Transform Calculus and Its Applications in Differential Equations
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Lecture – 16
Solution of Ordinary Differential Equations with variable coefficients using Laplace Transform

So, in this lecture, we will see what are the solution processes for various other types of ordinary differential equations and what kind of problems we may face.

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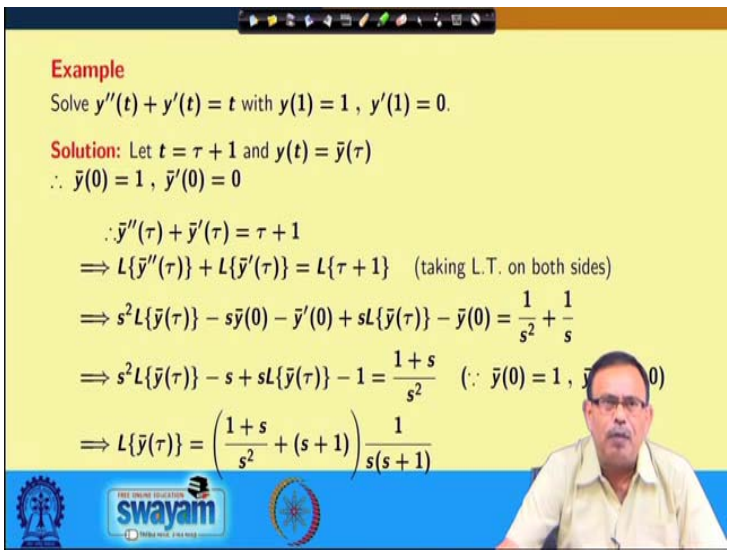

Consider this problem,

$$y''(t) + y'(t) = t \quad \text{with} \quad y(1) = 1, y'(1) = 0.$$

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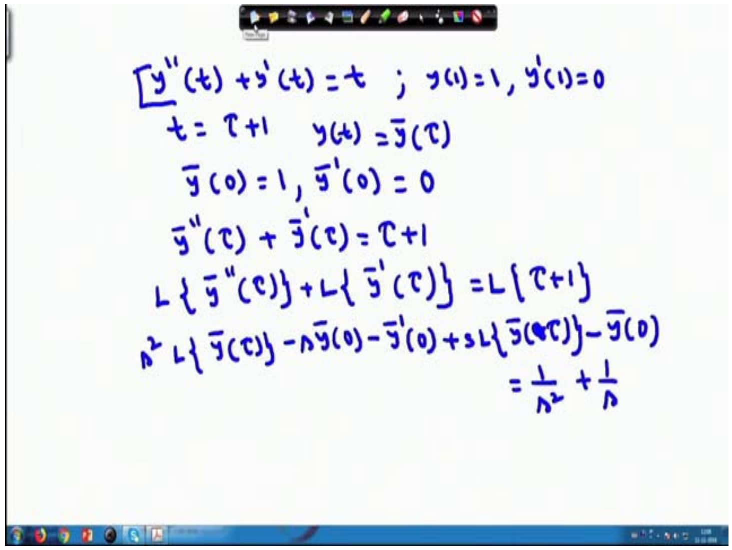

The conditions are given as, $y(1) = 1,\ y'(1) = 0$. Please note this, that in the earlier problem, we had the conditions at $t = 0$ only.

Whenever we are finding out the Laplace transform of $y'(t)$ or $y''(t)$, always it will involve $y(0)$ or $y'(0)$ but in this case, we know the values of $y(1)$ and $y'(1)$.

Whenever we are facing such type of problems, then we can make the change of parameter, that is we can substitute t by some other parameter in such a fashion that the conditions that are provided to us at $t = 1$ turn out to be conditions for the new variable at the point 0 instead, which we will actually require.

Suppose, in this case, if we substitute $t = \tau + 1$ so that $\tau = t - 1$. Then, at $t = 1$, we have $\tau = 0$. Once we are getting this, then the given conditions also will get changed and we will obtain the conditions at $\tau = 0$.

