Transform Calculus and Its Applications in Differential Equations
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Lecture – 15
Solution of Ordinary Differential Equations with constant coefficients using Laplace Transform

Now let us come to another application of Laplace transform: how to find out the solution of an ordinary differential equation using Laplace transform? In general, what we have observed is that, in order to find out the solution of an ordinary differential equation, usually we use the C.F.-P.I. method or some other known methods, depending upon the type of the problem.

Here we will see that whenever we have an ordinary differential equation, it can be transformed into one algebraic equation only, using Laplace transform. And from the solution of the algebraic equation, we can get back the solution of the original problem using inverse Laplace transform.

So, in this way, it becomes very easy to solve an ODE. Let us go through the involved procedure.

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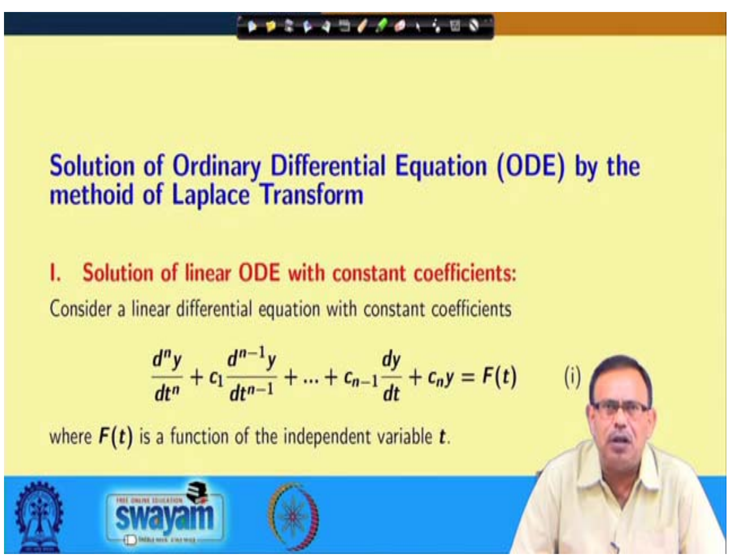

First, we will solve an ODE with constant coefficients, then we will continue with the one having variable coefficients as well. Then we will see the solution of linear simultaneous ODE.

The first one is the solution of ordinary differential equation with constant coefficients by the method of Laplace transform. What do we mean by linear ODE with constant coefficients? We consider the following as a linear differential equation with constant coefficients namely $c_1, c_2, \ldots, c_n$. Thus the general form of an ODE with constant coefficient is

$$\frac{d^n y}{dx^n} + c_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + c_{n-1} \frac{dy}{dx} + c_n y = F(t)$$

where F is a function of the only independent variable t. Here y is the dependent variable and t is the independent variable.

Whenever we try to find out the solution in normal classical approach, we use the complementary function (C.F.) and we use the particular integral (P.I.). We find out C.F., we find out P.I. and from there we try to find out the solution of the arbitrary constants using the given conditions.

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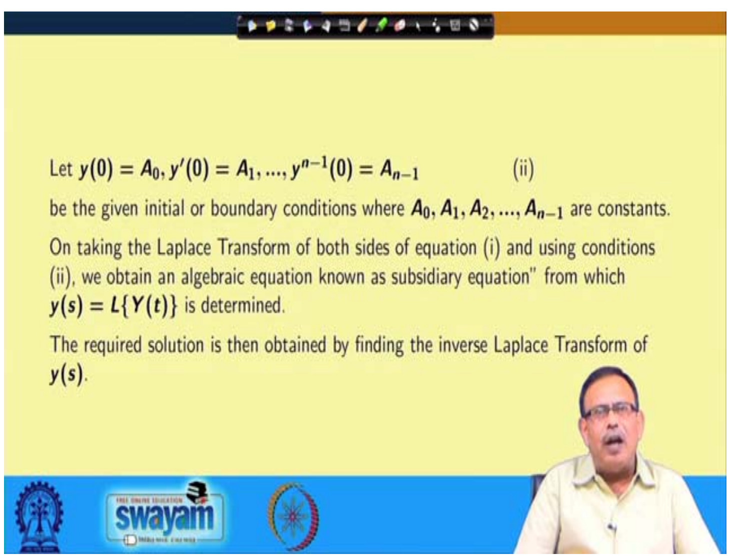

So, we assume that a set of initial or boundary conditions are given like $y(0) = A_0$, $y'(0) = A_1, \cdots, y^{n-1}(0) = A_{n-1}$ where $A_0, A_1, A_2, \ldots, A_{n-1}$ are constants. So, effectively if we have the n^{th} derivative that is $\frac{d^n y}{dx^n}$, then number of constants should be n and n initial or boundary conditions will be appearing so that we can find out the values of those constants easily.

