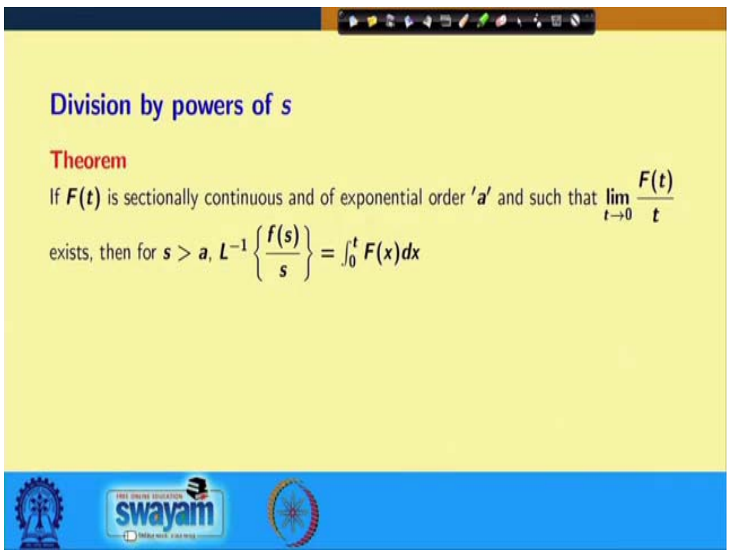
Transform Calculus and Its Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 13
Convolution and its Applications

So, in the last lecture what we have studied, is the Inverse Laplace transform, its properties and also the applications of those properties for solving various kinds of problems.

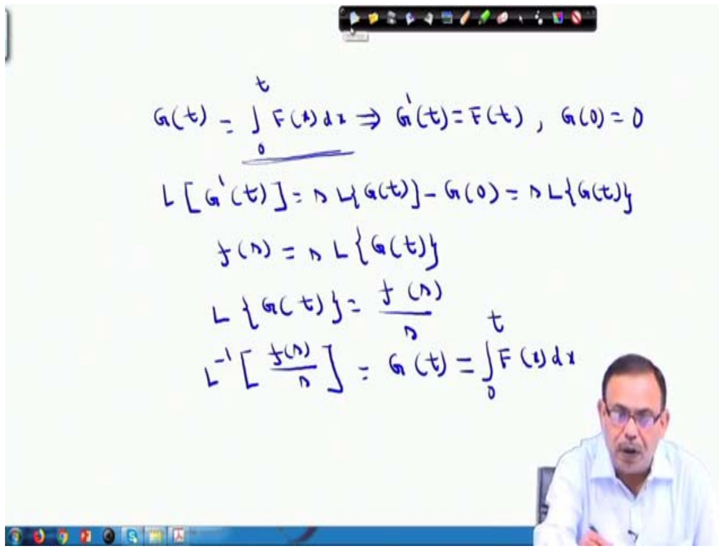
(Refer Slide Time: 00:39)



Now, let us see the next property i.e., division by powers of s. The theorem says that if $F(t)$ is sectionally continuous and of exponential order a such that $\lim_{t\to 0}\frac{F(t)}{t}$ exists, then for $s > a$,

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(x)dx.$$

(Refer Slide Time: 01:21)



Suppose $G(t) = \int_0^t F(x)dx$. So that from here, clearly we can tell $G'(t) = F(t)$ and $G(0) = 0$. Now, Laplace transform of $G'(t)$ is

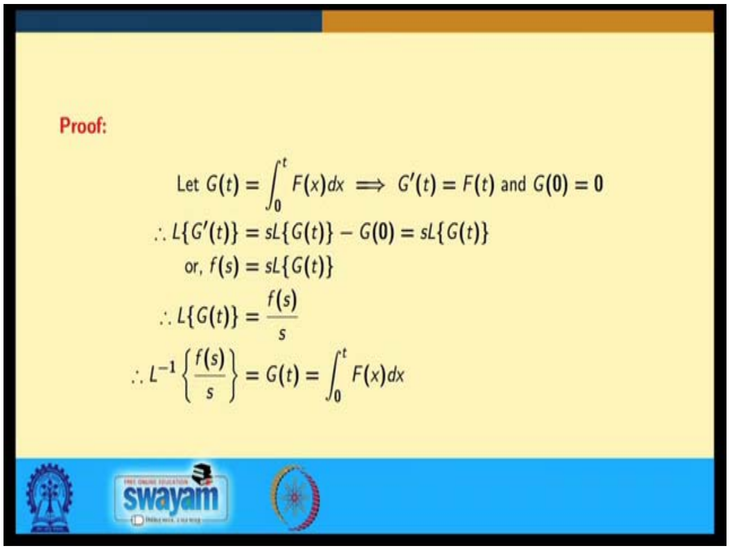
$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$
$$\Rightarrow f(s) = sL\{G(t)\} \quad ,(\because G'(t) = F(t) \text{ and } G(0) = 0\ ).$$

So that we can simply write

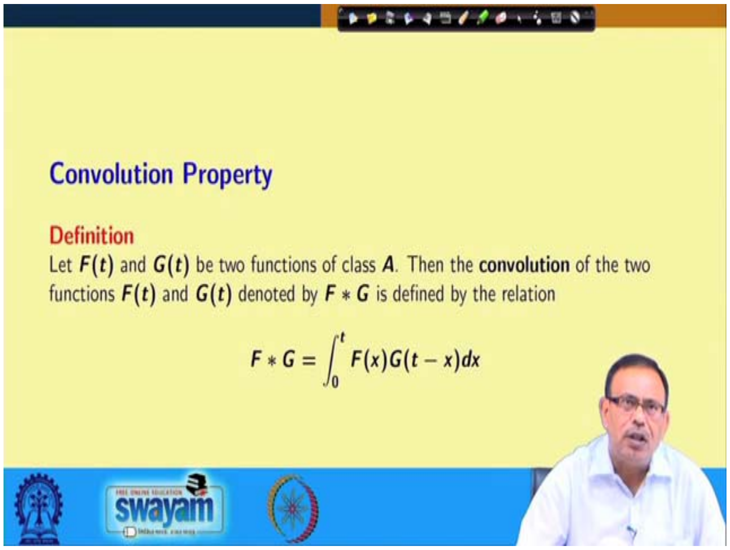
$$L\{G(t)\} = \frac{f(s)}{s}$$
$$\Rightarrow L^{-1}\left\{\frac{f(s)}{s}\right\} = G(t) = \int_0^t F(x)dx.$$

This completes the proof of this particular theorem.

