Transform Calculus and Its Applications in Differential Equation
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Lecture - 14
Evaluation of Integrals using Laplace Transform

In this particular lecture, what we are going to do is to find out the solution of some integrals using the concept of Laplace transform.

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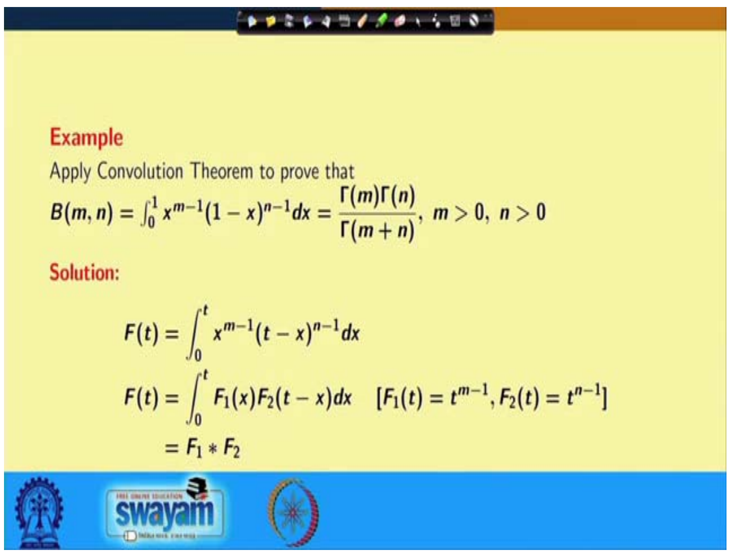

We will show that, $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ $, m > 0, n > 0$ using Convolution theorem, where $B(m,n)$ is well-known Beta function defined as,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

and $\Gamma(m)$ represents the Gamma function given by

$$\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}\,dx.$$

Now, we want to see, how using the Laplace transform, we can find out the value of $B(m,n)$ as $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ where, $m, n > 0$.

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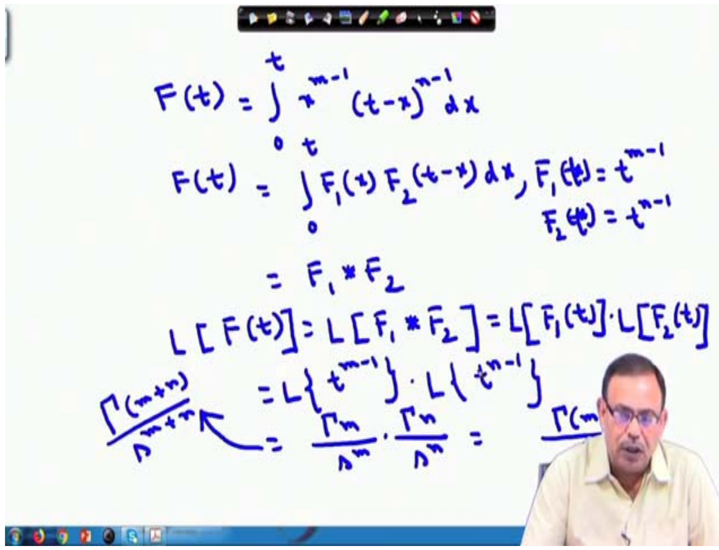

We assume a function $F(t)$ defined as,

$$F(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx. \qquad (1)$$

If we put $t = 1$, we will obtain the beta function that is $B(m,n)$. So, $F(t)$ can be written in the form of

$$F(t) = \int_0^t F_1(x)F_2(t-x)dx \text{ where } F_1(t) = t^{m-1}, F_2(t) = t^{n-1}$$

$$= F_1 * F_2 \quad \text{(by the definition of Convolution).}$$

We now take Laplace transform on both side of the equation, so that we will obtain