So the procedure goes as: We will take the Laplace transform on both the sides of the given ordinary differential equation. Using the properties of Laplace transform and the given initial or boundary conditions, we will then obtain an algebraic equation from it, known as subsidiary equation. And from that subsidiary equation, we can find out a solution in the form of $y(s)$ or Laplace transform of $Y(t)$. Once we are getting this, then using the inverse Laplace transform, we can obtain the value of $Y(t)$ easily i.e., the required solution is obtained by finding the inverse Laplace transform of $y(s)$.

So, basically the method is very simple. Whenever the original problem is given to us, we will find out the Laplace transform on both sides of the given equation. We know the Laplace transform of $\frac{d^n y}{dx^n}$ or Laplace transform of $t\frac{d^n y}{dx^n}$ and we will use the initial or boundary conditions, whatever is supplied to us, and using those conditions, always we can transform our original ordinary differential equation into an algebraic equation of the form $y(s) = L\{Y(t)\}$.

So that from there, if we take the inverse Laplace transform, then we can easily evaluate $Y(t)$. Please note one thing over here that all the values of the conditions are given at the point $t = 0$.

Because if we take any value of the function at any other point, we have to use some other substitution beforehand, because whenever we take Laplace transform of derivatives, always we will need values at the point $t = 0$.

So, let us see, how it works. Suppose we want to solve

$$\frac{d^2 y}{dt^2} + y = 0\,, \qquad y = 1,\;\; \frac{dy}{dt} = 0 \text{ when } t = 0.$$

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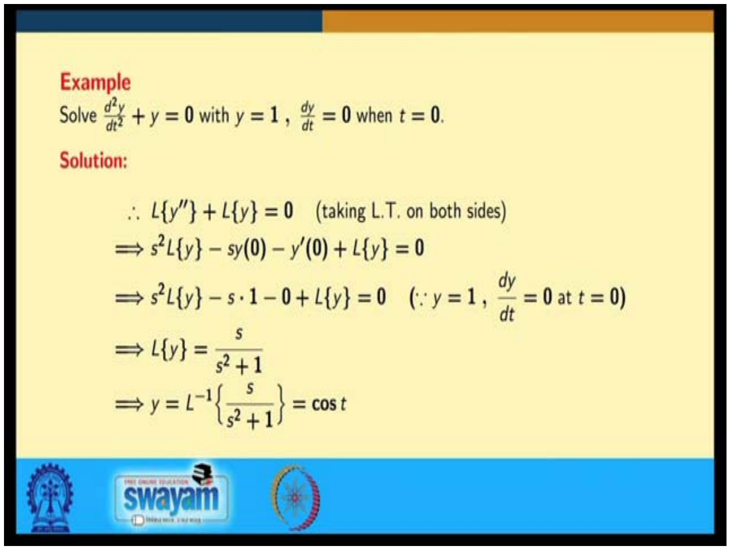

Let us see the solution process.

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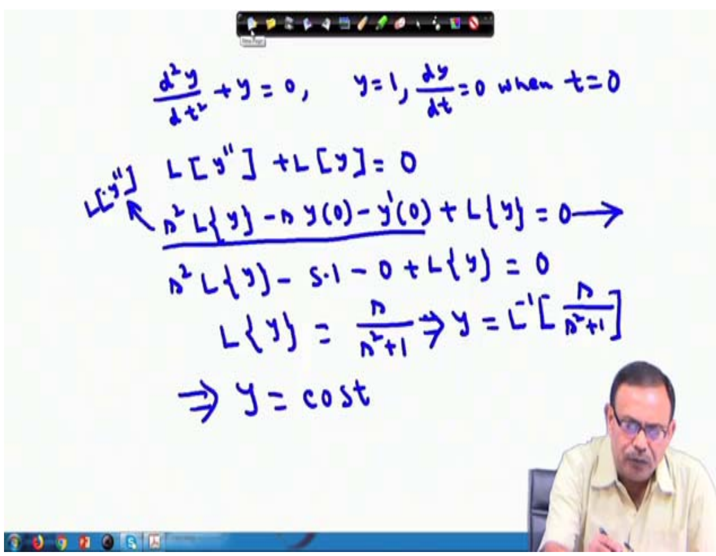