(Refer Slide Time: 03:29)



(Refer Slide Time: 04:03)



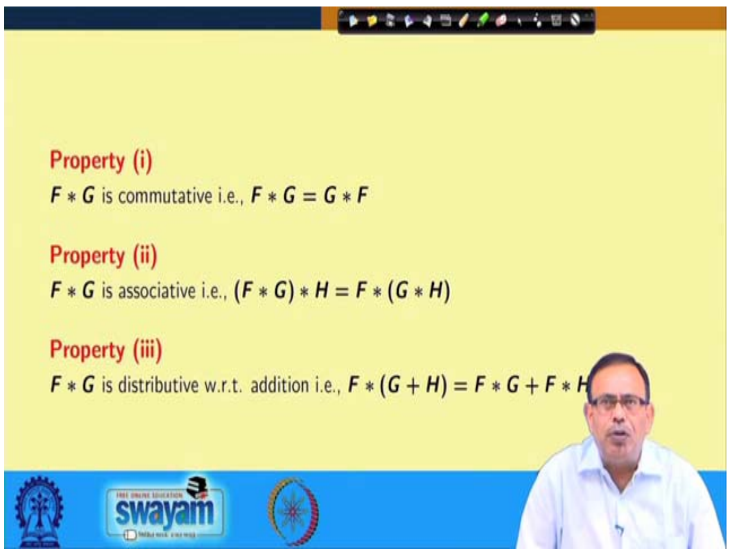
Now, let us come to a very important property of Laplace transform. We will discuss it with respect to the Laplace transform and later for the other transforms whenever we will discuss the same. This property is called the Convolution Property. Suppose $F(t)$ and $G(t)$ be two functions of class A. Please remember class A means they are piecewise continuous and they are of exponential order.

So, if we have two functions $F(t)$ and $G(t)$, then the convolution of two functions $F(t)$ and $G(t)$ denoted by $F * G$, read as F Convolution G, is defined as,

$$F * G = \int_0^t F(x)G(t-x)dx.$$

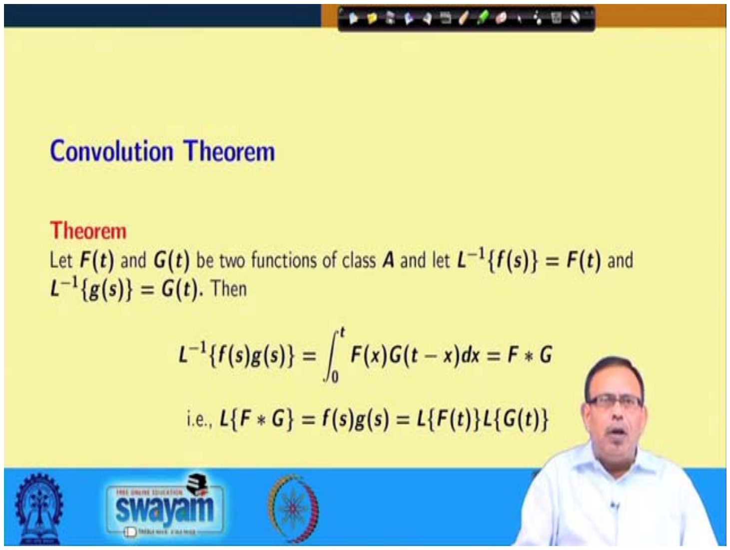
So, therefore, if we have two functions $F(t)$ and $G(t)$ such that they are of class A, means their Laplace transform exist, then the convolution of these two functions $F(t)$ and $G(t)$ denoted by $F * G$ is given by the above formula.

(Refer Slide Time: 05:29)



It has certain properties which can be proved very easily. $F * G$ is commutative that is $F * G = G * F$. The convolution with respect to the functions F, G and H is associative that is $(F * G) * H = F * (G * H)$. $F * G$ is distributive with respect to addition that is $F * (G + H) = F * G + F * H$. Please note that the convolution satisfies these three properties namely, convolution is commutative, associative and distributive with respect to addition only.

(Refer Slide Time: 06:55)



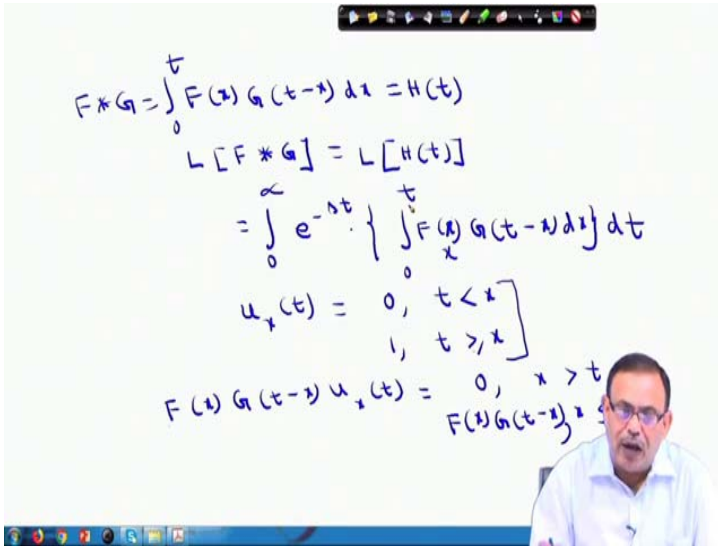
There is a well-known theorem which we call the Convolution Theorem. Let $F(t)$ and $G(t)$ be two functions of class A that is their Laplace transforms exist. And let $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$. Then,

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(x)G(t-x)dx = F * G$$

$$\text{i.e. } L\{F * G\} = f(s)g(s) = L\{F(t)\}\, L\{G(t)\}.$$

So, this one is very important that if we have the convolution of two functions, then Laplace transform of convolution of two functions is equal to the product of Laplace transform of those two individual functions. Let us see the proof of this.