So, whenever we are taking $t = \tau + 1$, a new function of τ will come into the scenario. We are denoting it as $\bar{y}(\tau)$ .Therefore $y(t)$ turns into $\bar{y}(\tau)$ after transformation. Since t and τ are linearly related, so $y'(t) = \bar{y}'(\tau),\ y''(t) = \bar{y}''(\tau)$. And our problem is now transformed to

$$\bar{y}''(\tau) + \bar{y}'(\tau) = \tau + 1 \text{ with } \bar{y}(0) = 1, \bar{y}'(0) = 0.$$

Solving this, when we will obtain the value of $\bar{y}(\tau)$, then by back substitution, we can evaluate $y(t)$ as well. So, let us try to find out the solution of the converted ordinary differential equation.

We take the Laplace transform on both sides to obtain

$$L\{\bar{y}''(\tau)\} + L\{\bar{y}'(\tau)\} = L\{\tau\} + L\{1\}$$

$$\Rightarrow [s^2 L\{\bar{y}(\tau)\} - s\bar{y}(0) - \bar{y}'(0)] + [sL\{\bar{y}(\tau)\} - \bar{y}(0)] = \frac{1}{s^2} + \frac{1}{s}.$$

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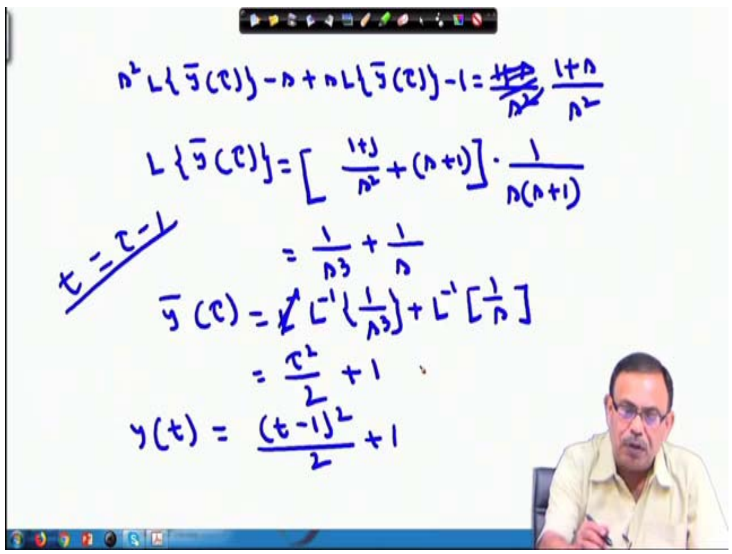

Now putting the values of $\bar{y}(0)$ and $\overline{y}'(0)$ and simplifying we get,

$$L\{\bar{y}(\tau)\} = \frac{1}{s^3} + \frac{1}{s}.$$

So now taking the inverse Laplace transform,

$$\bar{y}(\tau) = L^{-1}\left\{\frac{1}{s^3}\right\} + L^{-1}\left\{\frac{1}{s}\right\} = \frac{\tau^2}{2} + 1.$$

So, from here, we can obtain $y(t)$ because we know $\tau = t - 1$. So, substituting the value of τ,

