Transform Calculus and its Applications in Differential Equations
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Lecture - 12
Properties of Inverse Laplace Transform

We are continuing with the earlier lecture, where we have started the inverse Laplace transform. The last problem that we solved in the previous lecture was to find inverse Laplace transform of

$$\frac{s-1}{(s+3)(s^2+2s+2)}$$

and the solution was obtained as

$$L^{-1}\left\{\frac{s-1}{(s+3)(s^2+2s+2)}\right\} = -\frac{4}{5}e^{-3t} + \frac{1}{5}e^{-t}(4\cos t - 3\sin t).$$

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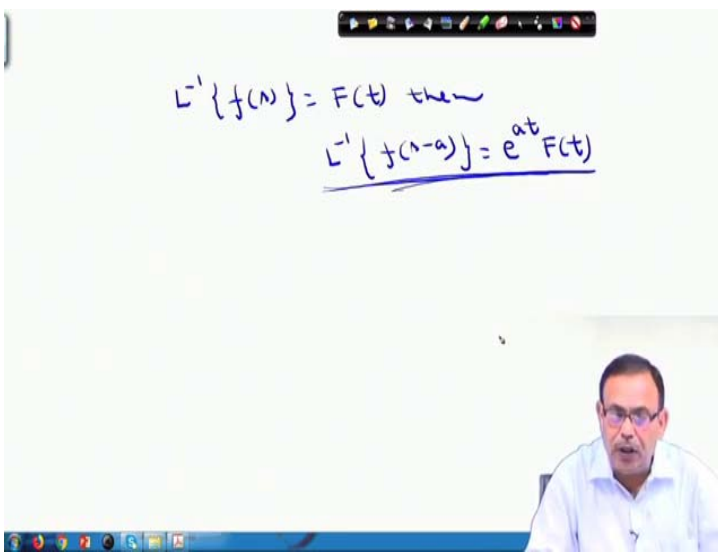

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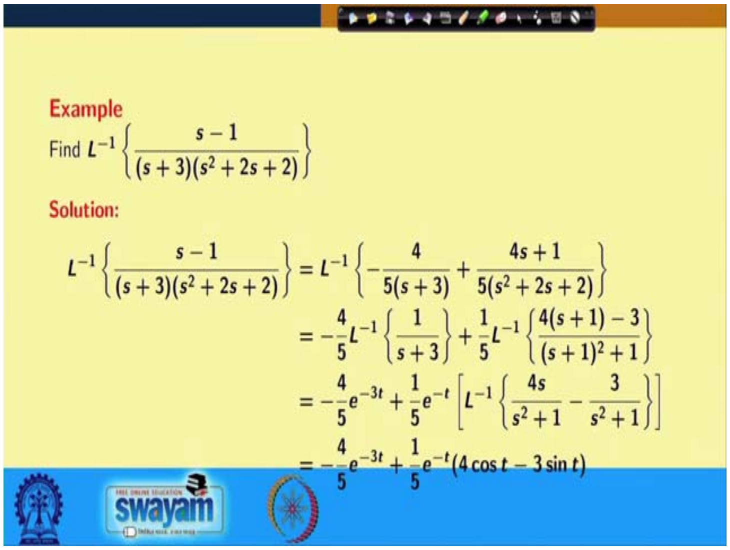

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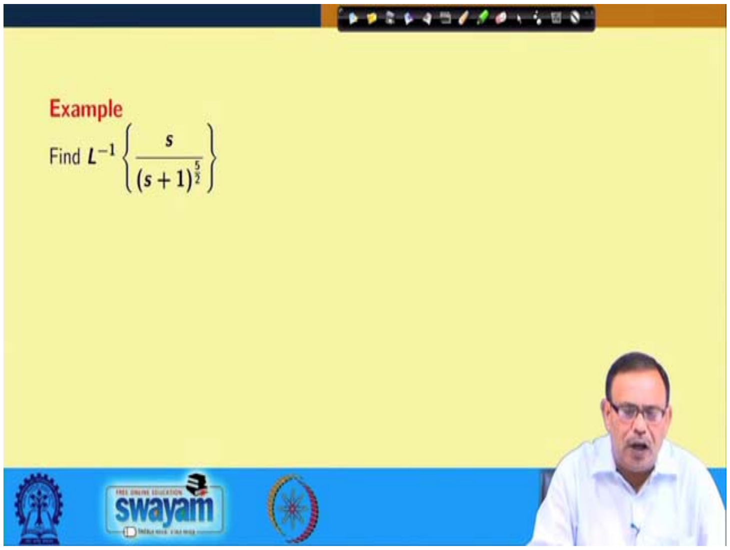

Now, let us see the next example, to find Laplace inverse of

$$\frac{s}{(s+1)^{5/2}}.$$

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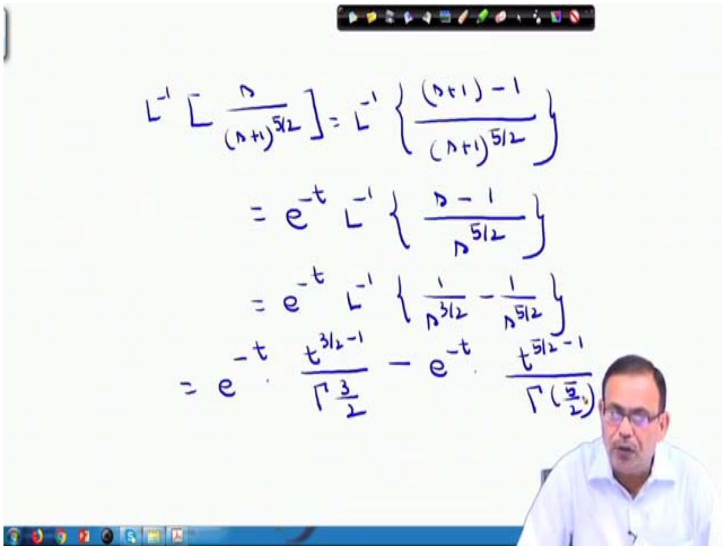