(Refer Slide Time: 08:01)



Let us assume that $\int_0^t F(x)G(t-x)dx = H(t)$. That is, we can say that

$$
\begin{aligned}
L\{F * G\} &= L\{H(t)\} \\
&= \int_0^\infty e^{-st} H(t)\, dt \\
&= \int_0^\infty e^{-st} \left\{ \int_0^t F(x)G(t-x)dx \right\} dt.
\end{aligned}
$$

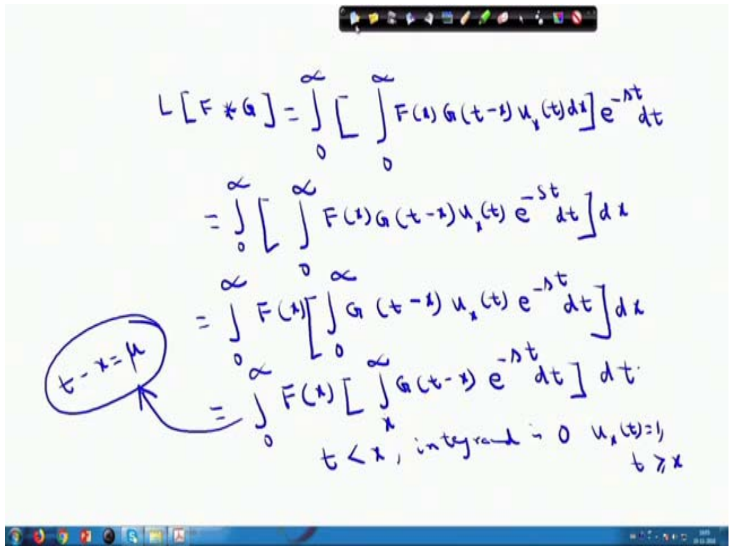
To evaluate this, let us introduce a new function $u_x(t)$ defined as,

$$
u_x(t) = \begin{cases} 0\ , & t < x \\ 1\ , & t \geq x \end{cases}
$$

$$
\therefore\ F(x)G(t-x)\, u_x(t) = \begin{cases} 0 & , \quad x > t \\ F(x)G(t-x) & , \quad x \leq t \end{cases}
$$

Now, the product vanishes for all values of $x > t$, then the limit can be extended from 0 to ∞.

(Refer Slide Time: 11:25)



So, therefore, we can right down using this new function

$$L\{F * G\} = \int_0^\infty \left[\int_0^\infty F(x)G(t-x)u_x(t)\, dx\right] e^{-st} dt.$$

Since both the integrals have the same limit i.e., 0 to ∞, so by changing the order of the integration, we can write down

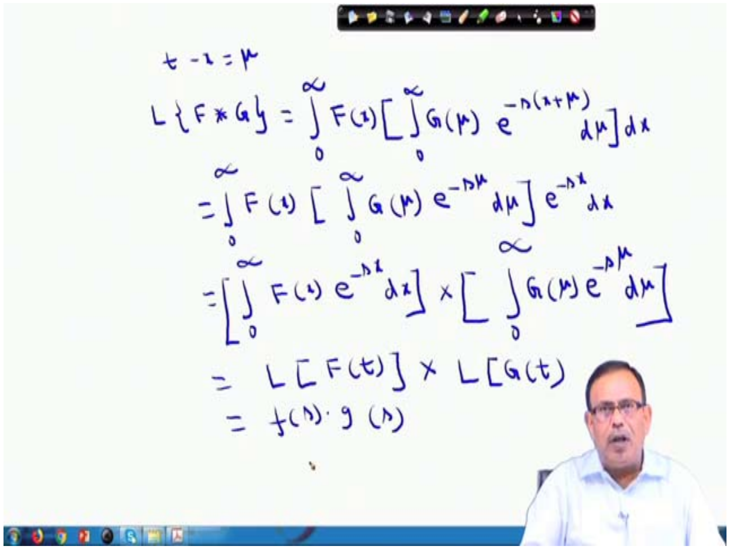
$$L\{F * G\} = \int_0^\infty F(x)\left[\int_0^\infty G(t-x)u_x(t)e^{-st}\, dt\right] dx.$$

$F(x)$ will come outside because this is independent of t. And this can be written as:

$$L\{F * G\} = \int_0^\infty F(x)\left[\int_x^\infty G(t-x)e^{-st}\, dt\right] dx$$

because for $t < x$, $u_x(t)$ is 0 So, for that reason, the range 0 to ∞ now can be broken into 0 to x and x to ∞.

(Refer Slide Time: 14:41)



Now, we substitute $t - x = \mu$ so that $dt = d\mu$ and the limits of integration will be changed to $[0, \infty)$. So, after substitution, $L\{F * G\}$ equals

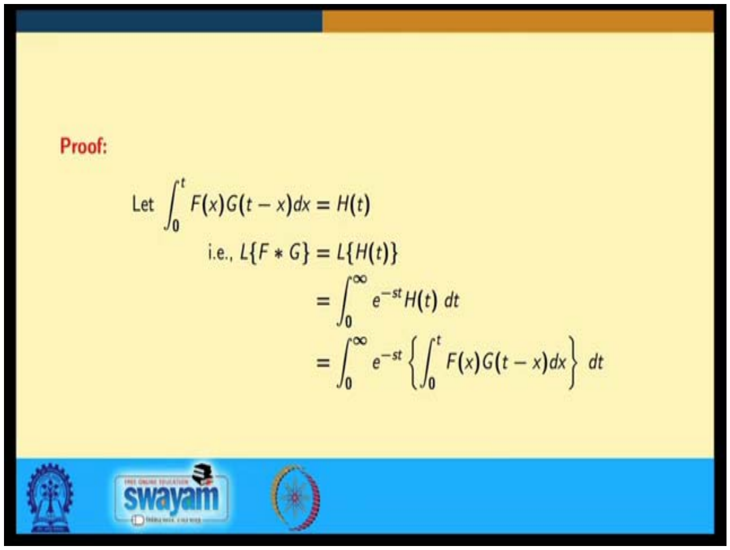
$$L\{F * G\} = \int_0^\infty F(x)\left[\int_0^\infty G(\mu)e^{-s(\mu+x)}\,d\mu\right]dx$$
$$= \int_0^\infty F(x)\left[\int_0^\infty G(\mu)e^{-s\mu}\,d\mu\right]e^{-sx}\,dx$$

