

**Transform Calculus and Its Applications in Differential Equation**  
**Prof. Adrijit Goswami**  
**Department of Mathematics**  
**Indian Institute of Technology, Kharagpur**

**Lecture - 14**  
**Evaluation of Integrals using Laplace Transform**

In this particular lecture, what we are going to do is to find out the solution of some integrals using the concept of Laplace transform.

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**Example**  
Apply Convolution Theorem to prove that  
 $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m > 0, n > 0$

**Solution:**

$$F(t) = \int_0^t x^{m-1}(t-x)^{n-1} dx$$
$$F(t) = \int_0^t F_1(x)F_2(t-x) dx \quad [F_1(t) = t^{m-1}, F_2(t) = t^{n-1}]$$
$$= F_1 * F_2$$

We will show that,  $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m > 0, n > 0$  using Convolution theorem, where  $B(m, n)$  is well-known Beta function defined as,

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

and  $\Gamma(m)$  represents the Gamma function given by

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx.$$

Now, we want to see, how using the Laplace transform, we can find out the value of  $B(m, n)$  as  $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$  where,  $m, n > 0$ .

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$$\begin{aligned}
 F(t) &= \int_0^t x^{m-1} (t-x)^{n-1} dx \\
 F(t) &= \int_0^t F_1(x) F_2(t-x) dx, \quad F_1(x) = t^{m-1} \\
 &\quad F_2(t) = t^{n-1} \\
 &= F_1 * F_2 \\
 L[F(t)] &= L[F_1 * F_2] = L[F_1(t)] \cdot L[F_2(t)] \\
 &= L\{t^{m-1}\} \cdot L\{t^{n-1}\} \\
 &= \frac{\Gamma(m)}{s^m} \cdot \frac{\Gamma(n)}{s^n} = \frac{\Gamma(m+n)}{s^{m+n}}
 \end{aligned}$$

We assume a function  $F(t)$  defined as,

$$F(t) = \int_0^t x^{m-1} (t-x)^{n-1} dx. \quad (1)$$

If we put  $t = 1$ , we will obtain the beta function that is  $B(m, n)$ . So,  $F(t)$  can be written in the form of

$$\begin{aligned}
 F(t) &= \int_0^t F_1(x) F_2(t-x) dx \quad \text{where } F_1(t) = t^{m-1}, F_2(t) = t^{n-1} \\
 &= F_1 * F_2 \quad (\text{by the definition of Convolution}).
 \end{aligned}$$

We now take Laplace transform on both side of the equation, so that we will obtain

$$L\{F(t)\} = L\{F_1 * F_2\}.$$

And using convolution theorem, we know that

$$\begin{aligned}
 L\{F(t)\} &= L\{F_1 * F_2\} = L\{F_1\} L\{F_2\} \\
 &= \frac{\Gamma(m)}{s^m} \times \frac{\Gamma(n)}{s^n} \\
 &= \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}.
 \end{aligned}$$

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The image shows a whiteboard with handwritten mathematical derivations. The first part shows the Laplace transform of a convolution integral:

$$F(t) = \int_0^t x^{m-1} (t-x)^{n-1} dx$$
$$= L^{-1} \left\{ \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \right\}$$
$$= \Gamma(m) \Gamma(n) L^{-1} \left\{ \frac{1}{s^{m+n}} \right\}$$
$$= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} t^{m+n-1}$$

The second part shows the evaluation of this integral at  $t=1$ , which is the Beta function:

$$\text{Put } t=1 \Rightarrow B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$
$$= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