This we will also try to bring to some known form as follows:

$$L^{-1}\left\{\frac{s}{(s+1)^{5/2}}\right\} = L^{-1}\left\{\frac{(s+1)-1}{(s+1)^{5/2}}\right\}$$
$$= e^{-t}L^{-1}\left\{\frac{s-1}{s^{5/2}}\right\}.$$

So that we can break it into two parts as

$$L^{-1}\left\{\frac{s}{(s+1)^{5/2}}\right\} = e^{-t}\left[L^{-1}\left\{\frac{1}{s^{3/2}}\right\} - L^{-1}\left\{\frac{1}{s^{5/2}}\right\}\right]$$
$$= e^{-t}\left[\frac{t^{\frac{3}{2}-1}}{\Gamma\left(\frac{3}{2}\right)} - \frac{t^{\frac{5}{2}-1}}{\Gamma\left(\frac{5}{2}\right)}\right]$$
$$= \frac{2}{3}e^{-t}(3-2t)\sqrt{\frac{t}{\pi}}.$$

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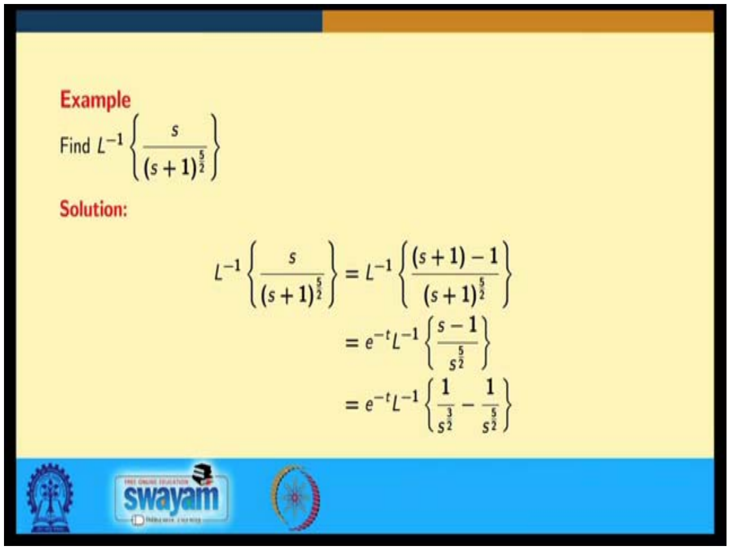

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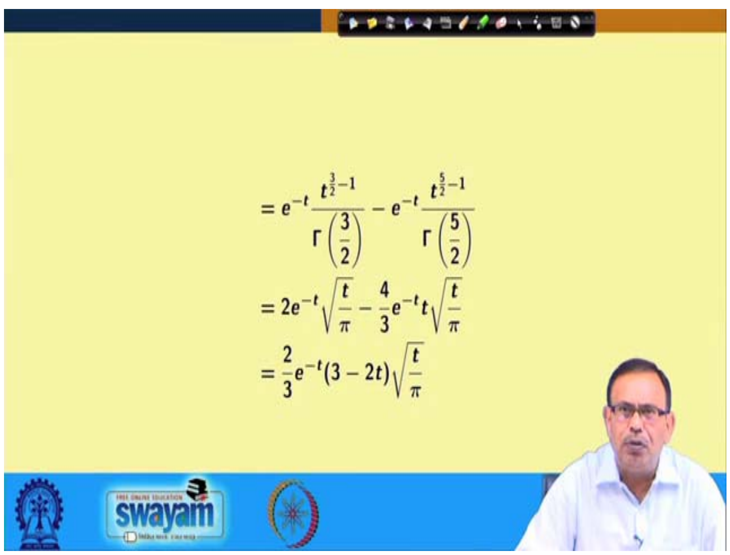

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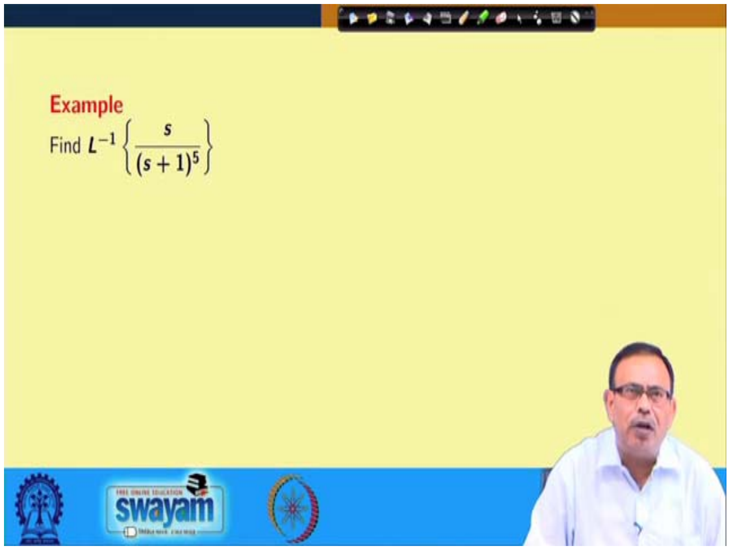