We will first take Laplace transform on both sides of the given equation. We know that, $L\left\{\frac{d^2y}{dt^2}\right\} = s^2L\{y\} - sy(0) - y'(0)$. So, taking Laplace transform on both sides we get,

$$s^2L\{y\} - sy(0) - y'(0) + L\{y\} = 0$$

$$\Rightarrow s^2 L\{y\} - s \cdot 1 - 0 + L\{y\} = 0$$

$$\Rightarrow L\{y\} = \frac{s}{s^2+1}.$$

Once we have obtained $L\{y\}$, we can now use the inverse Laplace technique to evaluate $y(t)$ as:

$$y(t) = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.$$

So, we have obtained the final solution. Thus, in very easy steps, without doing much calculations, we are taking the Laplace transform on both sides of the given equation and by that way, the ordinary differential equation is transformed into an algebraic equation as shown in the problem. Here, we have substituted the values which are provided to us i.e., initial conditions or boundary conditions, whatever it may be. Using those values, we are obtaining $L\{y\}$ and taking inverse Laplace transform, we will get the solution. Now at present, we have taken ODE with constant coefficients only, not variable coefficients. So, this is the basic mechanism to solve such problems.

Let us take another problem:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = 0 \ , \qquad y = 1, \frac{dy}{dt} = 1 \text{ when } t = 0.$$

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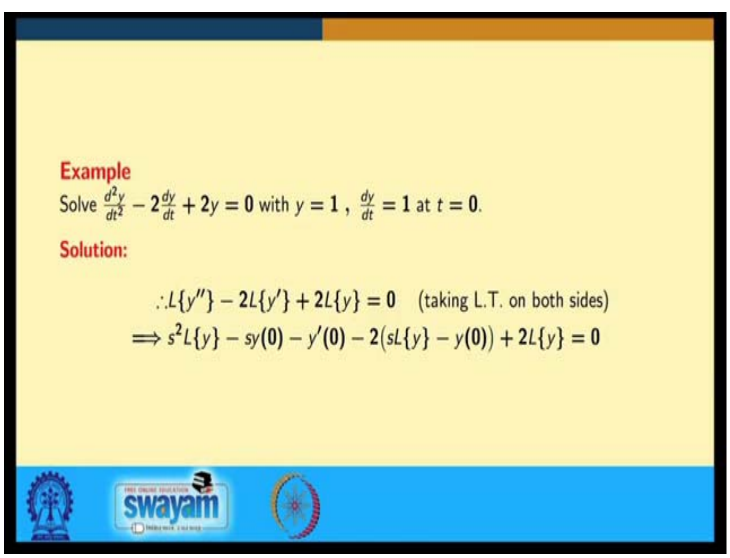

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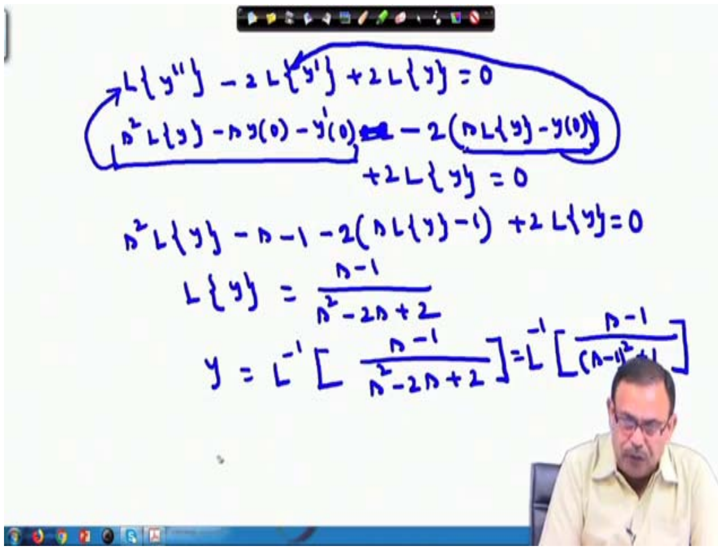