So, from (1), we can write down

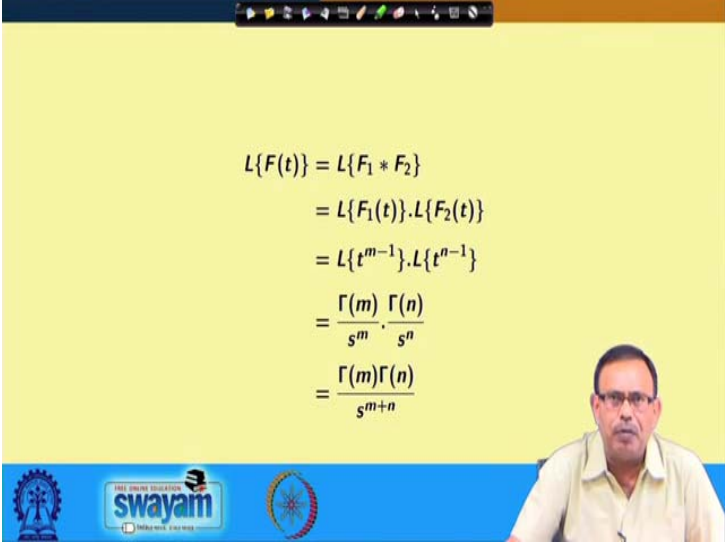
$$F(t) = \int_0^t x^{m-1} (t-x)^{n-1} dx$$
$$= F_1 * F_2$$
$$= L^{-1} \left\{ \frac{\Gamma(m) \Gamma(n)}{s^{m+n}} \right\}$$
$$= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} t^{m+n-1}.$$

So, now, we put  $t = 1$ . We will then obtain

$$F(1) = B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

So, using the convolution theorem, effectively we are able to derive the desired result.

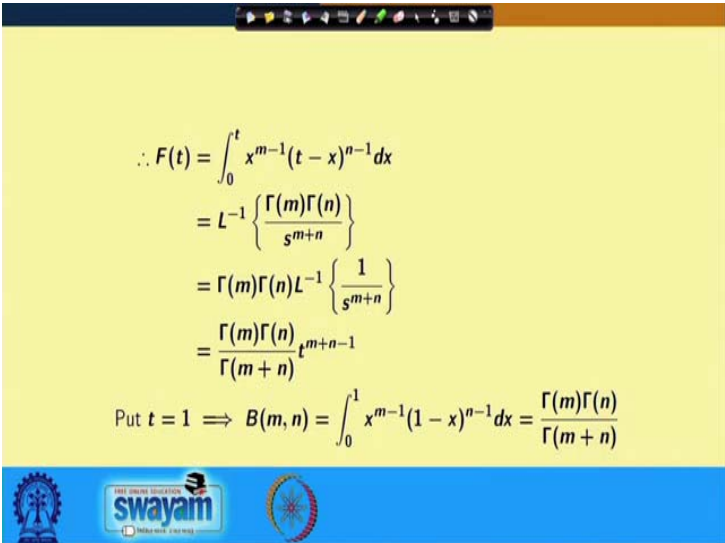
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A slide from a video lecture showing the Laplace transform of a convolution product. The slide has a yellow background and a blue footer with logos for Swamyam and the Indian Institute of Space Science and Technology. A small video inset of a man is visible in the bottom right corner.

$$\begin{aligned}L\{F(t)\} &= L\{F_1 * F_2\} \\ &= L\{F_1(t)\} \cdot L\{F_2(t)\} \\ &= L\{t^{m-1}\} \cdot L\{t^{n-1}\} \\ &= \frac{\Gamma(m)}{s^m} \cdot \frac{\Gamma(n)}{s^n} \\ &= \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}\end{aligned}$$

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A slide from a video lecture showing the derivation of the Beta function. The slide has a yellow background and a blue footer with logos for Swamyam and the Indian Institute of Space Science and Technology. A small video inset of a man is visible in the bottom right corner.

$$\begin{aligned}\therefore F(t) &= \int_0^t x^{m-1}(t-x)^{n-1} dx \\ &= L^{-1} \left\{ \frac{\Gamma(m)\Gamma(n)}{s^{m+n}} \right\} \\ &= \Gamma(m)\Gamma(n) L^{-1} \left\{ \frac{1}{s^{m+n}} \right\} \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} t^{m+n-1}\end{aligned}$$

Put  $t = 1 \Rightarrow B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

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$$\text{EX. } \int_0^{\infty} e^{-x^2} dx \quad F(t) = \int_0^{\infty} e^{-tx^2} dx$$

$$L[F(t)] = \int_0^{\infty} e^{-st} F(t) dt = \int_0^{\infty} e^{-st} \left[ \int_0^{\infty} e^{-tx^2} dx \right] dt$$

$$= \int_0^{\infty} \left[ \int_0^{\infty} \frac{e^{-st} \cdot e^{-tx^2}}{dt} dt \right] dx$$

$$= \int_0^{\infty} L[e^{-tx^2}] dx \quad L[e^{at}] = \frac{1}{s-a}$$

$$= \int_0^{\infty} \frac{dx}{s+tx^2}$$

Let us take another example, say we want to evaluate  $\int_0^{\infty} e^{-x^2} dx$ .