So, next example is to evaluate Laplace inverse of

$$\frac{s}{(s+1)^5}.$$

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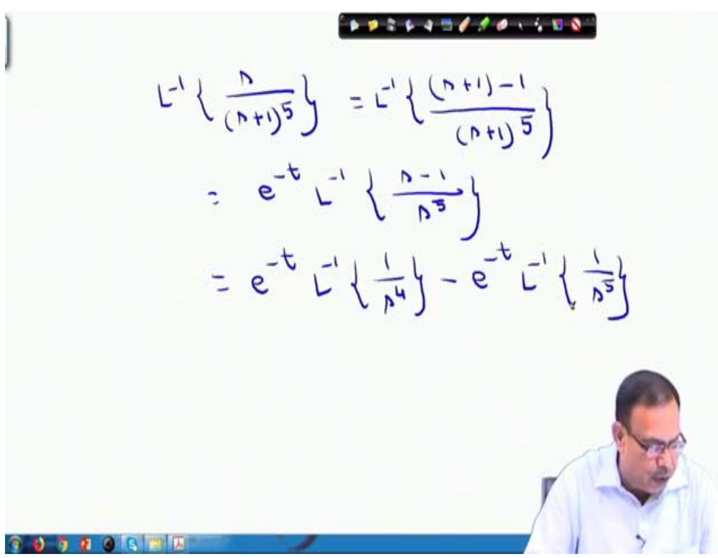

So obviously, again for this particular function, we have $(s+1)$. So, like earlier examples,

$$
\begin{aligned}
L^{-1}\left\{\frac{s}{(s+1)^5}\right\} &= L^{-1}\left\{\frac{(s+1)-1}{(s+1)^5}\right\} \\
&= e^{-t}\left[L^{-1}\left\{\frac{s-1}{s^5}\right\}\right] \\
&= e^{-t}\left[L^{-1}\left\{\frac{1}{s^4}\right\} - L^{-1}\left\{\frac{1}{s^5}\right\}\right] \\
&= \frac{e^{-t}}{24}t^3(4-t).
\end{aligned}
$$

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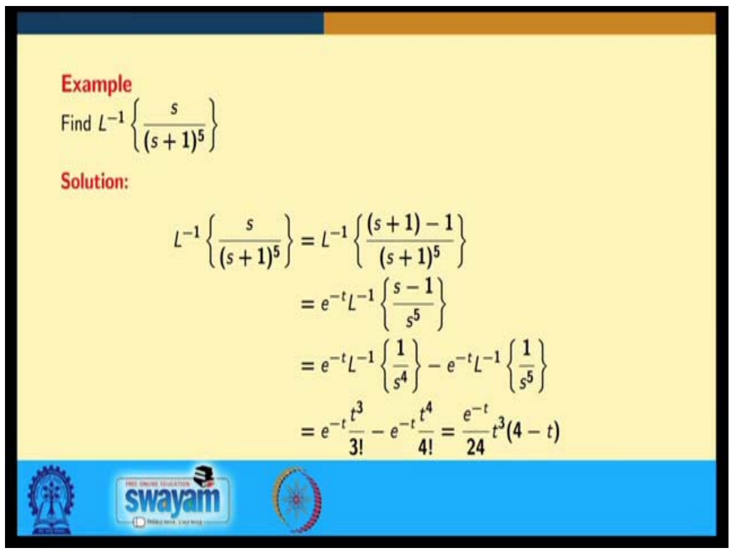

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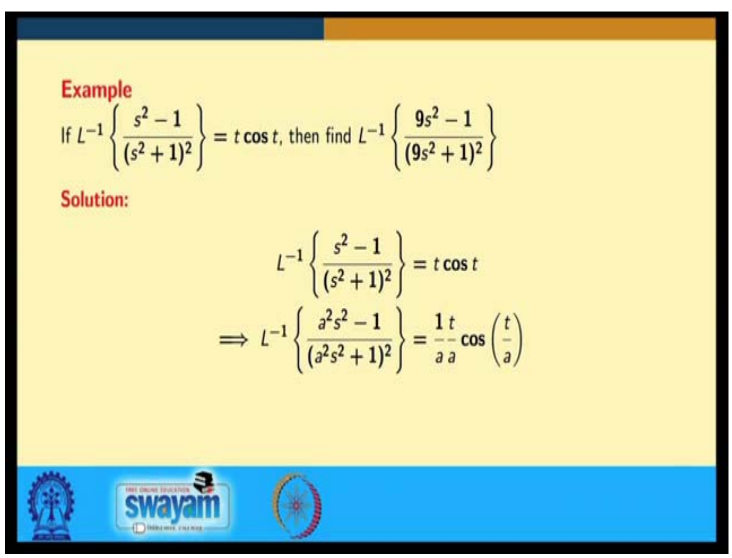

The next problem is suppose, we know,

$$L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$$

and we need to find out

$$L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)^2}\right\}$$

Or, in other sense if we know Laplace inverse of some function, then using that Laplace inverse knowledge, how to find out the Laplace inverse of some other function.

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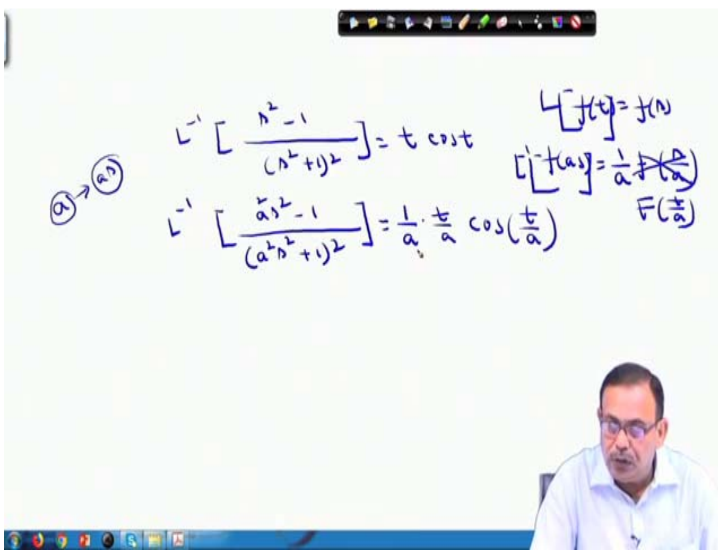

For this one,

$$L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$$

Then by using the change of scale property i.e., $L^{-1}\{f(as)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$, we will have

$$L^{-1}\left\{\frac{a^2s^2-1}{(a^2s^2+1)^2}\right\} = \frac{1}{a}\frac{t}{a}\cos\left(\frac{t}{a}\right)$$

where $F(t) = tcos\ t$ and $f(s) = \frac{s^2-1}{(s^2+1)^2}$. If we put $a = 3$, then it will become

$$L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)^2}\right\}=\frac{t}{9}\cos\left(\frac{t}{3}\right)$$

which gives the desired result.