$$L\{F(t)\} = L\{F_1 * F_2\}.$$

And using convolution theorem, we know that

$$L\{F(t)\} = L\{F_1 * F_2\} = L\{F_1\}\,L\{F_2\}$$

$$= \frac{\Gamma(m)}{s^m} \times \frac{\Gamma(n)}{s^n}$$

$$= \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}.$$

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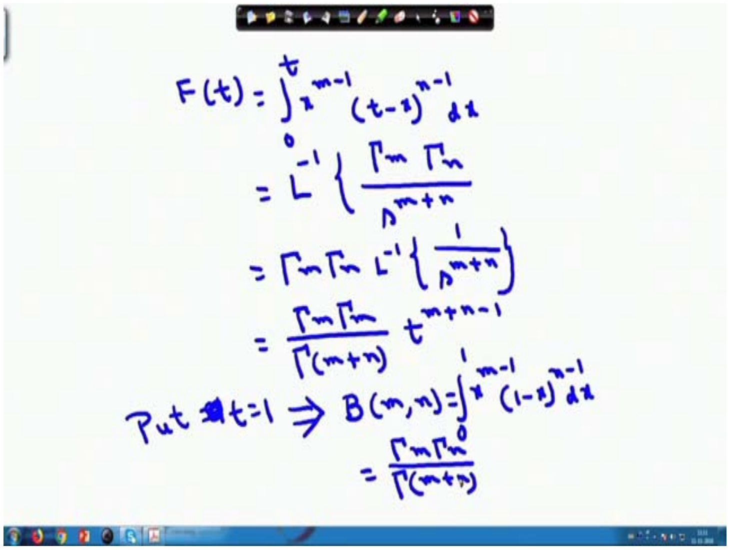

So, from (1), we can write down

$$\begin{aligned} F(t) &= \int_0^t x^{m-1}(t-x)^{n-1}\,dx \\ &= F_1 * F_2 \\ &= L^{-1}\left\{\frac{\Gamma(m)\Gamma(n)}{s^{m+n}}\right\} \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\,t^{m+n-1}. \end{aligned}$$

So, now, we put $t = 1$. We will then obtain

$$F(1) = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

So, using the convolution theorem, effectively we are able to derive the desired result.

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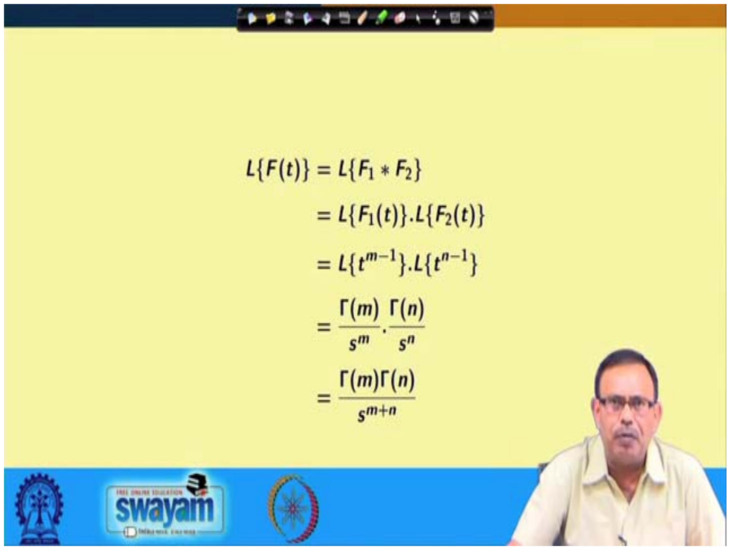