We assume a new function, say  $F(t) = \int_0^{\infty} e^{-tx^2} dx$ . So, whenever  $t$  will be equal to 1, then we will obtain  $\int_0^{\infty} e^{-x^2} dx$ . Now, let us see what is the Laplace transform of  $F(t)$ . From the definition of Laplace transform, we can write down,

$$L\{F(t)\} = \int_0^{\infty} e^{-st} \left[ \int_0^{\infty} e^{-tx^2} dx \right] dt.$$

Now, without losing the property, we can interchange the order of this integration to obtain

$$L\{F(t)\} = \int_0^{\infty} \left[ \int_0^{\infty} e^{-st} e^{-tx^2} dt \right] dx.$$

Now, we know

$$\int_0^{\infty} e^{-st} e^{-tx^2} dt = L\{e^{-tx^2}\} = \frac{1}{s+tx^2}.$$

So, we can write down

$$L\{F(t)\} = \int_0^{\infty} \frac{dx}{s+tx^2}.$$

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$$L\{F(t)\} = \left[ \frac{1}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}} \right]_{x=0}^{\infty} = \frac{\pi}{2\sqrt{s}}$$
$$F(t) = L^{-1} \left[ \frac{\pi}{2\sqrt{s}} \right] = \frac{\pi}{2} L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\} = \frac{\pi}{2} \cdot \frac{1}{\sqrt{\pi t}}$$
$$\int_0^{\infty} e^{-tx^2} dx = \frac{1}{2} \left( \frac{\pi}{t} \right)^{1/2}$$
$$\text{Put } t=1 \Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

So, the integral can be easily evaluated to obtain

$$L\{F(t)\} = \left[ \frac{1}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}} \right]_{x=0}^{\infty} = \frac{\pi}{2\sqrt{s}}$$

Since we got  $L\{F(t)\}$ , so from here, we can write down,

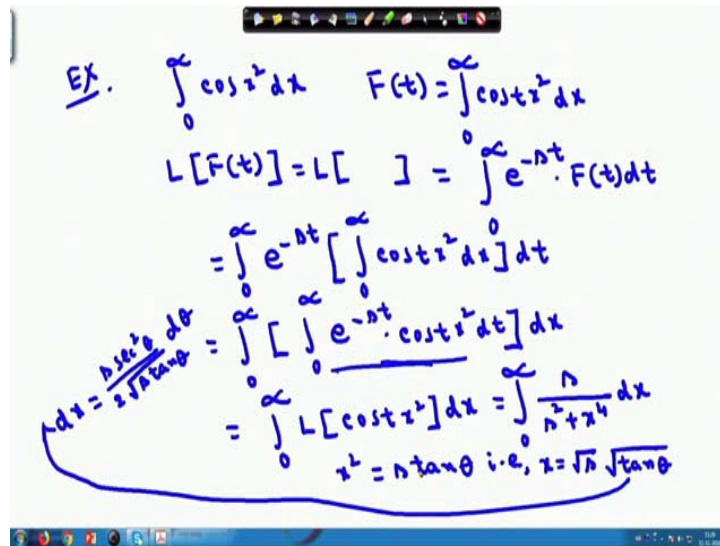
$$F(t) = \frac{\pi}{2} L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\} = \frac{1}{2} \sqrt{\frac{\pi}{t}}$$
$$\Rightarrow \int_0^{\infty} e^{-tx^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{t}}. \quad (2)$$

So, using the properties of Laplace Transform, we have obtained the value of  $F(t)$ . We need to evaluate  $F(1) = \int_0^{\infty} e^{-x^2} dx$ . So we put  $t = 1$  in (2), then,

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

This gives the required result. So, like this way, if we want we can evaluate the integrals also, using the concept of Laplace transform. Only thing that we have to remember is that we are constructing a new function and from that new function, we are taking the Laplace transform and finding out the value of the required integral for some particular value of the parameter.

