y\} = 0$$
$$s^2L\{y\} - sy(0) - y'(0) + L\{x\} + L\{y\} = 0$$

Please note that, here $x$ and $y$, both are functions of $t$. $x$ and $y$ are the dependent variables whose values we need to obtain in terms of the independent variable $t$.

Now, we have the conditions $x(0) = y(0) = y'(0) = 0, x'(0) = 2$. Since, we already know these values, let us substitute these to obtain the next set of equations that is

$$\begin{aligned} & s^2 L\{x\} - 2 - 3L\{x\} - 4L\{y\} = 0 \\ & s^2 L\{y\} + L\{x\} + L\{y\} = 0 \\ \Rightarrow\ & (s^2 - 3)L\{x\} - 4L\{y\} = 2 \\ & L\{x\} + (s^2 + 1)\, L\{y\} = 0 \end{aligned}$$

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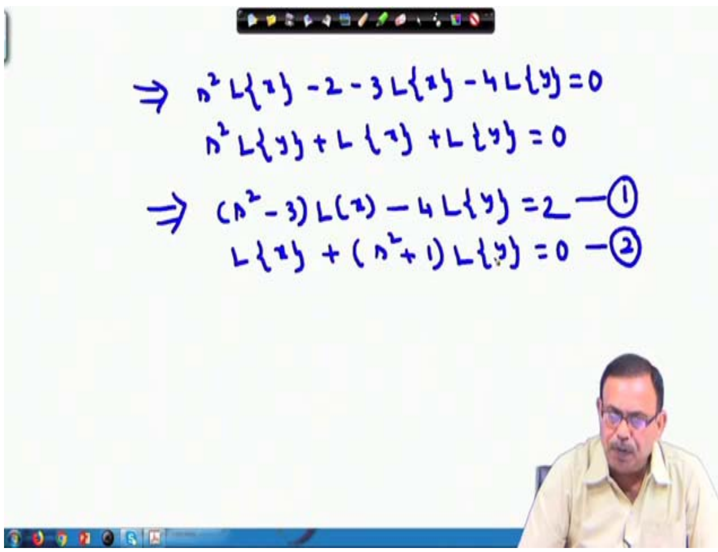


Therefore, we have basically two algebraic equations now in two variables namely, $L\{x\}$ and $L\{y\}$ which can be easily solved to obtain $L\{x\}$ and $L\{y\}$ separately as functions of $s$.

$$L\{x\} = \frac{2(s^2 + 1)}{(s^2 - 1)^2} = \frac{1}{(s - 1)^2} + \frac{1}{(s + 1)^2}$$
$$L\{y\} = -\frac{2}{(s + 1)^2 (s - 1)^2} = \frac{1}{2}\left[-\frac{1}{s - 1} + \frac{1}{s + 1} - \frac{1}{(s - 1)^2} - \frac{1}{(s + 1)^2}\right].$$

We have expressed $L\{x\}$ and $L\{y\}$ in such a way that it becomes easy to evaluate the inverse Laplace transform of these terms. Before taking inverse Laplace transform, we should always express the function of $s$ in some known form. So, now if we take the inverse Laplace transform of the set of equations, then we will get,

$$x(t) = te^{t} + te^{-t} = t(e^{t} + e^{-t})$$

$$y(t) = \frac{1}{2}[-e^{t} + e^{-t} - te^{t} - te^{-t}].$$

This gives the required result.