Similar to the previous problem, here also, we will take Laplace transform on both sides of the given ordinary differential equation to obtain

$$
\begin{aligned}
&[s^2L\{y\} - sy(0) - y'(0)] - 2[sL\{y\} - y(0)] + 2L\{y\} = 0 \\
\Rightarrow\ &[s^2L\{y\} - s - 1] - 2[sL\{y\} - 1] + 2L\{y\} = 0 \qquad (\because\ y(0) = 1, y'(0) = 1) \\
\Rightarrow\ &L\{y\} = \frac{s-1}{s^2 - 2s + 2} = \frac{s-1}{(s-1)^2 + 1}.
\end{aligned}
$$

So, once we are getting $L\{y\}$, we can easily obtain y from here by taking Laplace inverse. Therefore, we get

$$\Rightarrow y(t) = L^{-1}\left\{\frac{s-1}{(s-1)^2 + 1}\right\}$$

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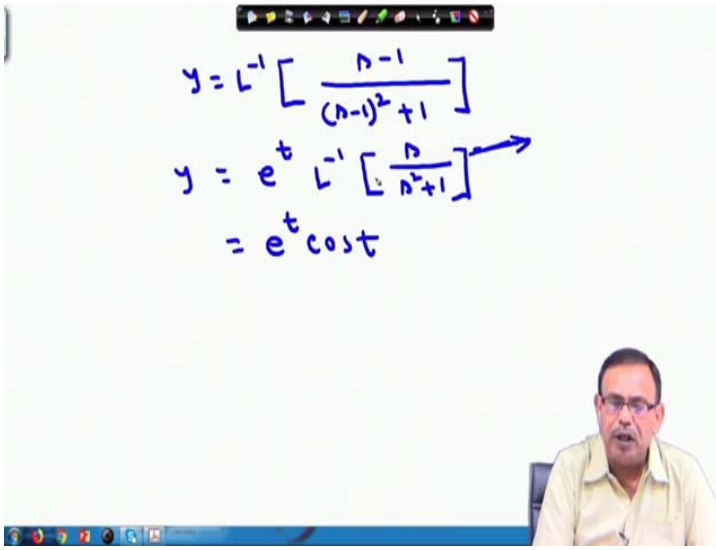

Using the first shifting theorem, we can write down

$$y(t) = e^t L^{-1}\left\{\frac{s}{s^2+1}\right\} = e^t \cos t.$$

So, by this way, we can find out the solution.

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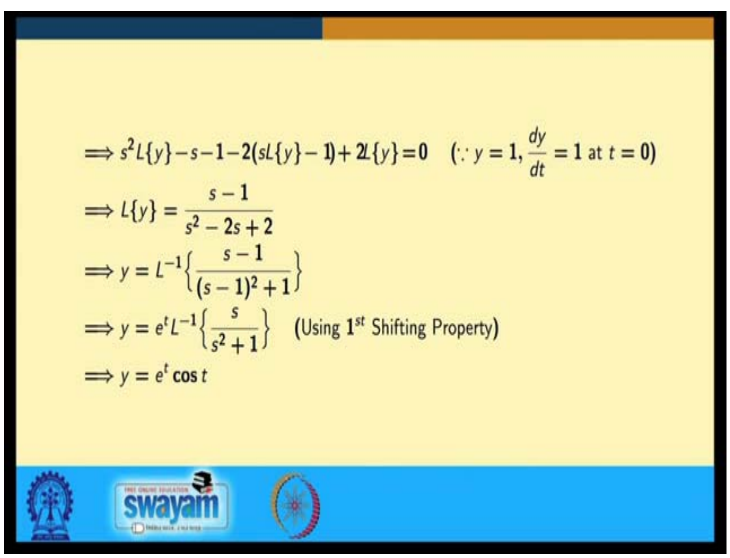

Now, in the earlier problems, we see that the right hand side was 0 instead of any function of t. We are now taking another problem where the right hand side is a function of the independent variable t. Suppose we need to solve the following:

$$\frac{d^2y}{dt^2} + y = t\cos 2t \quad , \qquad y = 0, \frac{dy}{dt} = 0 \text{ when } t = 0.$$