So, this simply now I am breaking it into two independent variables x and μ and this equals I can write

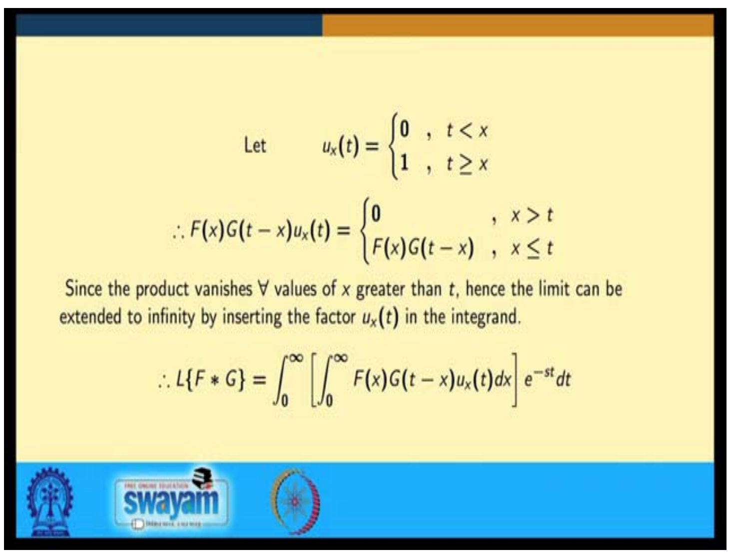
$$L\{F * G\} = \left[\int_0^\infty F(x)e^{-sx}\,dx\right] \times \left[\int_0^\infty G(\mu)e^{-s\mu}\,d\mu\right]$$
$$= L\{F(t)\}\,L\{G(t)\}$$
$$= f(s)g(s).$$

This completes the proof. So, we started with Laplace transform of the convolution of two functions and this equals $f(s)g(s)$. So, therefore, Laplace transform of convolution of two functions equals the product of the individual Laplace transform of the functions.

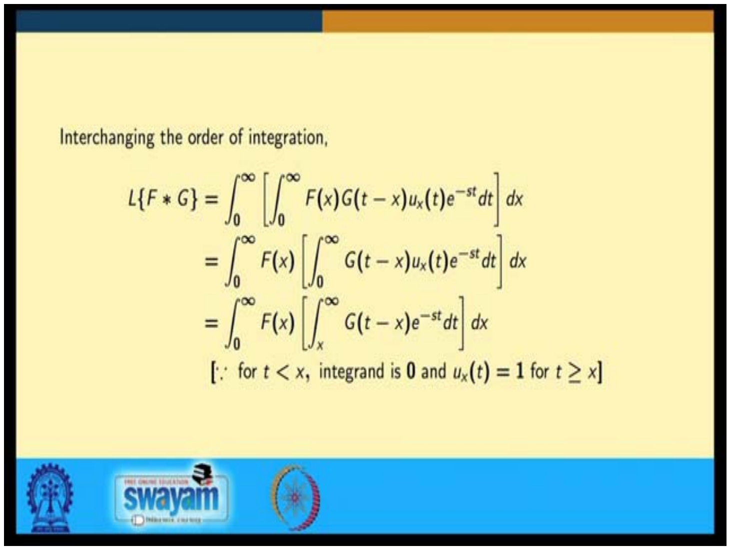
(Refer Slide Time: 17:09)



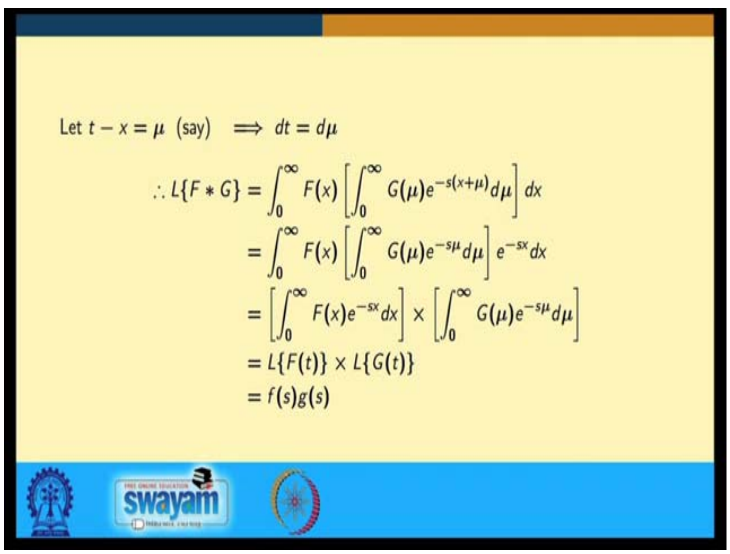
(Refer Slide Time: 17:19)



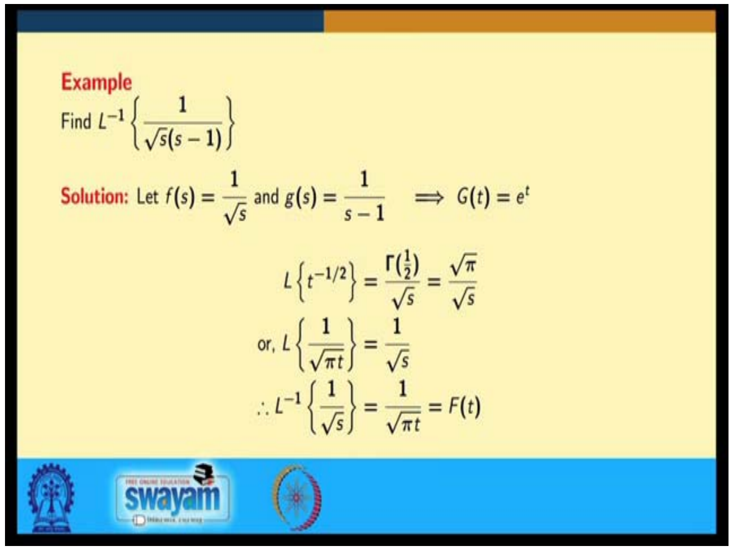
(Refer Slide Time: 17:41)