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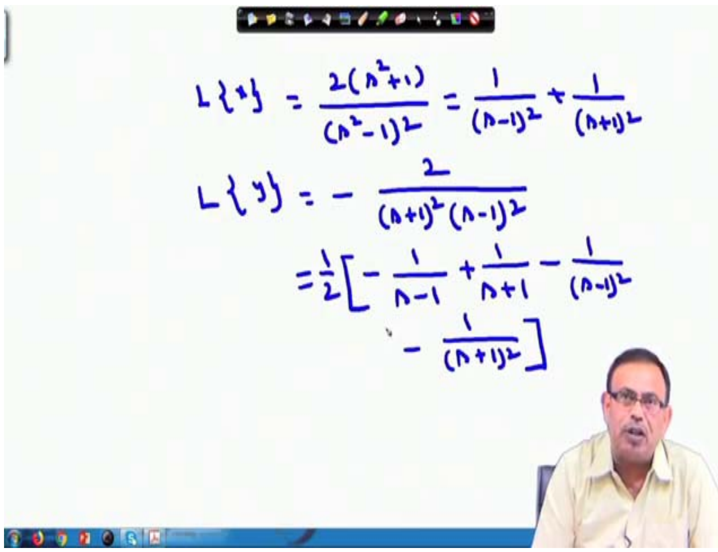


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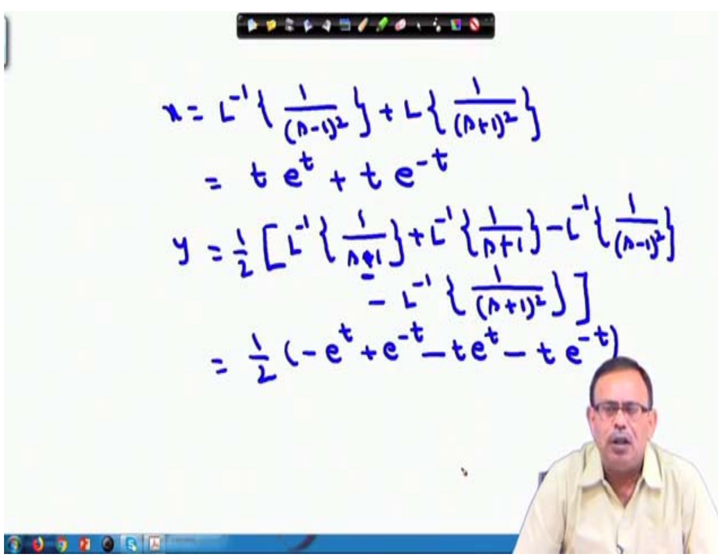


So, like this way, whenever we have two simultaneous ordinary differential equations in two dependent variables $x$ and $y$, applying the Laplace transform on both of them, we will get two simultaneous algebraic equations. Solving these two, we will get the Laplace transforms of $x$ and $y$ separately. Once we have their Laplace transforms, using inverse Laplace transform, always we can find out the values of $x$ and $y$. This is basically the solution procedure for simultaneous ODE using Laplace Transform.

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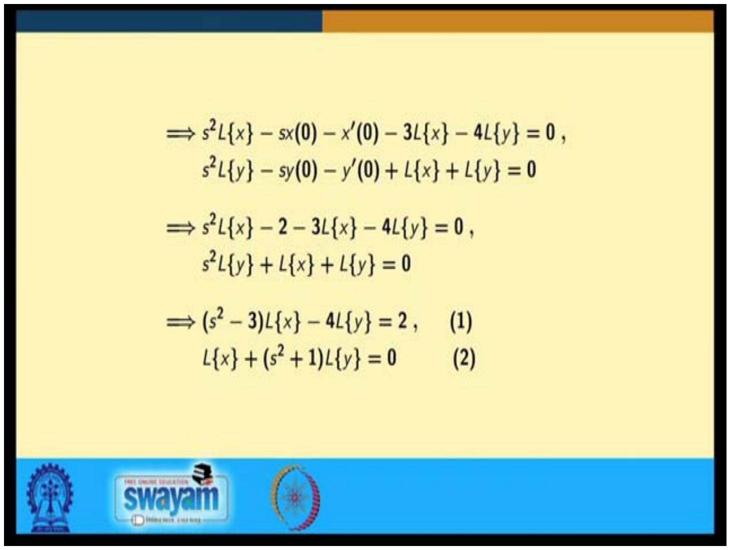