$$y(t) = \frac{1}{2}(t-1)^2 + 1 = \frac{t^2}{2} - t + \frac{3}{2}.$$

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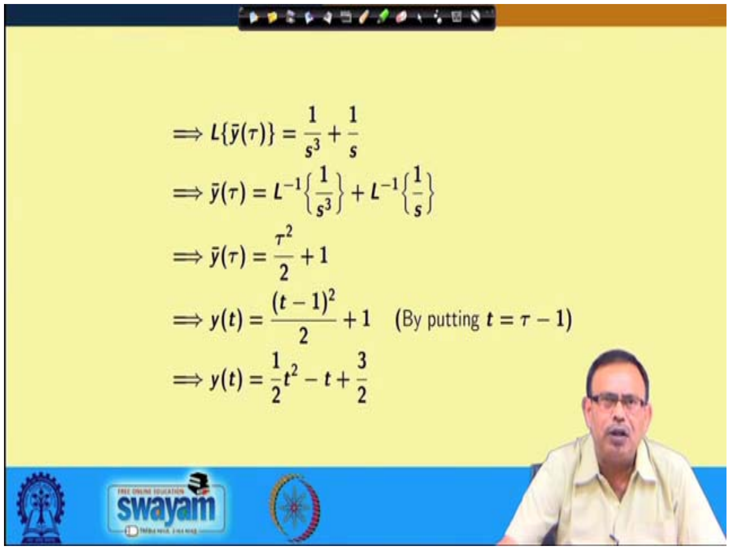

Let us see another similar type of problem:

$$y''(t) + 2y'(t) + y(t) = 2e^{1-t} \text{ with } y(1) = 1, y'(1) = -1.$$

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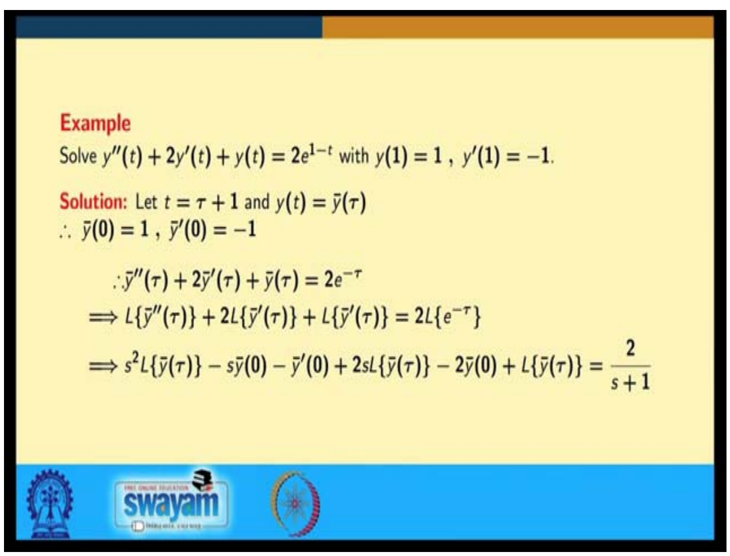

Here also the condition is provided at $t = 1$. So, we have to make the substitution over here.

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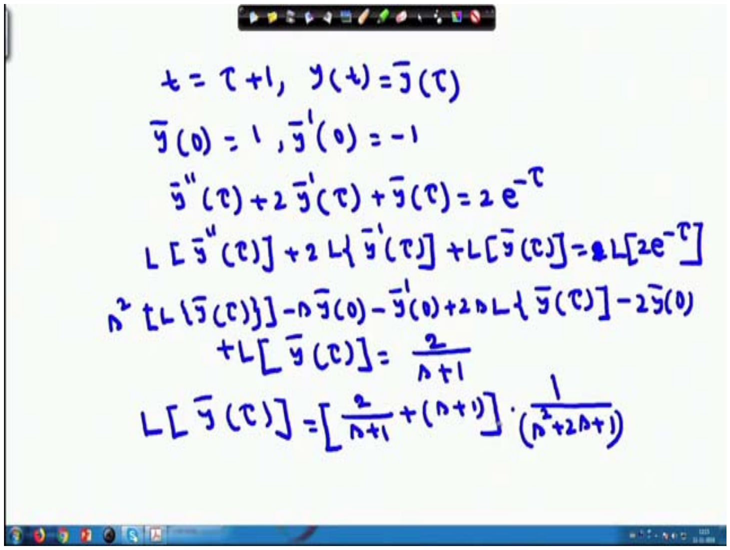

We substitute $t = \tau + 1$ so that $\tau = t - 1$ and $y(t) = \bar{y}(\tau)$ and so are the derivatives. Our conditions will be changed to $\bar{y}(0) = 1$, $\bar{y}'(0) = -1$.