(Refer Slide Time: 18:05)

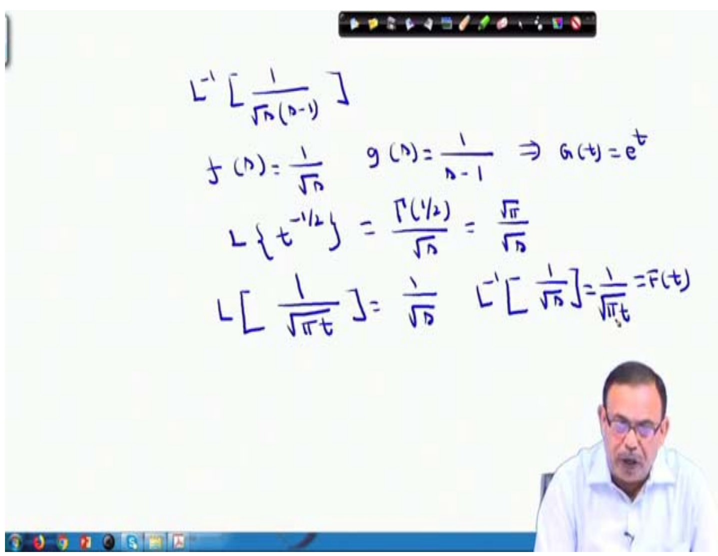


(Refer Slide Time: 18:27)



So, now let us see using example, how the convolution of functions turns useful in deriving the inverse Laplace transform. That is if we use convolution, how it becomes easy for us to find out the inverse Laplace transform of different functions.

(Refer Slide Time: 18:51)



So, we have to find out the Laplace inverse of

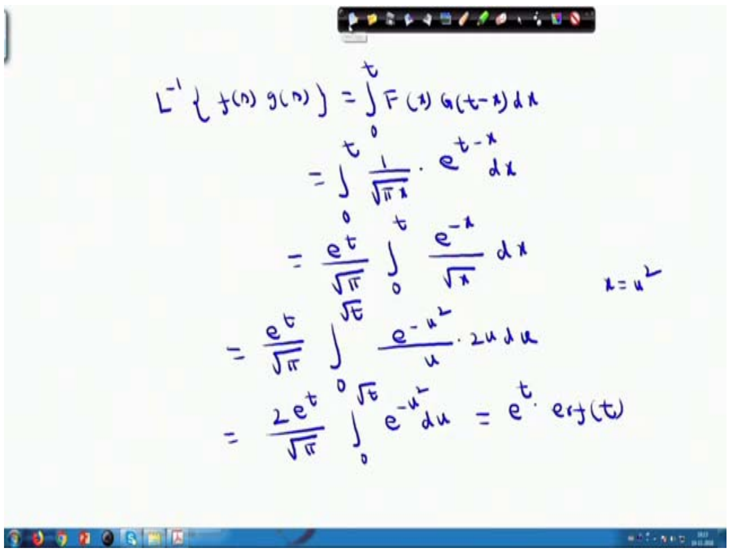
$$\frac{1}{\sqrt{s}(s-1)}$$

Now, we are assuming that $f(s) = \frac{1}{\sqrt{s}}$ and $g(s) = \frac{1}{s-1}$.

$$\therefore F(t) = L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} \text{ and } G(t) = L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$$

So, now, $F(t)$ and $G(t)$ are known to us.

(Refer Slide Time: 20:35)



So, by convolution theorem, we know,

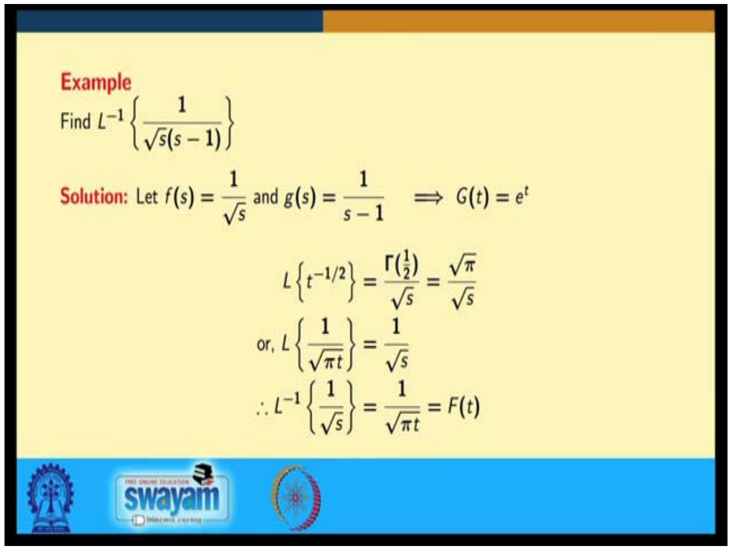
$$L^{-1}\{f(s)g(s)\} = \int_0^t F(x)G(t-x)dx$$
$$= \int_0^t \frac{1}{\sqrt{\pi x}} e^{t-x} dx.$$

So, effectively we have to now just evaluate an integral to find out the Laplace inverse of the given function. To evaluate the integral, we put $x = u^2$ so that $dx = 2udu$ and the limits of integration are changed to $[0, \sqrt{t}]$. Then this will become