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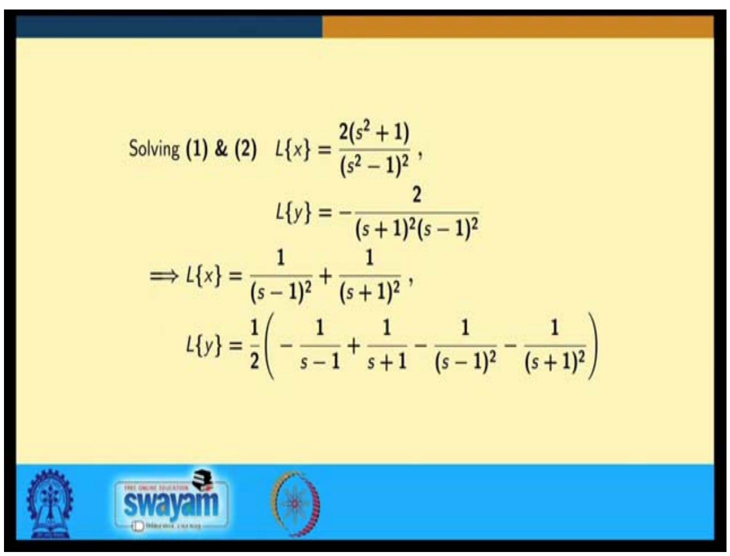

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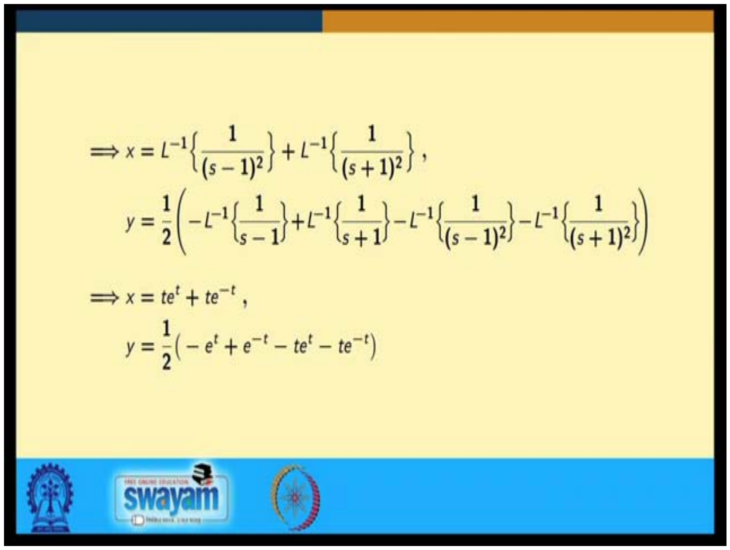


Let us solve one more example so that we can understand the procedure again in much better way.

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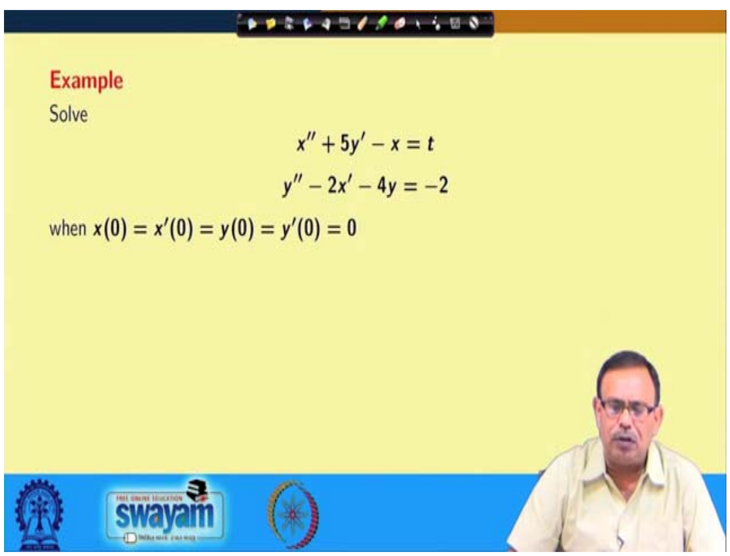


Suppose we need to solve the following problem,

$$x'' + 5y' - x = t$$
$$y'' - 2x' - 4y = -2$$

and the conditions are given as $x(0) = y(0) = x'(0) = y'(0) = 0$.