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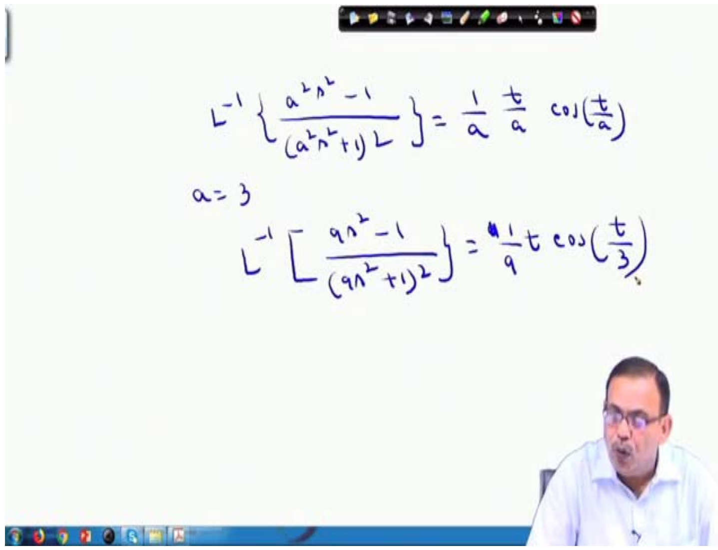

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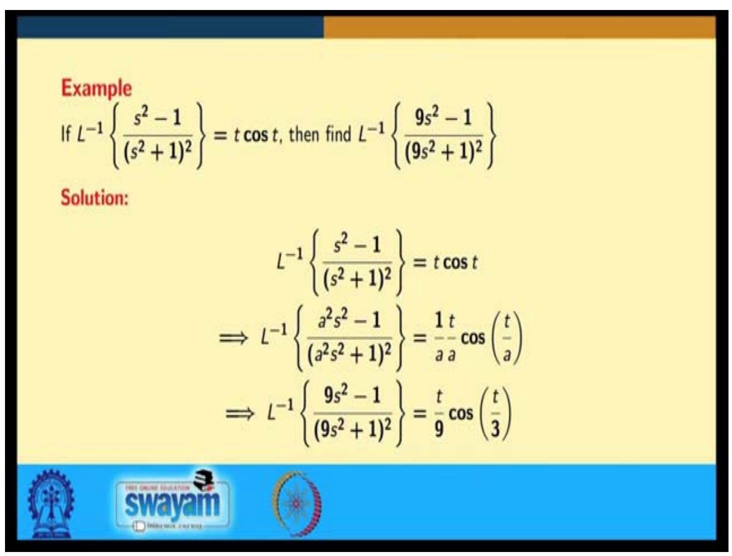

So, in some cases, by using the general parameter also, we can solve, like here we have done. And, then by substituting a particular value to the parameter, we can obtain the required result.

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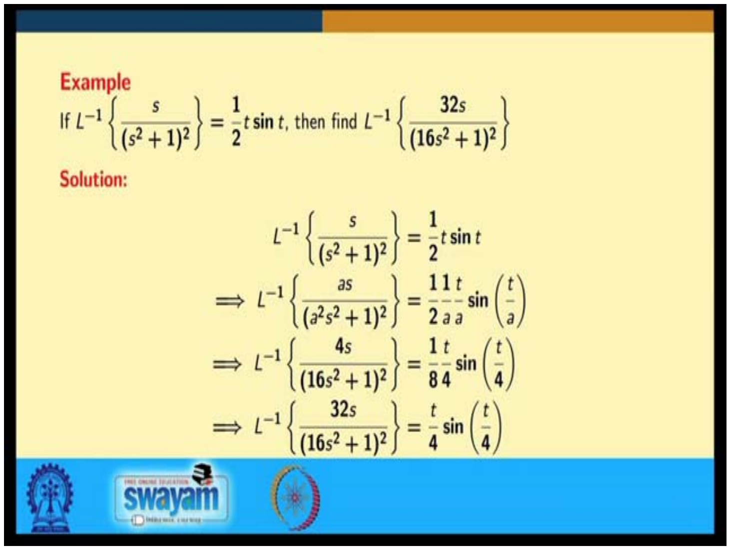

Next one is similar, suppose, we know,

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t$$

then we have to find out

$$L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\}.$$

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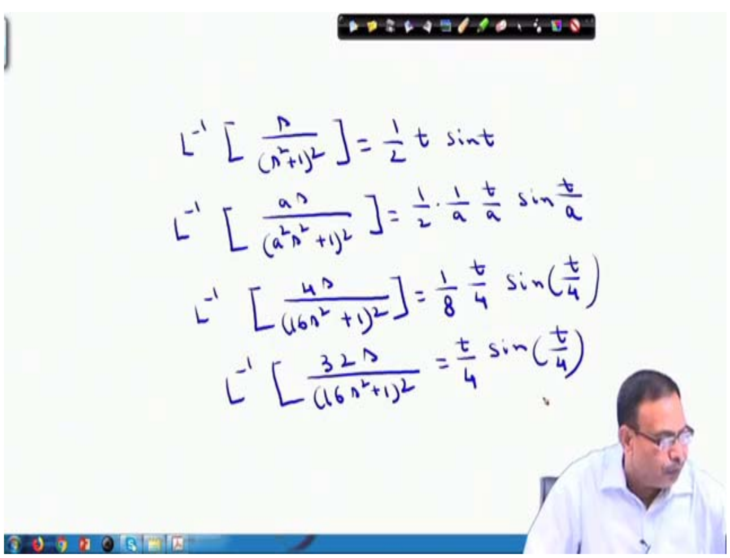