So, our original equation is transformed to

$$\bar{y}''(\tau) + 2\bar{y}'(\tau) + \bar{y}(\tau) = 2e^{-\tau} \text{ with } \bar{y}(0) = 1, \bar{y}'(0) = -1.$$

Now, we take the Laplace transform on both sides to get

$$[s^2 L\{\bar{y}(\tau)\} - s\bar{y}(0) - \bar{y}'(0)] + 2[sL\{\bar{y}(\tau)\} - \bar{y}(0)] + L\{\bar{y}(\tau)\} = \frac{2}{s+1}.$$

Now putting the values of $\bar{y}(0)$ and $\overline{y}'(0)$ and simplifying we get,

$$L\{\bar{y}(\tau)\} = \frac{2}{(s+1)^3} + \frac{1}{s+1}.$$

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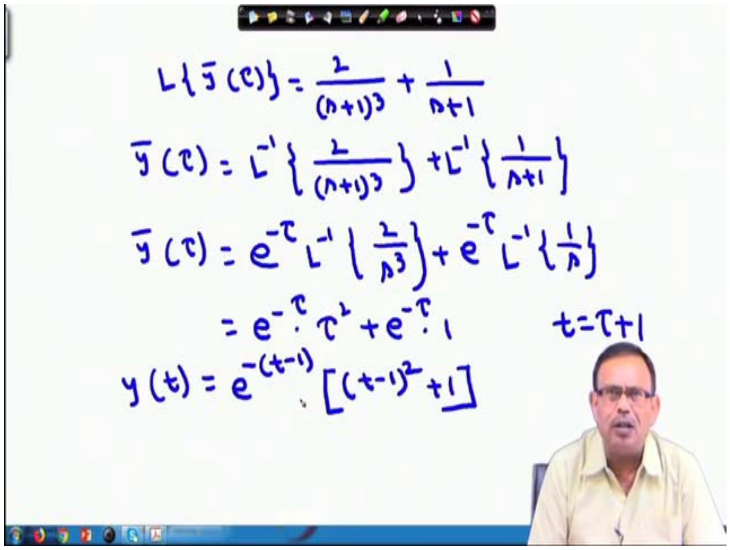

Now taking the inverse Laplace transform, we have,

$$\begin{aligned}\bar{y}(\tau) &= L^{-1}\left\{\frac{2}{(s+1)^3}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} \\ &= e^{-\tau}\left[L^{-1}\left\{\frac{2}{s^3}\right\} + L^{-1}\left\{\frac{1}{s}\right\}\right] \quad \text{(using first shifting theorem)} \\ &= e^{-\tau}[\tau^2 + 1].\end{aligned}$$

So, from here, we can obtain $y(t)$ because we know $\tau = t - 1$. So, substituting the value of τ, we get

$$y(t) = e^{1-t}[(t-1)^2 + 1].$$

So, by changing the scale of t, we can bring some new parameter and we can try to find out the solution by using the condition of $\bar{y}(0)$ or $\bar{y}'(0)$.

