Transform Calculus and its Applications in Differential Equations
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Lecture – 11
Introduction to Inverse Laplace Transform

Welcome back. In the last lecture, we were going through the null function and we have seen that, adding the null function will create certain new functions where the Laplace transform of two different functions will be same. We will discuss the consequence here. In this particular lecture, we will start with the Inverse Laplace Transform.

Till now we have done the Laplace transform of a function. Now, if we know the opposite one, that is if we know the Laplace Transform of a function and we need to find out the original function whose Laplace transform it is using the inverse Laplace transform technique, that we will study in this particular lecture.

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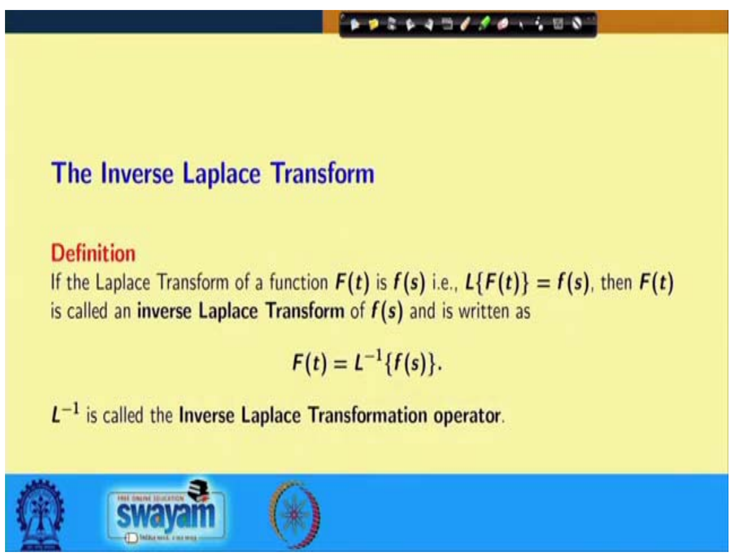

First see the definition of the inverse Laplace transform. The definition says that if the Laplace transform of a function $F(t)$ is $f(s)$ that is, $L\{F(t)\} = f(s)$, then $F(t)$ is called inverse Laplace transform of $f(s)$ and is written as $F(t) = L^{-1}\{f(s)\}$.

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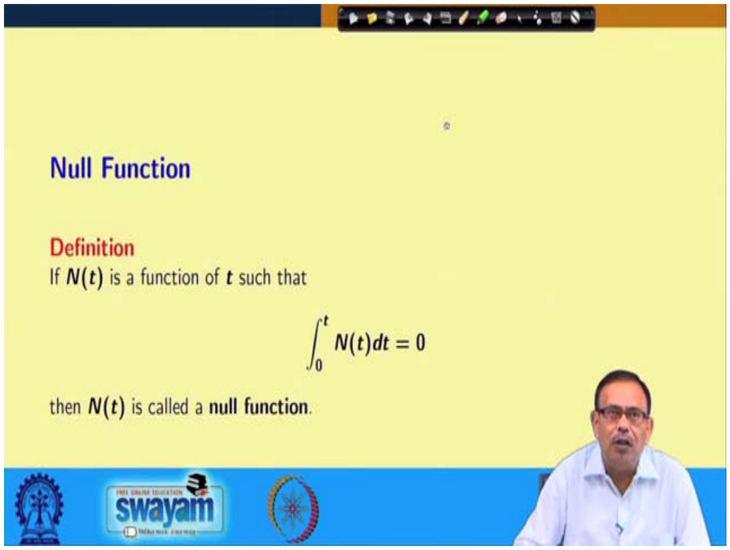