It is given that

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t.$$

Then similar to the previous problem, here also we use the change of scale property to get

$$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{a}\frac{1}{2}\frac{t}{a}\sin\left(\frac{t}{a}\right).$$

We now put $a = 4$, then it will become

$$L^{-1}\left\{\frac{4s}{(16s^2+1)^2}\right\} = \frac{1}{8}.\frac{t}{4}\sin\left(\frac{t}{4}\right)$$

$$\Rightarrow L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\} = \frac{t}{4}\sin\left(\frac{t}{4}\right)$$

which is the desired result.

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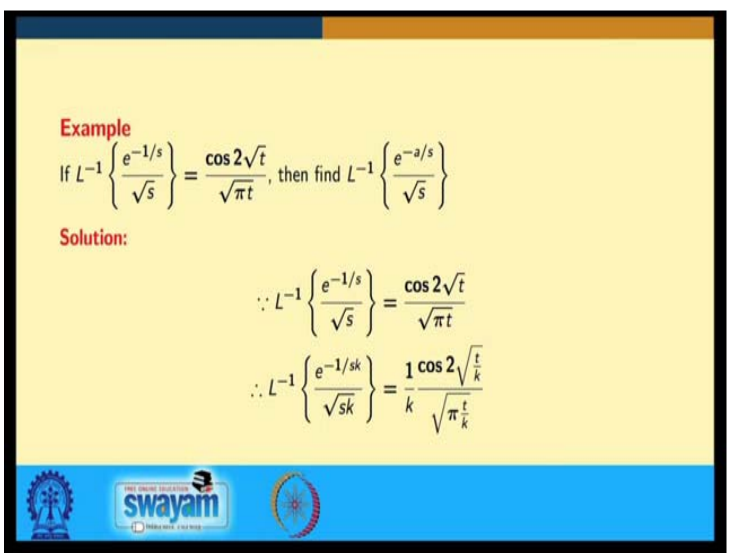

Now, next one is, if it is given that $L^{-1}\left\{\frac{e^{-1/s}}{\sqrt{s}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, then we have to find out

$$L^{-1}\left\{\frac{e^{-a/s}}{\sqrt{s}}\right\}$$

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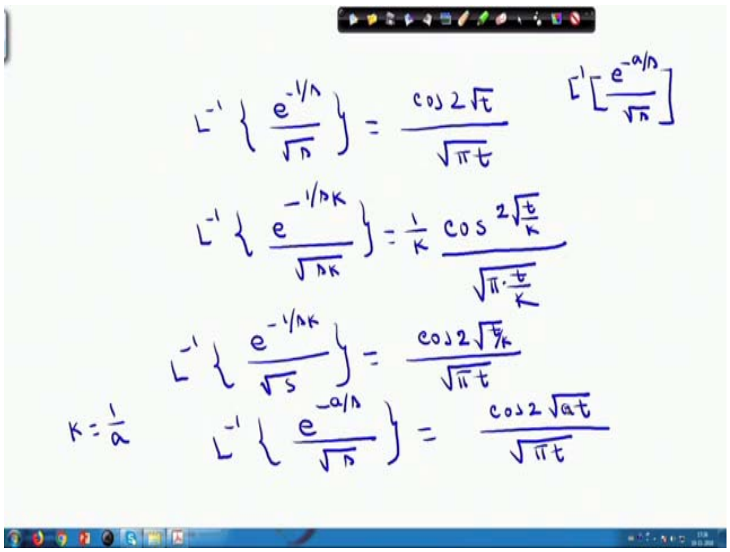

Using the change of scale property, we get

$$L^{-1}\left\{\frac{e^{-1/ks}}{\sqrt{ks}}\right\} = \frac{1}{k}\frac{\cos 2\sqrt{\frac{t}{k}}}{\sqrt{\frac{\pi t}{k}}}$$

We now put $k = \frac{1}{a}$, then it will become

$$L^{-1}\left\{\frac{\sqrt{a}\, e^{-a/s}}{\sqrt{s}}\right\} = a\frac{\cos 2\sqrt{at}}{\sqrt{a\pi t}}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-a/s}}{\sqrt{s}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}.$$

So, we are getting the required result over here.

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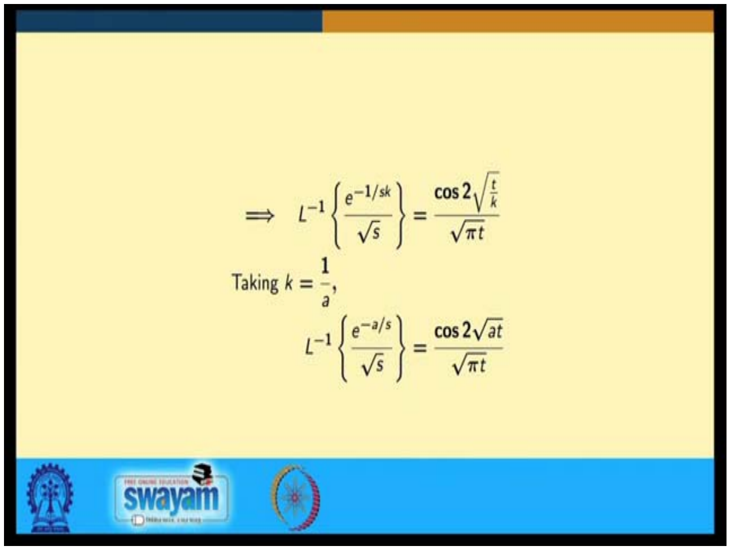