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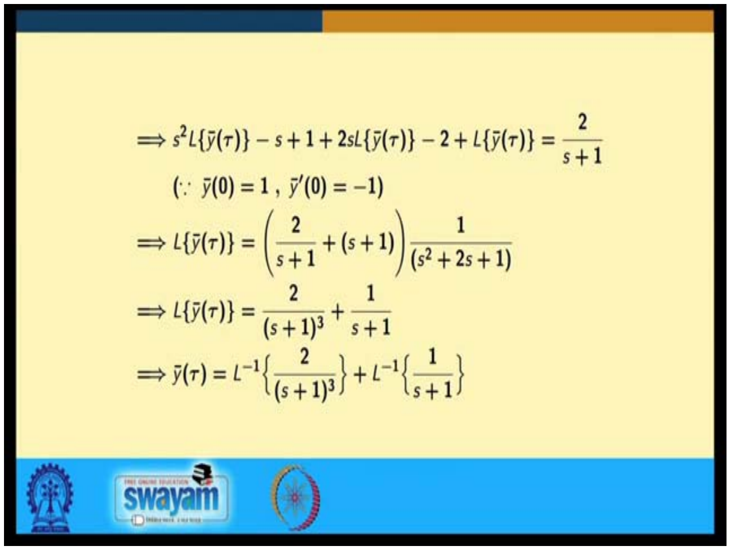

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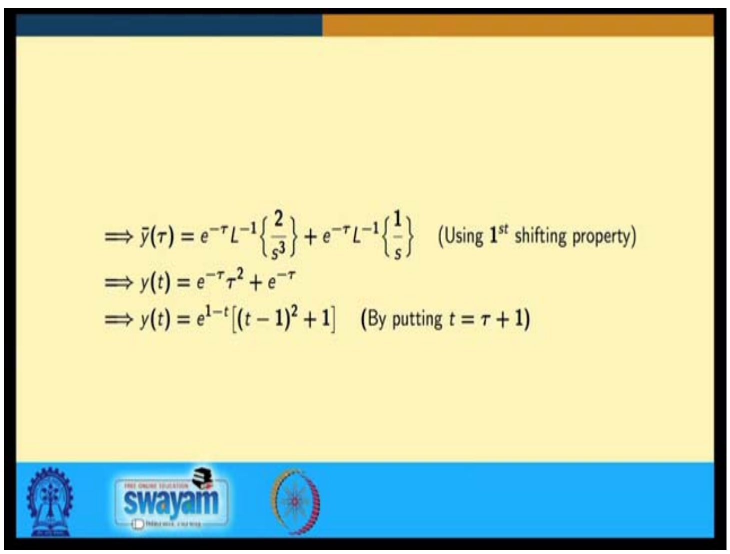

Now, let us see another problem:

$$ty'' + y' + 4ty = 0 \quad \text{with } y(0) = 3,\ \ y'(0) = 0.$$

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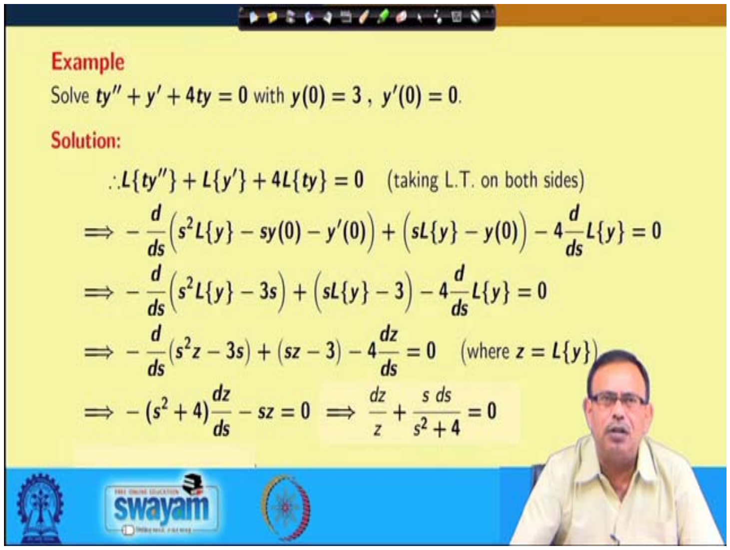

Please note that here, the coefficients of y'' and y' are not constant. Let us see the solution process for this type of problem.