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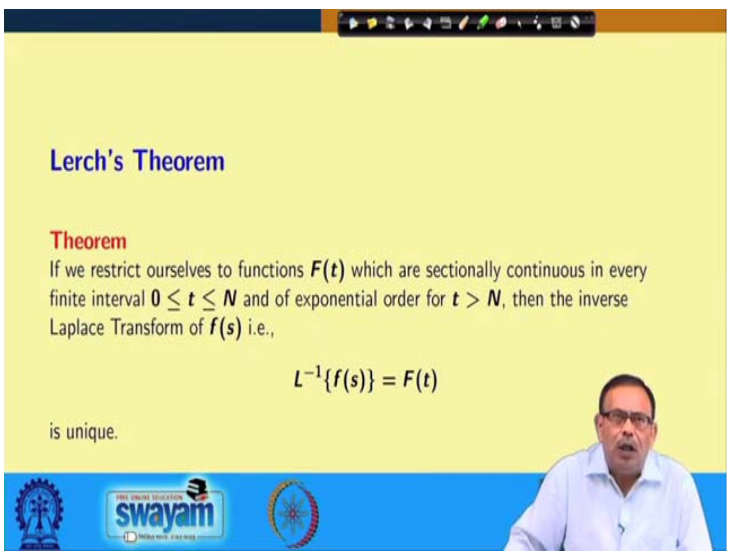

Now, we see this theorem: Lerch's theorem which says that, if we restrict ourselves to functions $F(t)$ which are sectionally continuous in every finite interval $0 \leq t \leq N$ and are of exponential order for $t > N$, then the inverse Laplace transform of $f(s)$ that is $L^{-1}\{f(s)\} = F(t)$ is unique. It means, if we have the Laplace transform of a function, then the inverse Laplace transform of that function will be unique if we restrict the functions to this only, that is if we do not consider the null functions.

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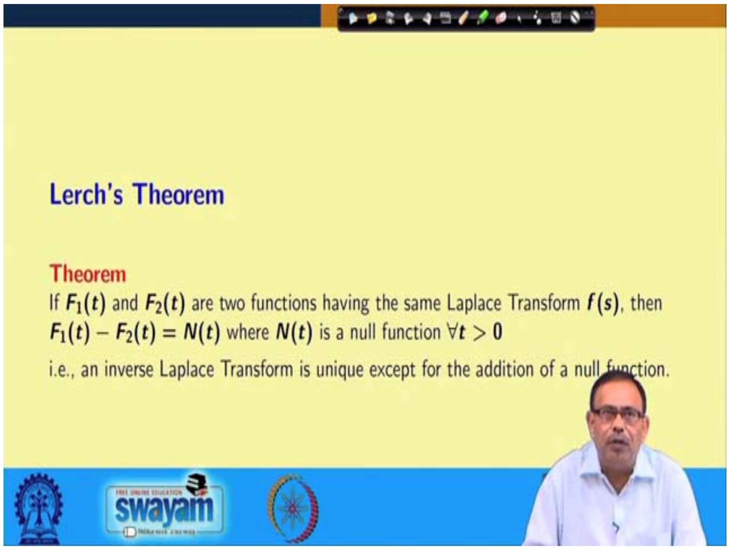

And there is the other form of Lerch's theorem that is, if $F_1(t)$ and $F_2(t)$ are two functions having the same Laplace transform $f(s)$, then $F_1(t) - F_2(t) = N(t)$, where $N(t)$ is a null function i.e., an inverse Laplace transform is unique except for addition of a null function. So, if the function is not a null function, always the inverse Laplace transform will be unique.

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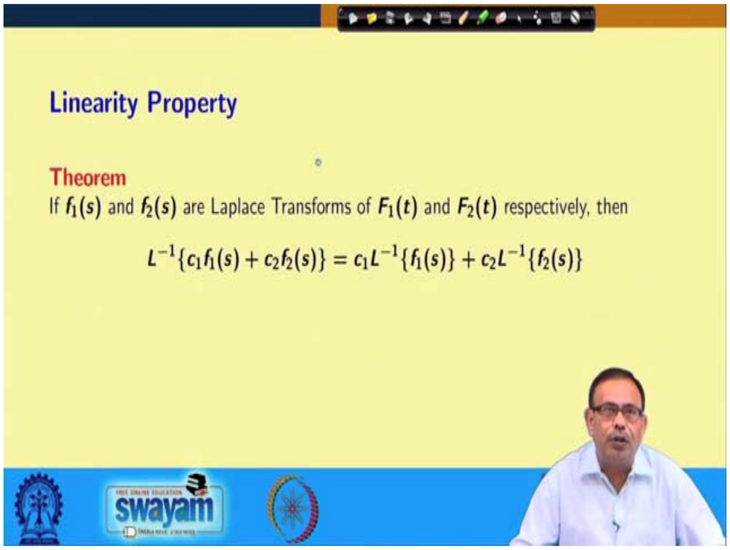