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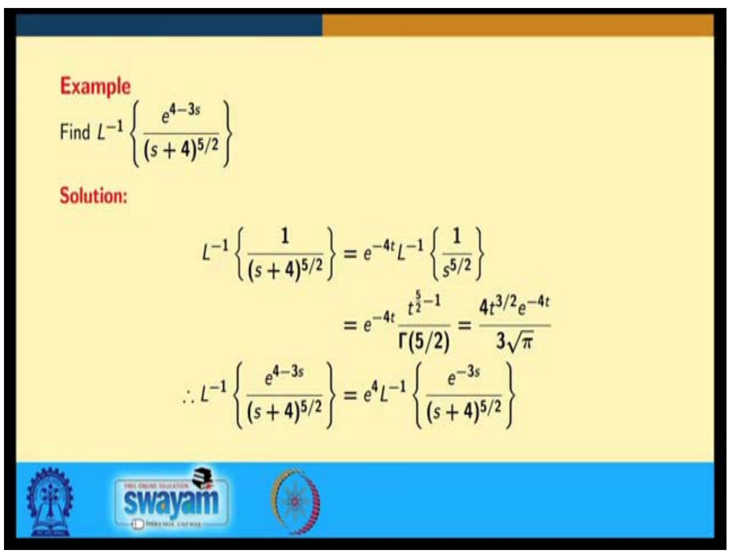

Next example is to evaluate the Laplace inverse of

$$\frac{e^{4-3s}}{(s+4)^{5/2}}.$$

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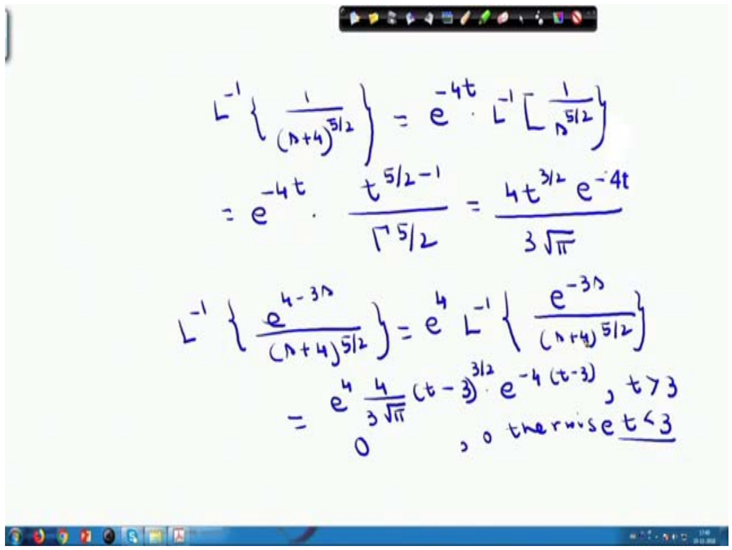

Let us start with

$$L^{-1}\left\{\frac{1}{(s+4)^{5/2}}\right\} = e^{-4t}L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$$

$$= \frac{e^{-4t}t^{\frac{5}{2}-1}}{\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{4t^{3/2}e^{-4t}}{3\sqrt{\pi}}.$$

Therefore, using second shifting theorem, we have,

$$L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^{4}L^{-1}\left\{\frac{e^{-3s}}{(s+4)^{5/2}}\right\}$$

$$= \begin{cases} e^{4}\dfrac{4(t-3)^{3/2}e^{-4(t-3)}}{3\sqrt{\pi}}, & \text{if } t > 3 \\ 0, & \text{if } t < 3 \end{cases}$$

$$= \frac{4}{3\sqrt{\pi}}(t-3)^{3/2}e^{-4(t-4)}\,H(t-3)$$

where, $H(t)$ is Heaviside unit step function.

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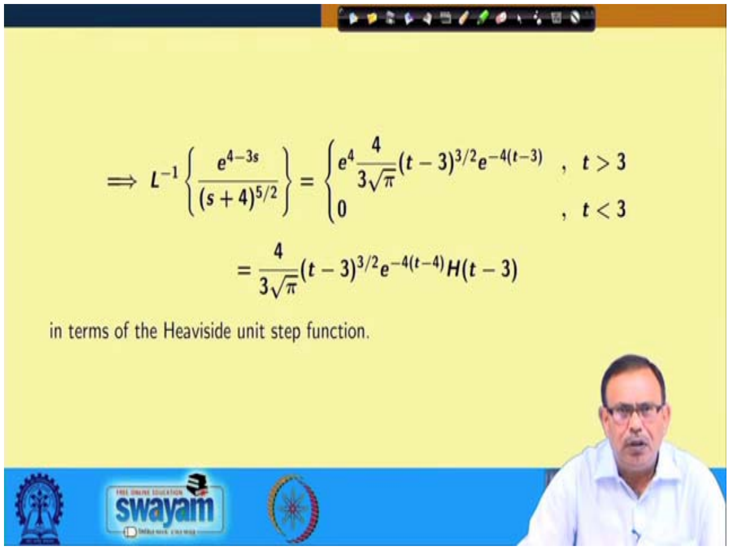