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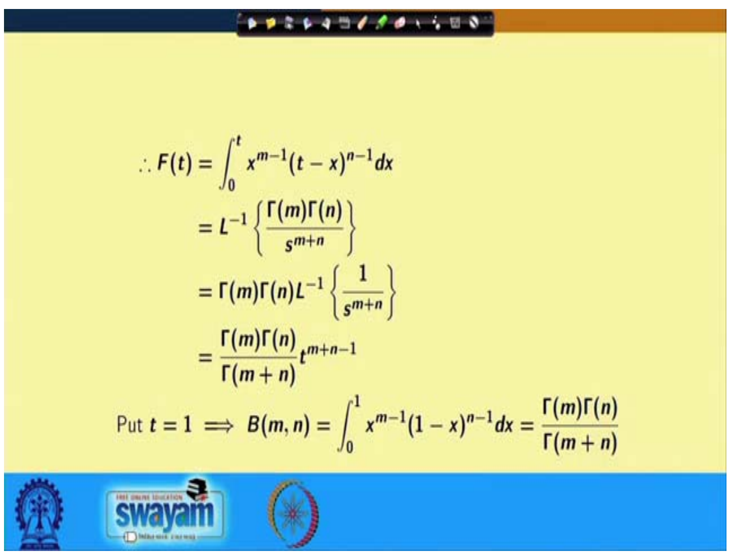

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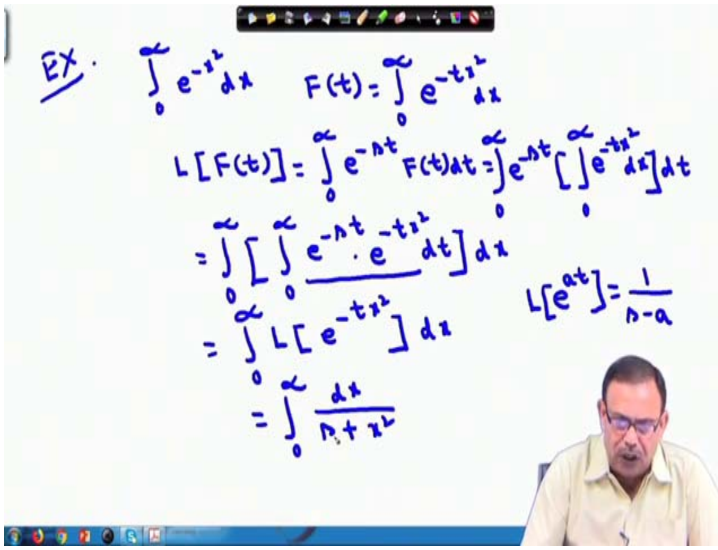

Let us take another example, say we want to evaluate $\int_0^\infty e^{-x^2} dx$.

We assume a new function, say $F(t) = \int_0^\infty e^{-tx^2} dx$. So, whenever t will be equal to 1, then we will obtain $\int_0^\infty e^{-x^2} dx$. Now, let us see what is the Laplace transform of $F(t)$. From the definition of Laplace transform, we can write down,

$$L\{F(t)\} = \int_0^\infty e^{-st} \left[\int_0^\infty e^{-tx^2} dx\right] dt.$$

Now, without losing the property, we can interchange the order of this integration to obtain

$$L\{F(t)\} = \int_0^\infty \left[\int_0^\infty e^{-st} e^{-tx^2} dt\right] dx.$$

Now, we know

$$\int_0^\infty e^{-st} e^{-tx^2} dt = L\{e^{-tx^2}\} = \frac{1}{s + x^2}.$$

So, we can write down

$$L\{F(t)\} = \int_0^\infty \frac{dx}{s + x^2}.$$

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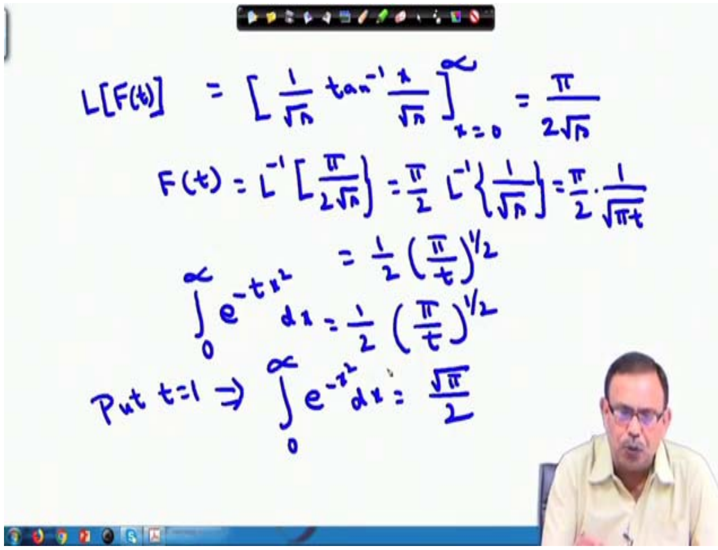