And so, let us go to some properties. The first property is the linearity property. Just like in the Laplace transform we had the linearity property, in inverse Laplace transform also we have the linearity property. If $f_1(s)$ and $f_2(s)$ are Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$$L^{-1}\{c_1f_1(s) + c_2f_2(s)\} = c_1L^{-1}\{f_1(s)\} + c_2L^{-1}f_2(s)\}.$$

Let us see the proof of this one.

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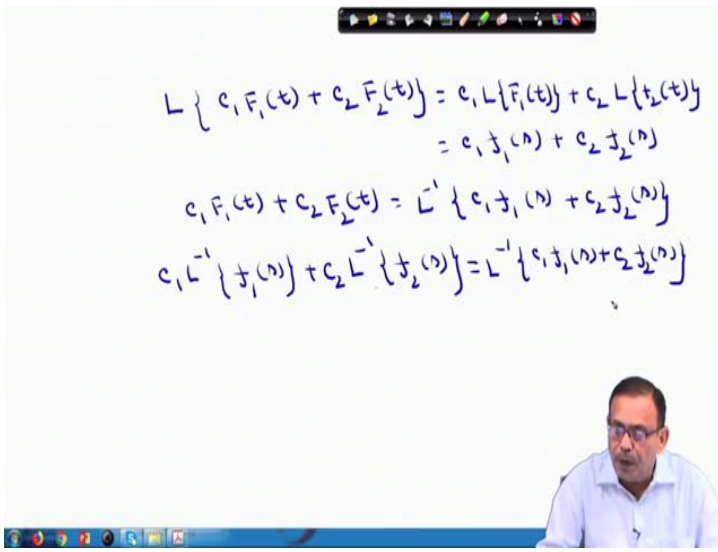

Since, Laplace Transform is linear, so we have,

$$\begin{aligned} L\{c_1F_1(t) + c_2F_2(t)\} &= c_1L\{F_1(t)\} + c_2L\{F_2(t)\} \\ &= c_1f_1(s) + c_2f_2(s). \end{aligned}$$

This implies that

$$c_1F_1(t) + c_2F_2(t) = L^{-1}\{c_1f_1(s) + c_2f_2(s)\}$$
$$\Rightarrow c_1L^{-1}\{f_1(s)\} + c_2L^{-1}f_2(s)\} = L^{-1}\{c_1f_1(s) + c_2f_2(s)\}.$$

And therefore, this completes the proof of our theorem.

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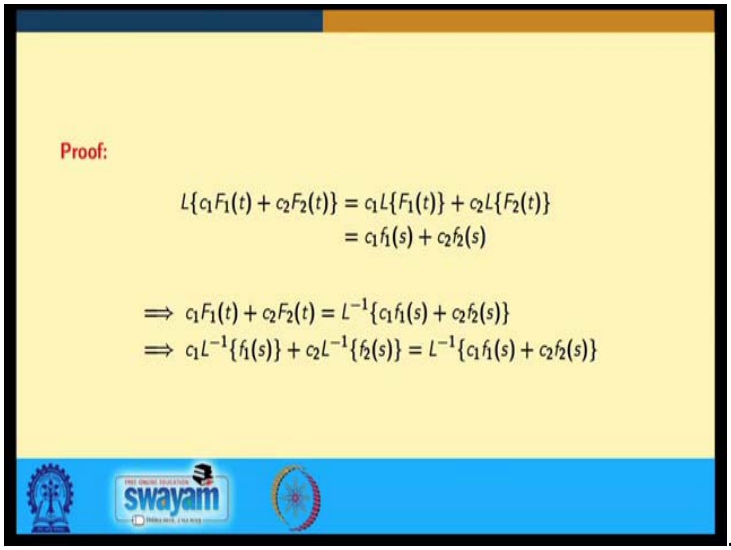