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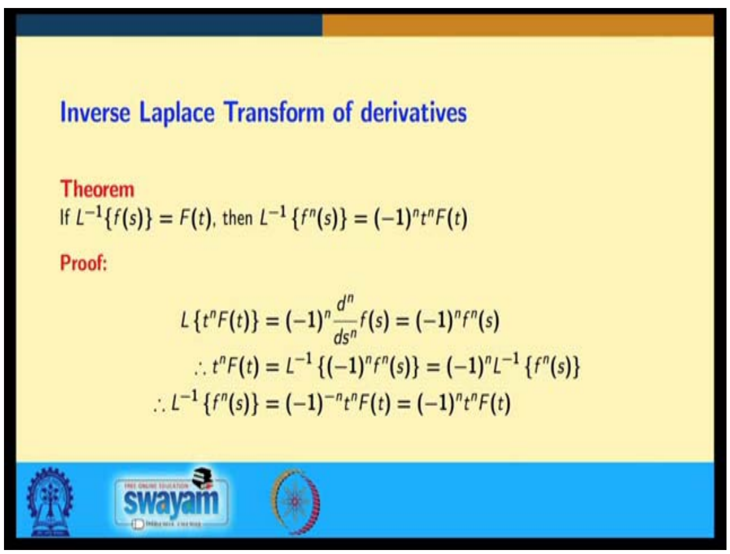

Now, let us see the inverse Laplace transform of derivatives of a function. We have seen that if we know the Laplace transform of a function, then from the theorem, we can tell what would be the Laplace transform of derivatives or in particular the n^{th} derivative of a function. Similarly, here also we can find out the inverse Laplace transform of derivatives of a given function that is if $L^{-1}\{f(s)\} = F(t)$, then, $L^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$. So, let us see how it works over here.

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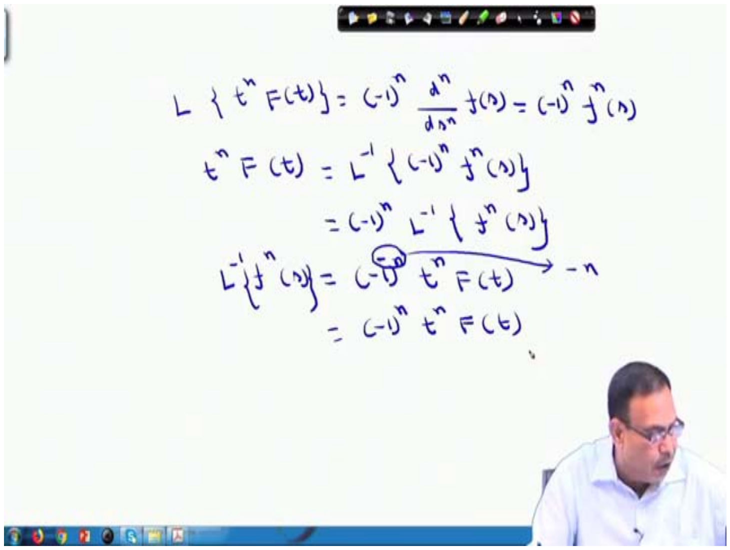

Using the theorem of multiplication by powers of t in Laplace transform, we know the following:

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^n(s)$$
$$\therefore (-1)^n L\{t^n F(t)\} = f^n(s)$$

Therefore, we can obtain

$$L^{-1}\{f^n(s)\} = (-1)^n t^n F(t).$$

This completes the proof.

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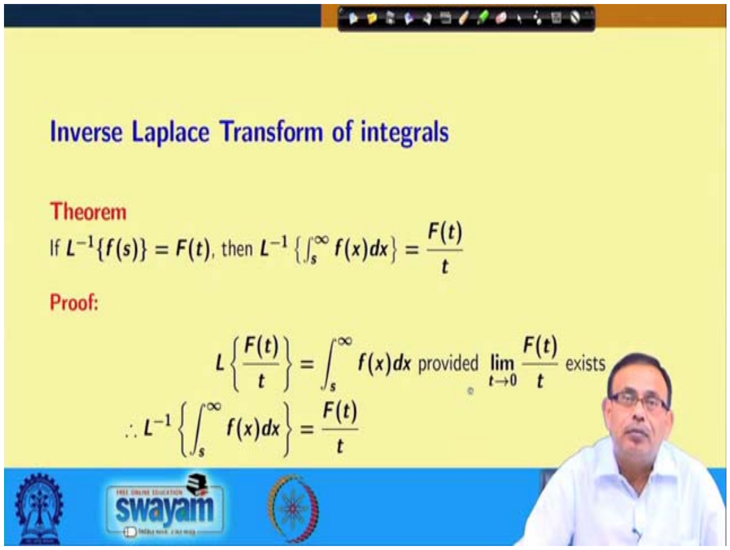

Next is inverse Laplace transform of integrals. If $L^{-1}\{f(s)\} = F(t)$, then, $L^{-1}\{\int_s^\infty f(x)dx\} = \frac{F(t)}{t}$ i.e., if we are integrating the Laplace transform $f(s)$ from s to ∞, then the Laplace inverse of that function will be equal to $\frac{F(t)}{t}$.