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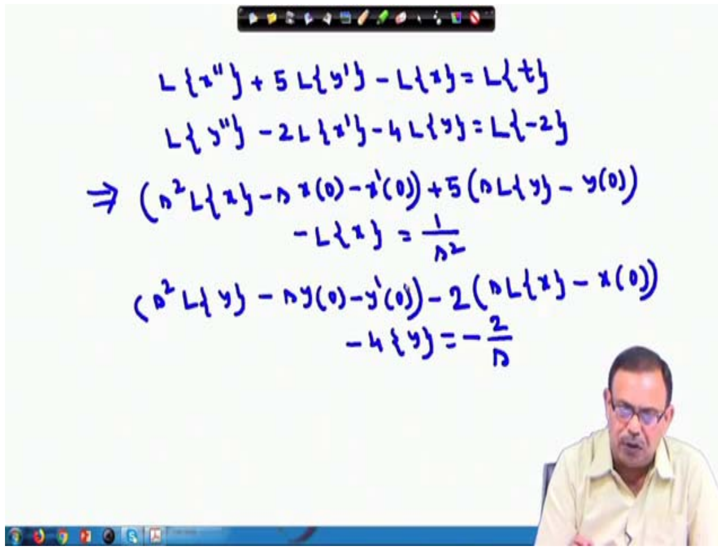


As in the previous case, here also, we take the Laplace transform on both sides of the given equations to obtain,

$$L\{x''\} + 5L\{y'\} - L\{x\} = L\{t\}$$

$$L\{y''\} - 2L\{x'\} - 4L\{y\} = -L\{2\}$$

$$\Rightarrow s^2L\{x\} - sx(0) - x'(0) + 5[sL\{y\} - y(0)] - L\{x\} = \frac{1}{s^2}$$

$$s^2L\{y\} - sy(0) - y'(0) - 2[sL\{x\} - x(0)] - 4L\{y\} = -\frac{2}{s}$$

Now, we have the conditions $x(0) = y(0) = x'(0) = y'(0) = 0$. Since, we already know these values, let us substitute these to obtain the next set of equations that is

$$s^2L\{x\} + 5sL\{y\} - L\{x\} = \frac{1}{s^2}$$

$$s^2L\{y\} - 2sL\{x\} - 4L\{y\} = -\frac{2}{s}$$

$$\Rightarrow (s^2 - 1)L\{x\} + 5sL\{y\} = \frac{1}{s^2}$$

$$-2sL\{x\} + (s^2 - 4)\,L\{y\} = -\frac{2}{s}$$

Therefore, we have basically two algebraic equations now in two variables namely, $L\{x\}$ and $L\{y\}$ which can be easily solved to obtain $L\{x\}$ and $L\{y\}$ separately as functions of $s$.

$$L\{x\} = \frac{11s^2 - 4}{s^2(s^2+1)(s^2+4)} = -\frac{1}{s^2} + \frac{5}{s^2+1} - \frac{4}{s^2+4}$$

$$L\{y\} = \frac{-2s^2 + 4}{s(s^2+1)(s^2+4)} = \frac{1}{s} - \frac{2s}{s^2+1} + \frac{s}{s^2+4}.$$