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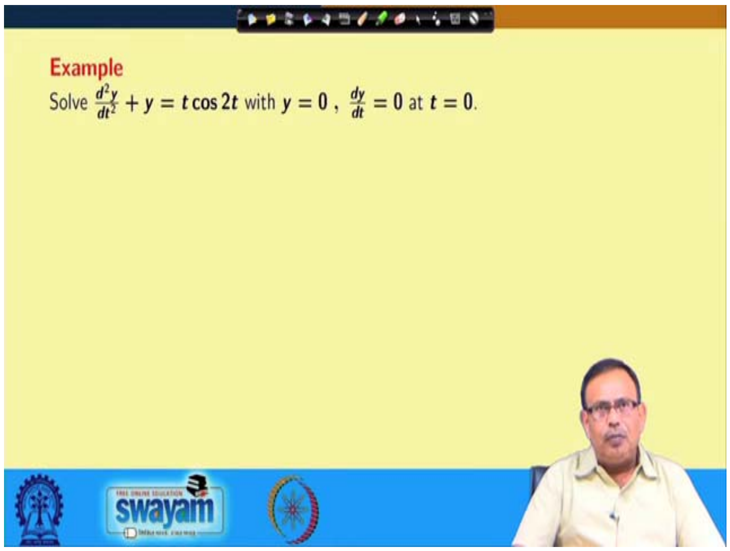

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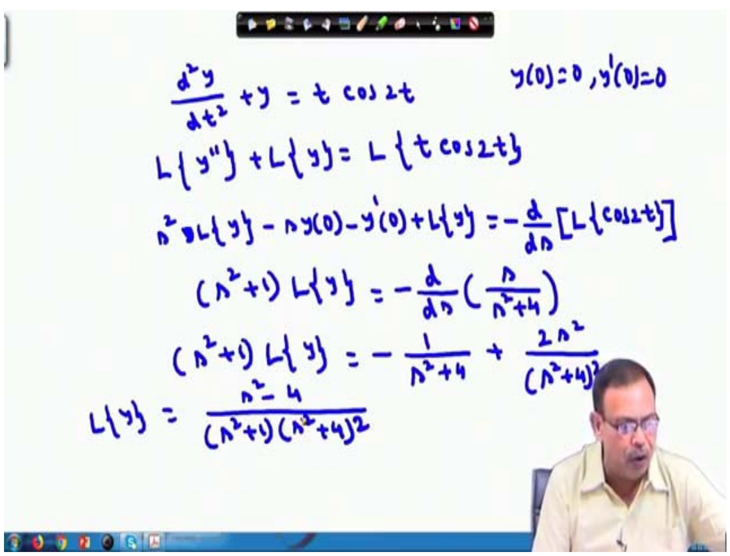

We take the Laplace transform on both sides of the given equation as usual.

$$\therefore L\left\{\frac{d^2y}{dt^2}\right\} + L\{y\} = L\{t\cos 2t\}.$$

This again using the property, we can write,

$$s^2L\{y\} - sy(0) - y'(0) + L\{y\} = -\frac{d}{ds}[L\{\cos 2t\}]$$

$$\Rightarrow s^2L\{y\} - 0 - 0 + L\{y\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$\Rightarrow (s^2+1)L\{y\} = \frac{s^2-4}{(s^2+4)^2}$$

$$\Rightarrow L\{y\} = \frac{s^2-4}{(s^2+1)(s^2+4)^2}.$$

Now we have to break it in such a way that in the denominator, we have only one factor using the normal procedure as discussed in the previous lectures.

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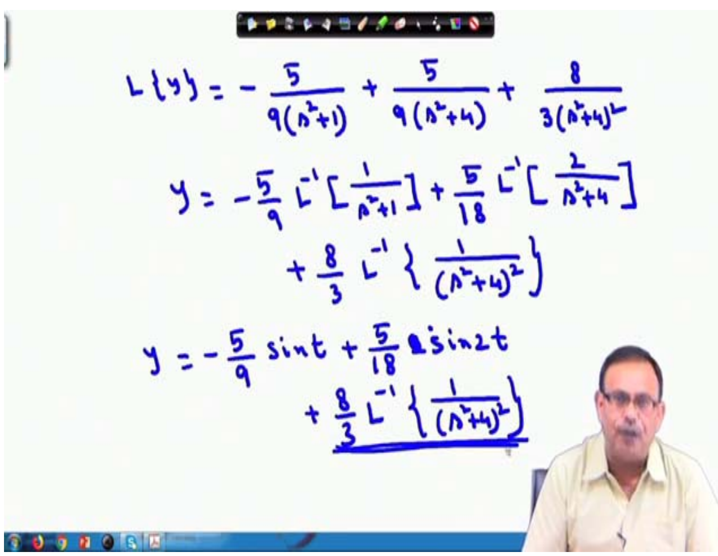

We will finally obtain it as

$$L\{y\} = -\frac{5}{9(s^2+1)} + \frac{5}{9(s^2+4)} + \frac{8}{3(s^2+4)^2}$$

So, now, we have to find out the Laplace inverse of the above in order to obtain $y(t)$:

$$\therefore y(t) = -\frac{5}{9}L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{5}{18}L^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}.$$

We cannot evaluate the Laplace inverse of $\frac{1}{(s^2+4)^2}$ directly, but we have solved such types of problems earlier using convolution. So, now we will find out the Laplace inverse of $\frac{1}{(s^2+4)^2}$ first.