So, the integral can be easily evaluated to obtain

$$L\{F(t)\} = \left[\frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\right]_{x=0}^{\infty} = \frac{\pi}{2\sqrt{s}}.$$

Since we got $L\{F(t)\}$, so from here, we can write down,

$$F(t) = \frac{\pi}{2}L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

$$\Rightarrow \int_0^{\infty} e^{-tx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}. \qquad (2)$$

So, using the properties of Laplace Transform, we have obtained the value of $F(t)$. We need to evaluate $F(1) = \int_0^{\infty} e^{-x^2}dx$. So we put $t = 1$ in (2), then,

$$\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

This gives the required result. So, like this way, if we want we can evaluate the integrals also, using the concept of Laplace transform. Only thing that we have to remember is that we are constructing a new function and from that new function, we are taking the Laplace transform and finding out the value of the required integral for some particular value of the parameter.

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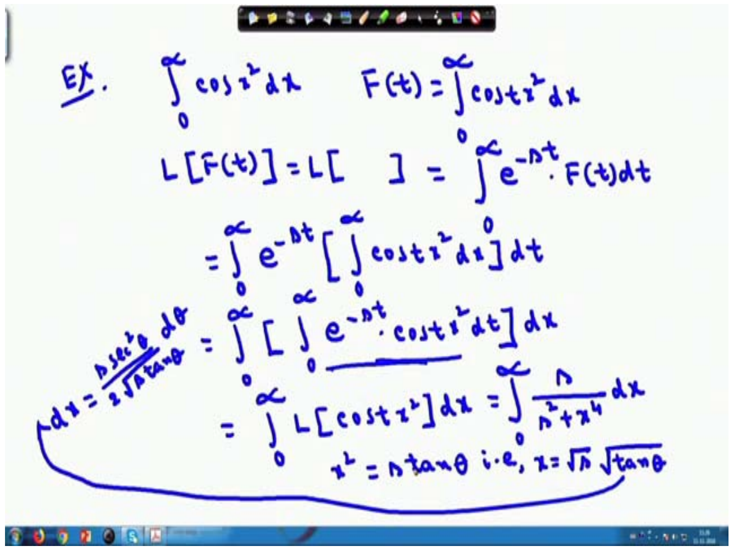

In the next problem, we want to evaluate $\int_0^\infty \cos x^2\, dx$. So, like earlier cases, we have to take a new parameter t. So, obviously, we will take a new function $F(t) = \int_0^\infty \cos tx^2\, dx$ so that afterwards if we substitute $t = 1$, we will get back the original integral.

Now, we take Laplace transform on both sides and using the definition, we obtain

$$L\{F(t)\} = \int_0^\infty e^{-st} \left[\int_0^\infty \cos tx^2\, dx\right] dt.$$

So, now, we can change the order of the integration as

$$L\{F(t)\} = \int_0^\infty \left[\int_0^\infty e^{-st} \cos tx^2\, dt\right] dx.$$

Again we see, $\int_0^\infty e^{-st} \cos tx^2\, dt$ is the Laplace transform of $\cos tx^2$ from the definition a Laplace transform. So we have,

$$L\{F(t)\} = \int_0^\infty \frac{s\, dx}{s^2 + x^4}. \qquad \left(\because L\{\cos tx^2\} = \frac{s}{s^2 + x^4}\right)$$

Now we have to evaluate this integral. Let us substitute $x^2 = s\tan\theta$ so that $2xdx = s\,sec^2\theta d\theta$,

$$\therefore dx = \frac{s\sec^2\theta}{2\sqrt{s\tan\theta}}d\theta$$

and the limits of integration will be changed to $\left[0,\frac{\pi}{2}\right]$.