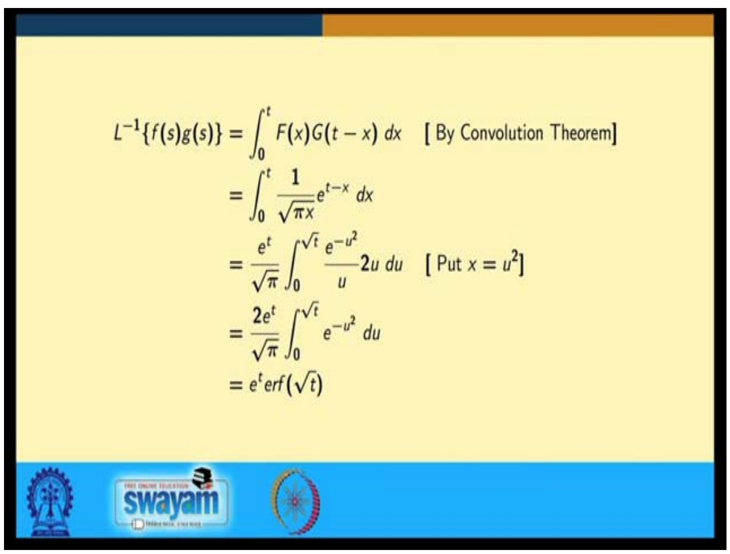
$$L^{-1}\{f(s)g(s)\} = e^t \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du = e^t \operatorname{erf}(\sqrt{t}).$$

So, $L^{-1}\{f(s)g(s)\}$ is coming as $e^t \operatorname{erf}(\sqrt{t})$.

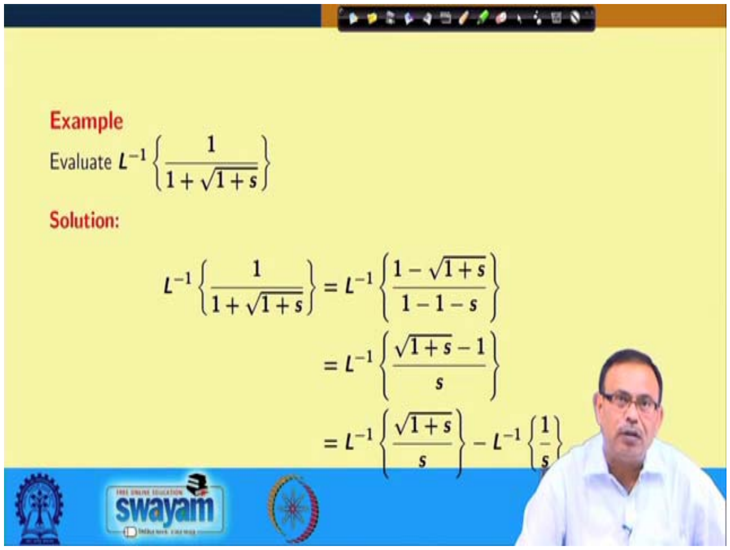
(Refer Slide Time: 22:38)



(Refer Slide Time: 23:01)



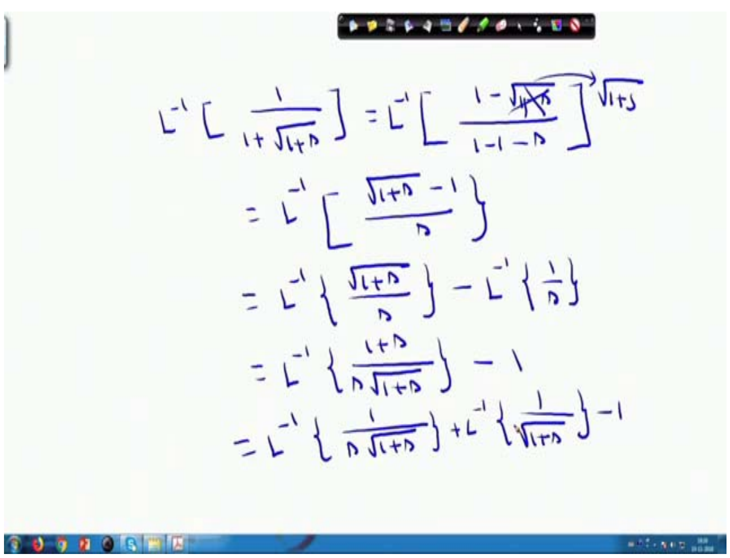
(Refer Slide Time: 23:25)



Now, let us see the next example that is to evaluate the Laplace inverse of

$$\frac{1}{1+\sqrt{1+s}}$$

(Refer Slide Time: 23:35)



For this case obviously, this denominator we cannot handle. So, we have to rationalize it.

So that it becomes,

$$L^{-1}\left\{\frac{1}{1+\sqrt{1+s}}\right\} = L^{-1}\left\{\frac{1-\sqrt{1+s}}{1-1-s}\right\}$$

$$= L^{-1}\left\{\frac{\sqrt{1+s}}{s}\right\} - L^{-1}\left\{\frac{1}{s}\right\}$$

$$= L^{-1}\left\{\frac{1+s}{s\sqrt{1+s}}\right\} - 1$$

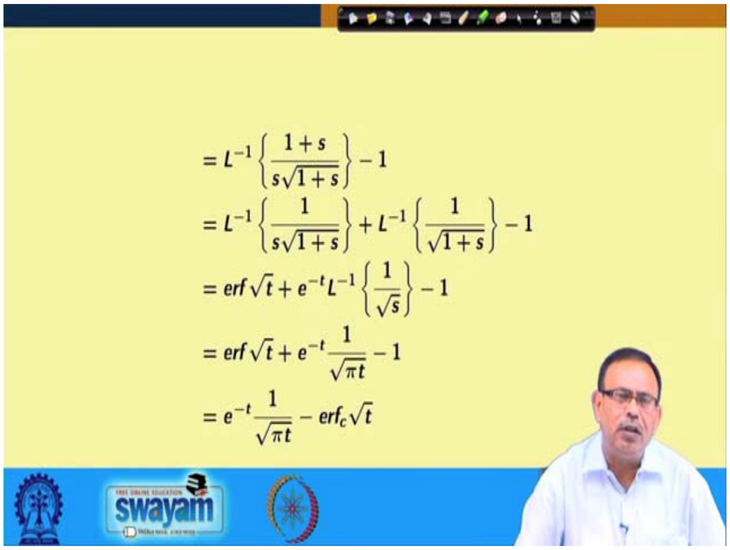
$$= L^{-1}\left\{\frac{1}{s\sqrt{1+s}}\right\} + L^{-1}\left\{\frac{1}{\sqrt{1+s}}\right\} - 1$$