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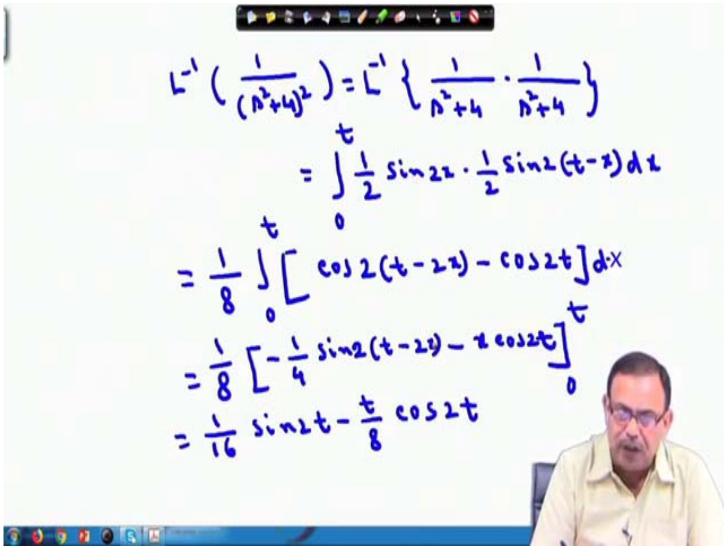

Using Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = L^{-1}\left\{\left(\frac{1}{2}\frac{2}{s^2+4}\right)\cdot\left(\frac{1}{2}\frac{2}{s^2+4}\right)\right\}$$

$$= \int_0^t \frac{1}{2}\sin 2x \cdot \frac{1}{2}\sin 2(t-x)\,dx \qquad \left[\because \mathrm{L}\{\sin 2t\} = \frac{2}{s^2+4}\right]$$

$$= \frac{1}{8}\int_0^t [\cos 2(t-2x) - \cos 2t]dx$$

If we evaluate the integral, we will obtain

$$L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \frac{1}{16}\sin 2t - \frac{t}{8}\cos 2t.$$

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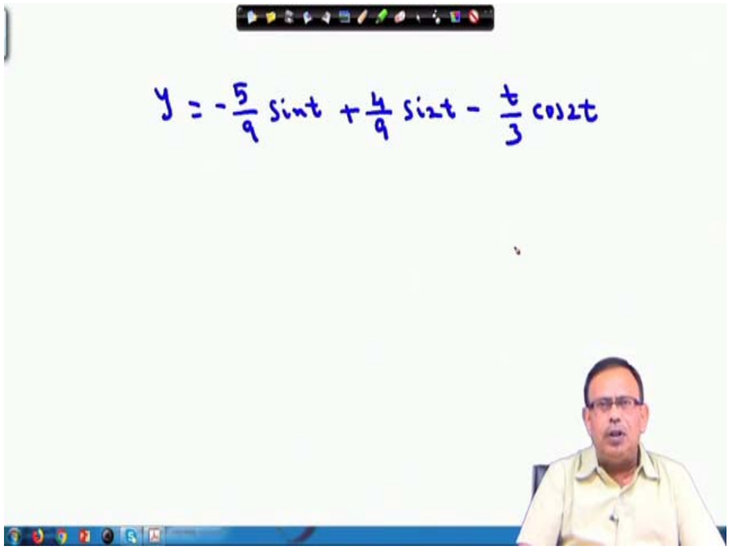

So that, now we can tell the value of $y(t)$ as

$$\begin{aligned} y(t) &= -\frac{5}{9}L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{5}{18}L^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} \\ &= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\left(\frac{1}{16}\sin 2t - \frac{t}{8}\cos 2t\right) \\ &= -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{t}{8}\cos 2t. \end{aligned}$$

This is the required solution of the given ODE.

So, this is the solution process whenever we are dealing with ODE with constant coefficients. In the next lecture also, we will continue with the solution of some other types of ordinary differential equations. Thank you.