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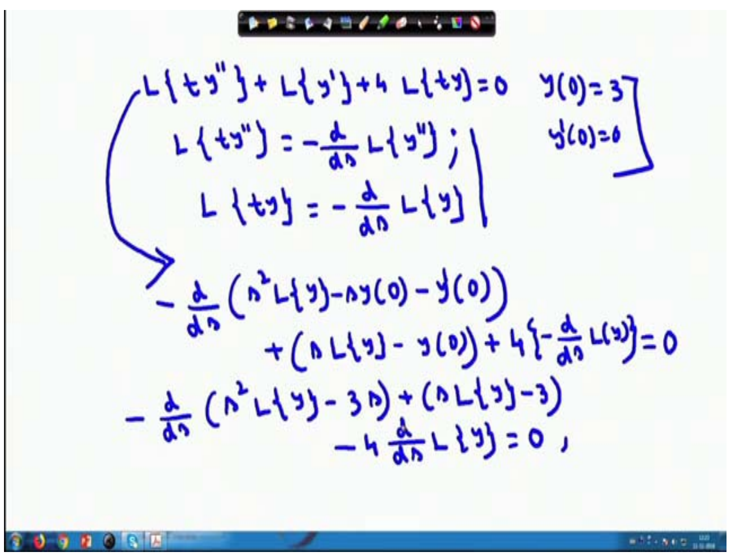

We take Laplace transform on both sides of the given equation,

$$L\{ty''\} + L\{y'\} + 4L\{ty\} = 0.$$

Now we use the well-known properties of Laplace transform (multiplication by powers of t) as required to obtain

$$-\frac{d}{ds}(L\{y''\}) + L\{y'\} - 4\frac{d}{ds}L\{y\} = 0$$

$$\Rightarrow -\frac{d}{ds}[s^2L\{y\} - sy(0) - y'(0)] + [sL\{y\} - y(0)] - 4\frac{d}{ds}L\{y\} = 0$$

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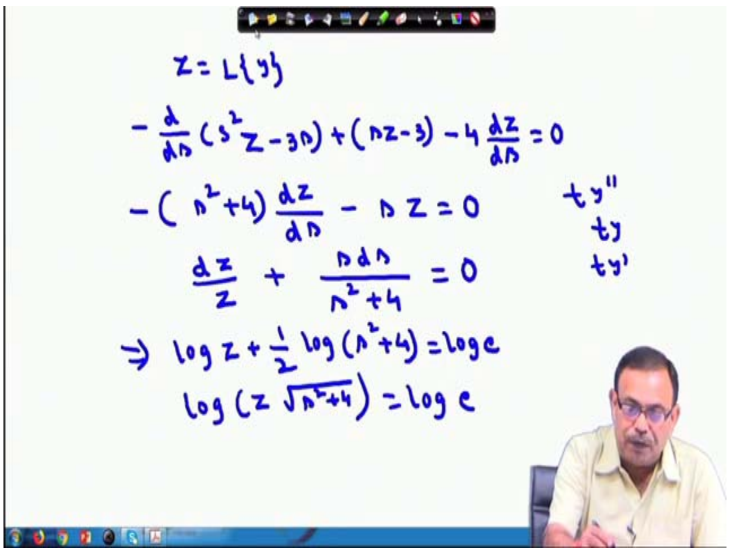

Now suppose $z(s) = L\{y\}$. And if we put the values of $y(0)$ and $y'(0)$, then we get

$$-\frac{d}{ds}(s^2z - 3s) + (sz - 3) - 4\frac{dz}{ds} = 0$$

$$\Rightarrow -(s^2 + 4)\frac{dz}{ds} - sz = 0$$

$$\Rightarrow \frac{dz}{z} + \frac{s\,ds}{s^2 + 4} = 0.$$

So, whenever we are having this ty'' or ty' or ty, then effectively, we are not getting any algebraic equation, but we are going back to some simplified ODE only. Please note this change.