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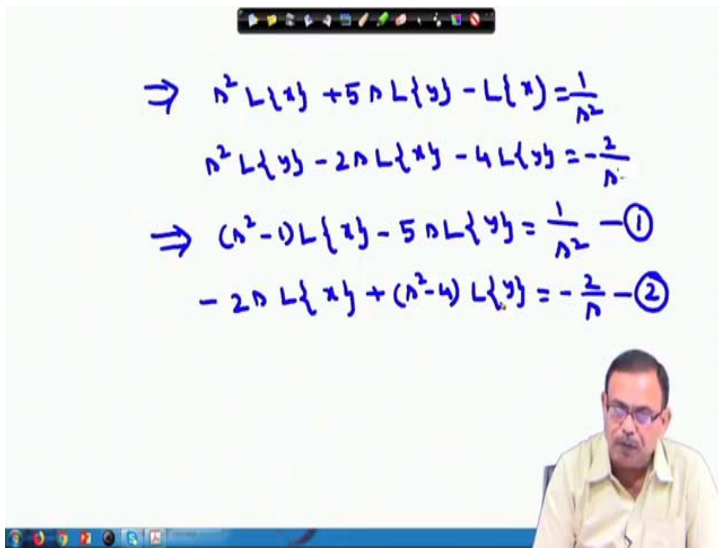


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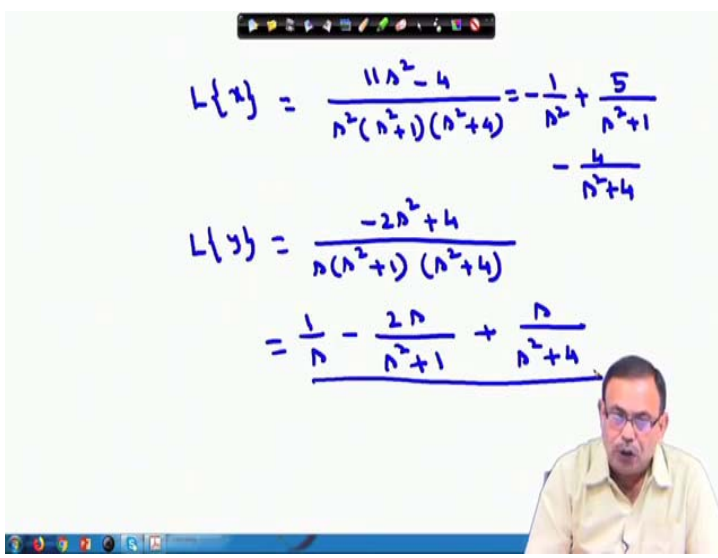

So, now if we take the inverse Laplace transform of the set of equations, then we will get,

$$x(t) = -t + 5\sin t - 2\sin 2t$$
$$y(t) = 1 - 2\cos t + \cos 2t.$$

This gives the required result.

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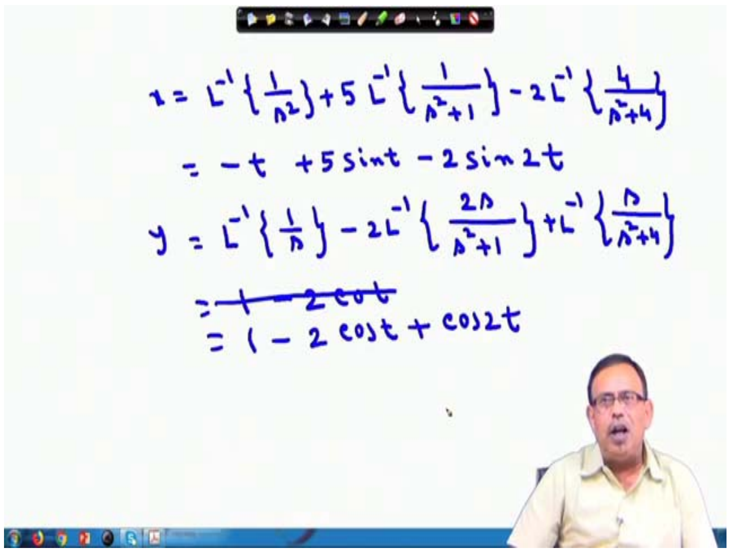


So, we have covered almost all the things of ODE, how to solve one ODE using Laplace transform with constant coefficients, with variable coefficients. If the initial condition at $t = 0$ is not provided, then also how to handle it, that also we have done. We have also gone through simultaneous ODE. So, in the next lecture, we will see the next part that is solution of integral equations using Laplace transform. Thank you.