$$= \mathrm{erf}\left(\sqrt{t}\right) + \frac{e^{-t}}{\sqrt{\pi t}} - 1$$

$$= \frac{e^{-t}}{\sqrt{\pi t}} - \mathrm{erf_c}(\sqrt{t})$$

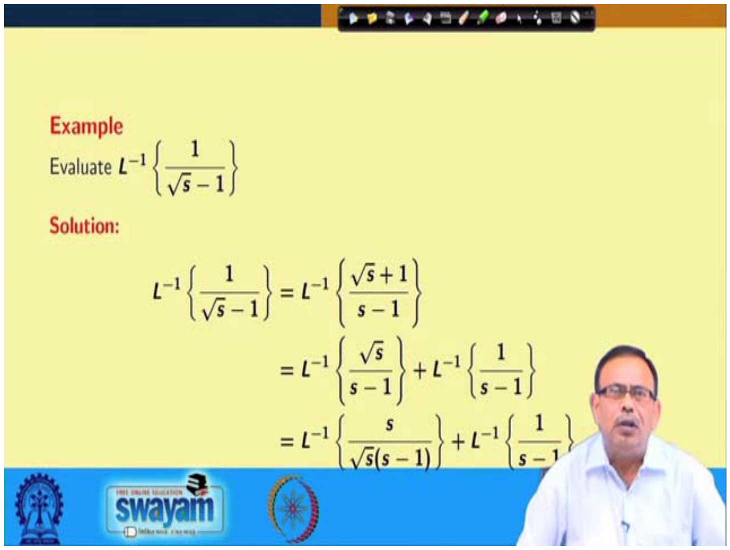
Thus very easily we can find out the solution of this.

(Refer Slide Time: 25:41)



In the next example also, we need to evaluate the Laplace inverse of $\frac{1}{\sqrt{s}-1}$.

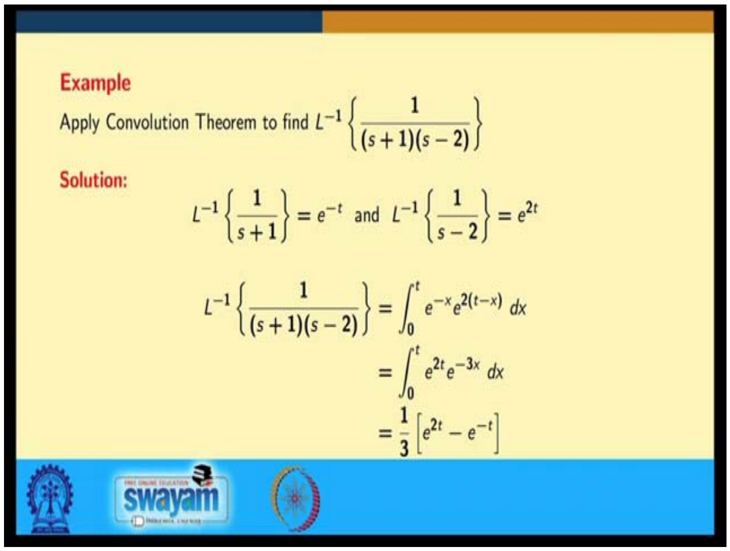
(Refer Slide Time: 26:23)



In a similar way, we can rationalize the denominator here and proceed as earlier to obtain the required solution as:

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{\sqrt{s}-1}\right\} &= L^{-1}\left\{\frac{\sqrt{s}+1}{s-1}\right\} \\
&= L^{-1}\left\{\frac{\sqrt{s}}{s-1}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} \\
&= L^{-1}\left\{\frac{s-1+1}{\sqrt{s}(s-1)}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} \\
&= L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} + L^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} \\
&= \frac{1}{\sqrt{\pi t}} + e^{t}\,\mathrm{erf}(\sqrt{t}) + e^{t} \\
&= \frac{1}{\sqrt{\pi t}} + e^{t}\left[\mathrm{erf}(\sqrt{t}) + 1\right].
\end{aligned}
$$

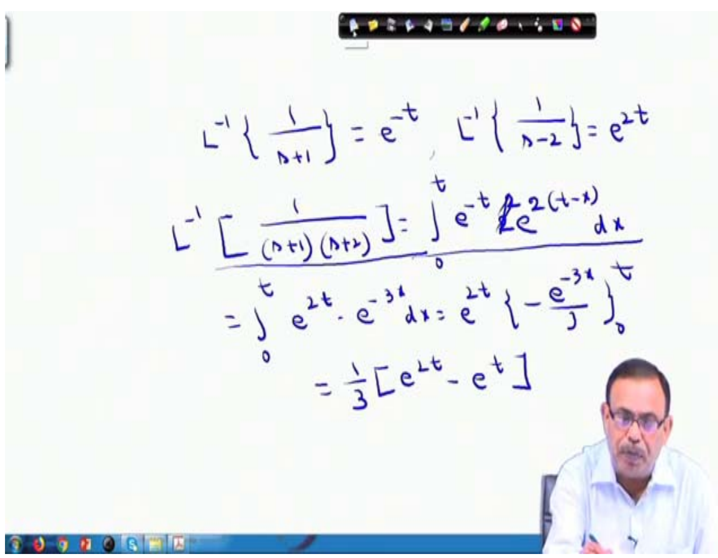
(Refer Slide Time: 26:27)



The next problem says to apply convolution theorem to find the value of

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}.$$

(Refer Slide Time: 26:39)



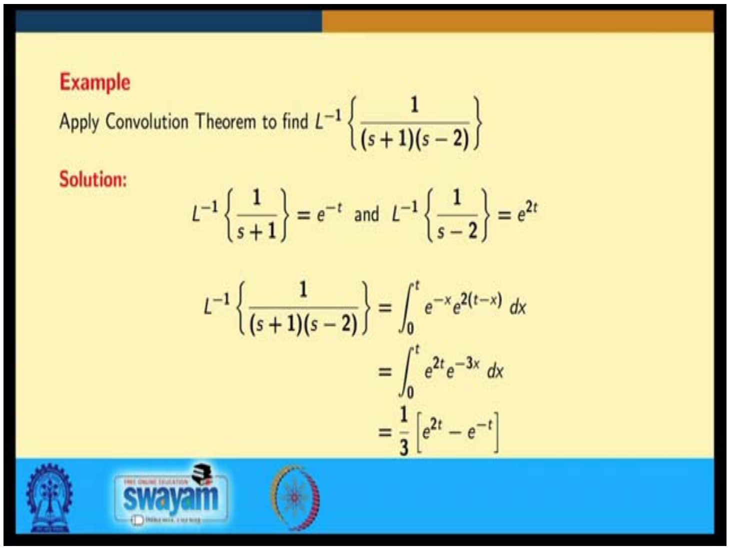
We know,

$$L^{-1}\left\{\frac{1}{(s+1)}\right\} = e^{-t} \text{ and } L^{-1}\left\{\frac{1}{(s-2)}\right\} = e^{2t}.$$