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$$\text{Ex. } \int_0^{\infty} \cos x^2 dx \quad F(t) = \int_0^{\infty} \cos tx^2 dx$$

$$L\{F(t)\} = L\left[ \int_0^{\infty} \cos tx^2 dx \right] = \int_0^{\infty} e^{-st} \cdot F(t) dt$$

$$= \int_0^{\infty} e^{-st} \left[ \int_0^{\infty} \cos tx^2 dx \right] dt$$

$$= \int_0^{\infty} \left[ \int_0^{\infty} e^{-st} \cdot \cos tx^2 dt \right] dx$$

$$= \int_0^{\infty} L\{\cos tx^2\} dx = \int_0^{\infty} \frac{n}{n^2 + x^4} dx$$

$$x^2 = n^2 \tan^2 \theta \text{ i.e., } x = \sqrt{n} \sqrt{\tan \theta}$$

In the next problem, we want to evaluate  $\int_0^{\infty} \cos x^2 dx$ . So, like earlier cases, we have to take a new parameter  $t$ . So, obviously, we will take a new function  $F(t) = \int_0^{\infty} \cos tx^2 dx$  so that afterwards if we substitute  $t = 1$ , we will get back the original integral.

Now, we take Laplace transform on both sides and using the definition, we obtain

$$L\{F(t)\} = \int_0^{\infty} e^{-st} \left[ \int_0^{\infty} \cos tx^2 dx \right] dt.$$

So, now, we can change the order of the integration as

$$L\{F(t)\} = \int_0^{\infty} \left[ \int_0^{\infty} e^{-st} \cos tx^2 dt \right] dx.$$

Again we see,  $\int_0^{\infty} e^{-st} \cos tx^2 dt$  is the Laplace transform of  $\cos tx^2$  from the definition a Laplace transform. So we have,

$$L\{F(t)\} = \int_0^{\infty} \frac{s dx}{s^2 + x^4}. \quad \left( \because L\{\cos tx^2\} = \frac{s}{s^2 + x^4} \right)$$

Now we have to evaluate this integral. Let us substitute  $x^2 = s \tan \theta$  so that  $2x dx = s \sec^2 \theta d\theta$ ,

$$\therefore dx = \frac{s \sec^2 \theta}{2\sqrt{s \tan \theta}} d\theta$$

and the limits of integration will be changed to  $\left[0, \frac{\pi}{2}\right]$ .

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$$\begin{aligned}
 L\{F(t)\} &= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan \theta}} \\
 &= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{1/2} \theta \cos^{1/2} \theta d\theta \\
 &= \frac{1}{2\sqrt{s}} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{1}{2\sqrt{s}} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(1 - \frac{1}{4}\right)}{2} \\
 &= \frac{1}{4\sqrt{s}} \frac{\pi}{\sin \frac{\pi}{4}} \quad \Gamma_m \Gamma(1-m) = \frac{\pi}{\sin m\pi} \\
 &= \frac{\pi}{2\sqrt{2s}}
 \end{aligned}$$

Thus, after substitution we get,

$$\begin{aligned}
 L\{F(t)\} &= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan \theta}} \\
 &= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta d\theta \\
 &= \frac{1}{2\sqrt{s}} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} \\
 &= \frac{\pi}{2\sqrt{2s}}
 \end{aligned}$$



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$$F(t) = \frac{\pi}{2\sqrt{2}} L^{-1} \left[ \frac{1}{\sqrt{s}} \right]$$
$$= \frac{\pi}{2\sqrt{2}} \frac{\sqrt{s-1} (\sqrt{t})^{-1}}{\Gamma(\frac{1}{2})}$$
$$= \frac{\pi}{2\sqrt{2}} \frac{1}{\sqrt{\pi t}} = \frac{1}{2} \sqrt{\frac{\pi}{2t}}$$

Put  $t=1 \Rightarrow \int_0^{\infty} \cos x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$

So, from here, we can have

$$F(t) = \frac{\pi}{2\sqrt{2}} L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\}$$
$$\Rightarrow \int_0^{\infty} \cos tx^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2t}}$$

So, once we are obtaining  $F(t)$ , now we put  $t = 1$  to obtain,

$$\int_0^{\infty} \cos x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

which is the desired result. So, this way, we can evaluate different integrals also, using the concept of Laplace transform, convolution theorem and we can use other properties of Laplace transform as required.

Thank you.