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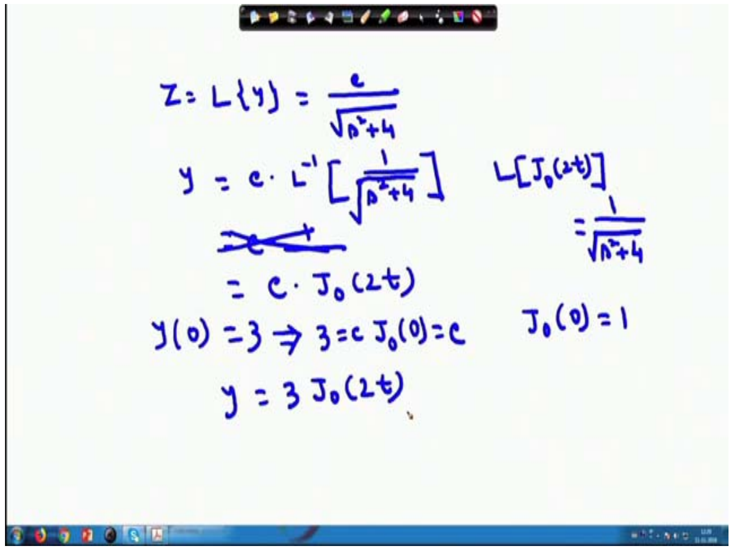

Now integrating both sides we get,

$$\log z + \frac{1}{2}\log(s^2 + 4) = \log c$$
$$\Rightarrow z = L\{y\} = \frac{c}{\sqrt{s^2 + 4}}.$$

where c is integrating constant.

Now, taking the Laplace inverse, we can write down y as:

$$y(t) = c\, L^{-1}\left\{\frac{1}{\sqrt{s^2 + 4}}\right\}$$
$$= c\, J_0(2t). \qquad \left(\because L\{J_0(2t)\} = \frac{1}{\sqrt{s^2 + 4}}\right)$$

But we know that,

$$y(0) = 3$$
$$\Rightarrow c\, J_0(0) = 3$$
$$\Rightarrow c = 3 \quad [\because J_0(0) = 1]$$

Therefore, the final solution is obtained as

$$y(t) = 3J_0(2t).$$

So, in this case, it is little complex that we are not directly obtaining any algebraic equation, but we are obtaining one ordinary differential equation again. And by finding the solution of the ODE, we are getting $L\{y\}$ as the solution. From there, we are finding out the final solution taking the inverse Laplace transform.

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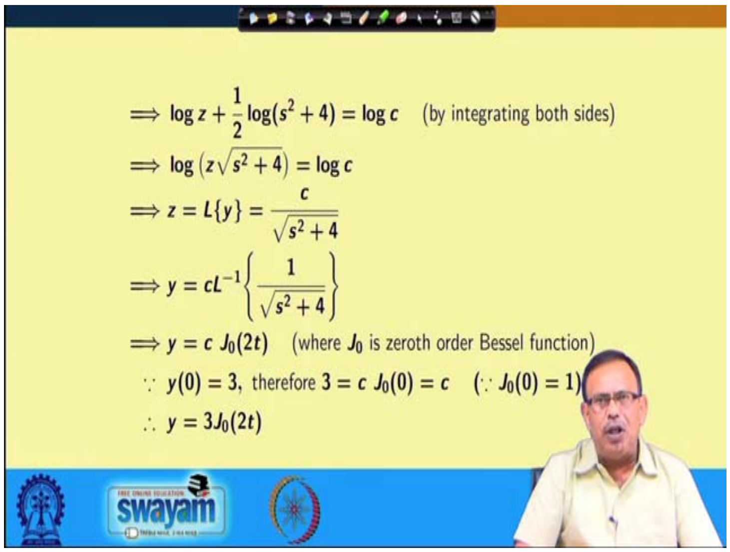

In the next lecture, we will take some more examples on this, so that the solution procedure gets more clarified. Thank you.