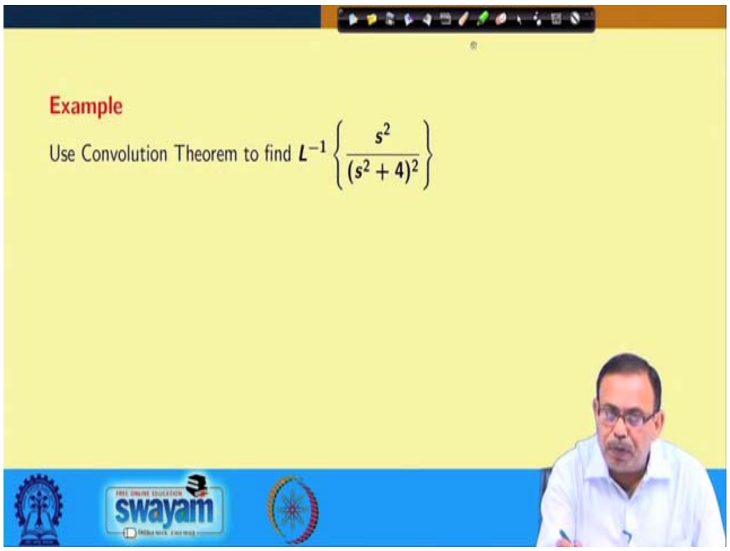
So, using Convolution theorem, we get,

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \int_0^t e^{-x} e^{2(t-x)} dx = \frac{1}{3}[e^{2t} - e^{-t}].$$

So, using convolution, very easily, we can obtain the solution.



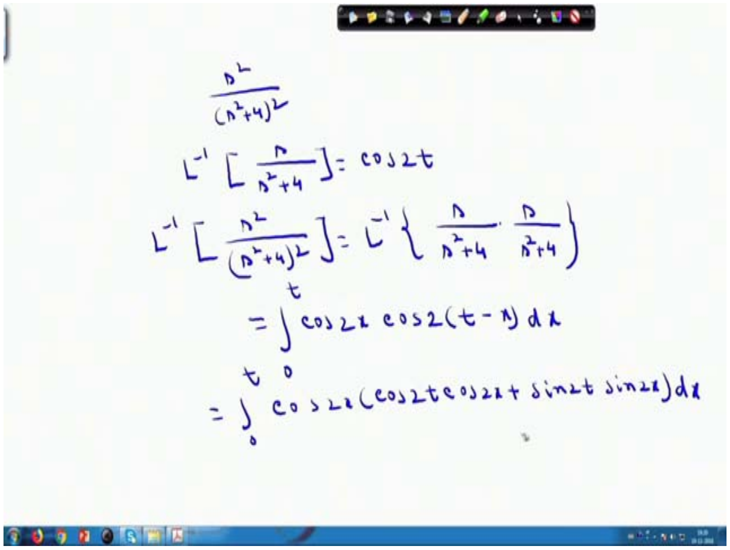
(Refer Slide Time: 28:17)



And let us see one more example that is Laplace inverse of

$$\frac{s^2}{(s^2+4)^2}.$$

(Refer Slide Time: 28:27)



We know $L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$. Therefore, we can simply write

$$L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+4}\cdot\frac{s}{s^2+4}\right\}$$

So, using the convolution theorem we get,

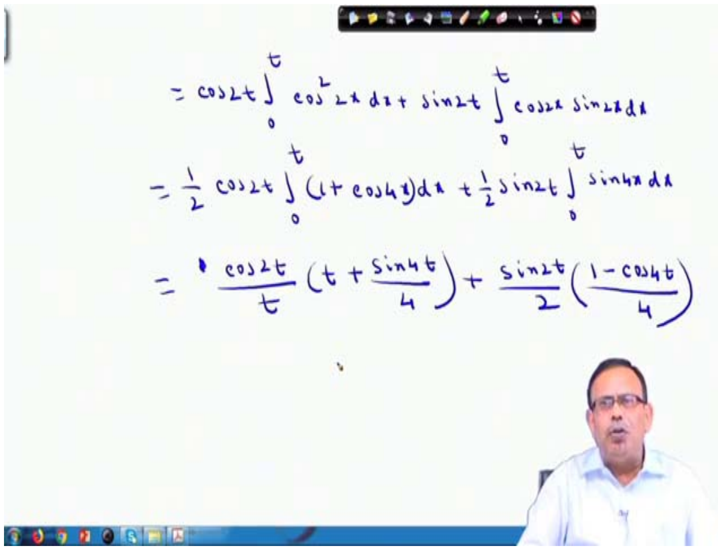
$$L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \int_0^t \cos 2x \cos 2(t-x)\,dx.$$

And now we have to evaluate this particular integral. We write it as,

$$L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \int_0^t \cos 2x\,[\cos 2t \cos 2x + \sin 2t \sin 2x]dx$$

$$= \frac{1}{2}\cos 2t \int_0^t (1+\cos 4x)\,\mathrm{dx} + \frac{1}{2}\sin 2t \int_0^t \sin 4x\ \mathrm{dx}.$$

(Refer Slide Time: 29:51)



So that we can now evaluate the integrals easily. After simplification, this will be equal to

$$L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \frac{t\cos 2t}{2} + \frac{\sin 2t}{4}.$$

So, in a similar way, whenever we have the product of two Laplace transforms, always we can use the convolution theorem to simply evaluate that integral to obtain the Laplace inverse of some product of functions.

Thank you.