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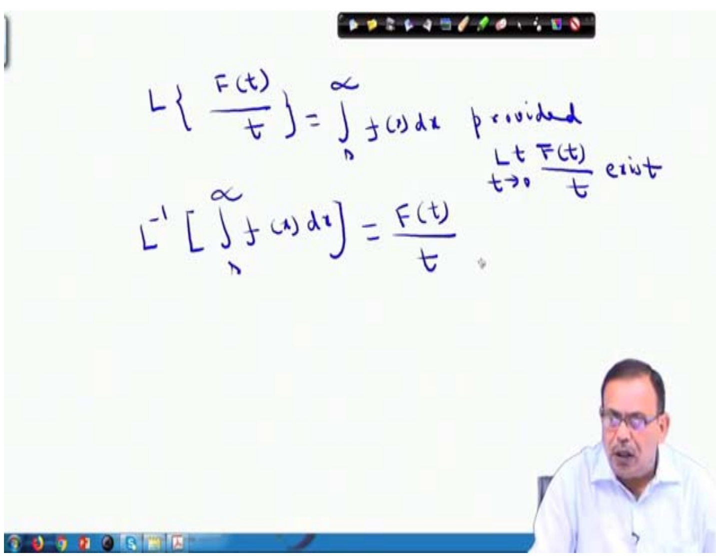

The result directly follows from the property of division by t in Laplace transform. As we know,

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)dx \quad , \qquad \text{provided} \lim_{t\to 0}\frac{F(t)}{t} \text{ exists.}$$

So therefore, we can write down

$$L^{-1}\left\{\int_s^\infty f(x)dx\right\} = \frac{F(t)}{t}.$$

This completes the proof.

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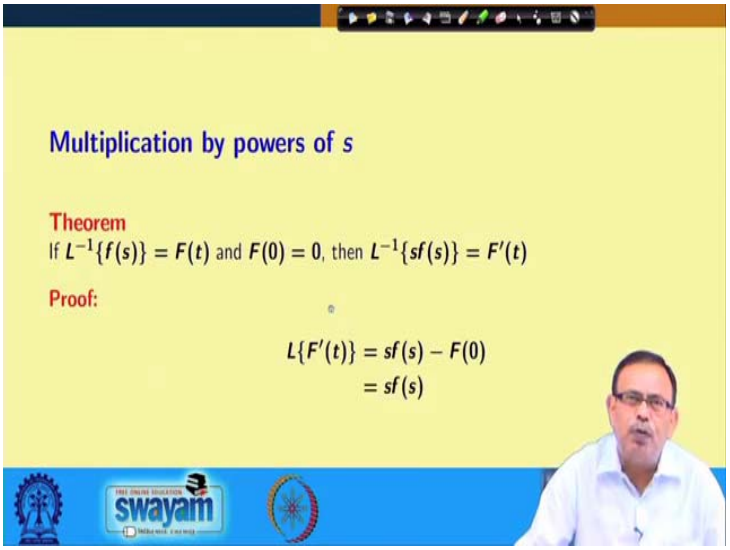

Now, we come to Multiplication by powers of s in Inverse Laplace transform. If $L^{-1}\{f(s)\} = F(t)$, and $F(0) = 0$ then, $L^{-1}\{sf(s)\} = F'(t)$.

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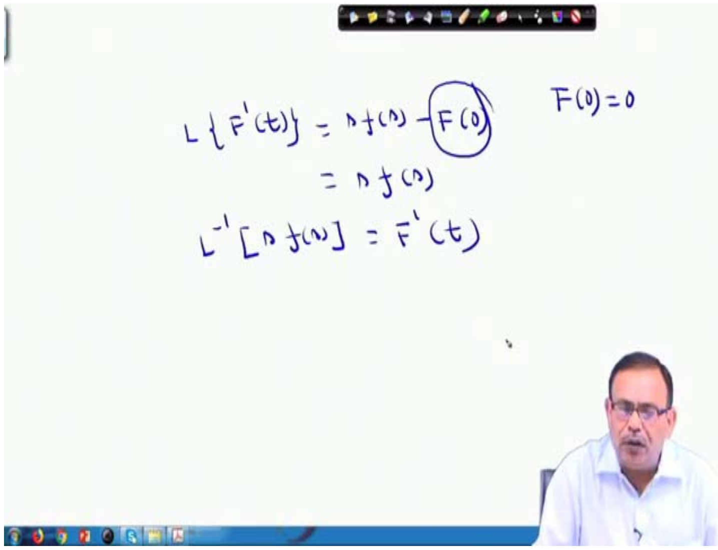

Again, just we are using the properties of Laplace transform. As we know from the Laplace transform of derivative of a function:

$$L\{F'(t)\} = sf(s) - F(0) = sf(s) \quad [\because \text{F}(0) = 0 \text{ given}]$$
$$\therefore \; L^{-1}\{sf(s)\} = F'(t).$$

This completes the proof of this particular theorem.

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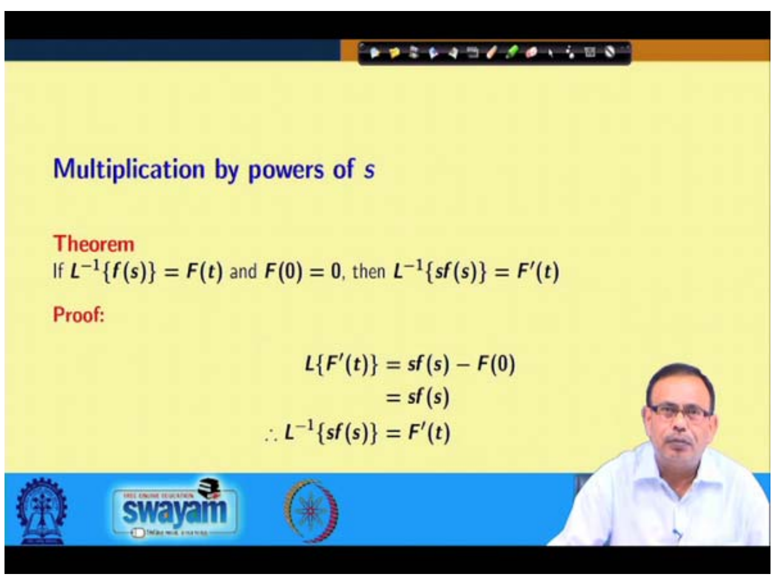

So, let us stop here. In the next lecture, we will see the division by power of s also in Inverse Laplace transform. Thank you.