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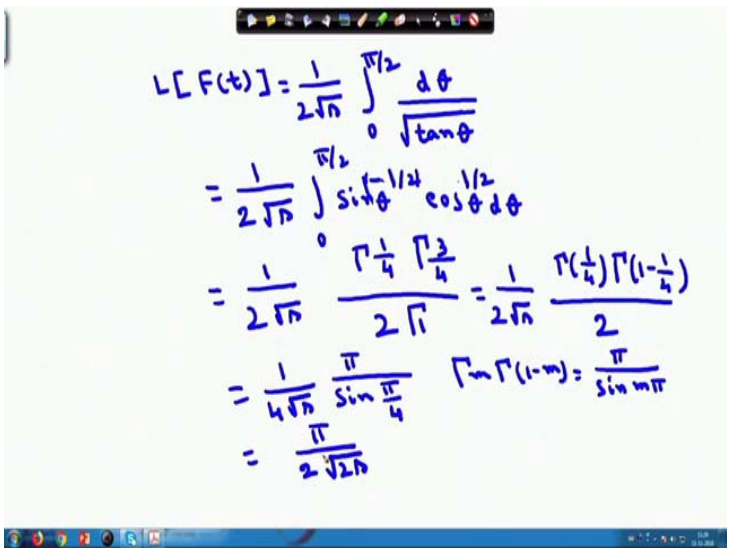

Thus, after substitution we get,

$$\begin{aligned} L\{F(t)\} &= \frac{1}{2\sqrt{s}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\tan\theta}} \\ &= \frac{1}{2\sqrt{s}}\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\theta\ \cos^{\frac{1}{2}}\theta\ d\theta \\ &= \frac{1}{2\sqrt{s}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} \\ &= \frac{\pi}{2\sqrt{2s}}. \end{aligned}$$

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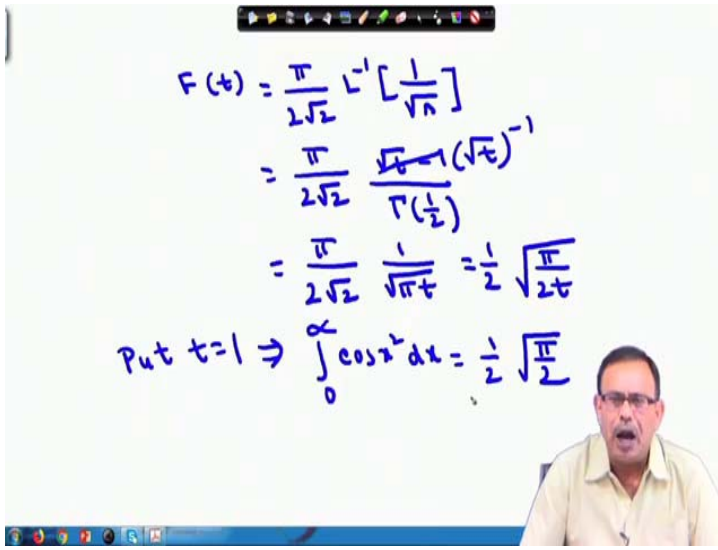

So, from here, we can have

$$F(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left\{\frac{1}{\sqrt{s}}\right\}$$

$$\Rightarrow \int_0^\infty \cos tx^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2t}}.$$

So, once we are obtaining $F(t)$, now we put $t = 1$ to obtain,

$$\int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

which is the desired result. So, this way, we can evaluate different integrals also, using the concept of Laplace transform, convolution theorem and we can use other properties of Laplace transform as required.

Thank you.