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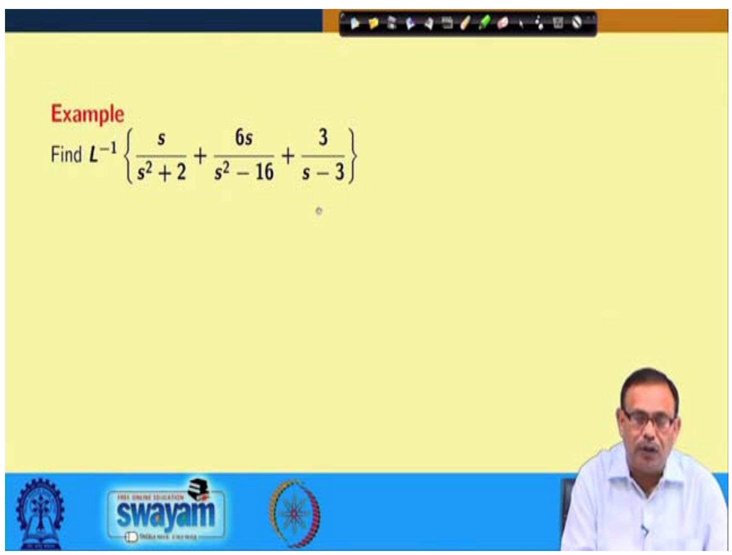

Let us take an example, how to find out the Laplace inverse of

$$f(s) = \frac{s}{s^2+2}+\frac{6s}{s^2-16}+\frac{3}{s-3}.$$

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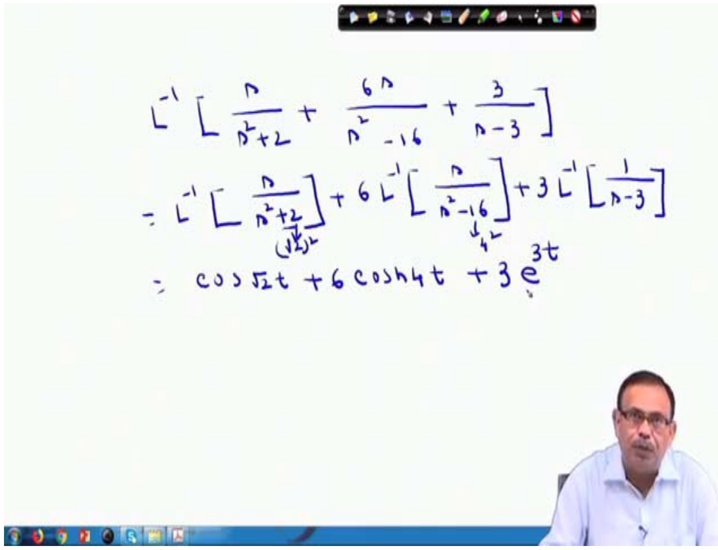

So, using the linearity property, we can write down

$$L^{-1}\left\{\frac{s}{s^2+2}\right\} + L^{-1}\left\{\frac{6s}{s^2-16}\right\} + L^{-1}\left\{\frac{3}{s-3}\right\}$$
$$= \cos\sqrt{2}\,t + 6\cosh 4t + 3e^{3t}.$$

So, in this way, once we know the Laplace transform of a function, using the inverse Laplace transform we can easily tell what is the corresponding function whose Laplace transform it is. And, using linearity property, effectively we are finding out this particular value of the function.

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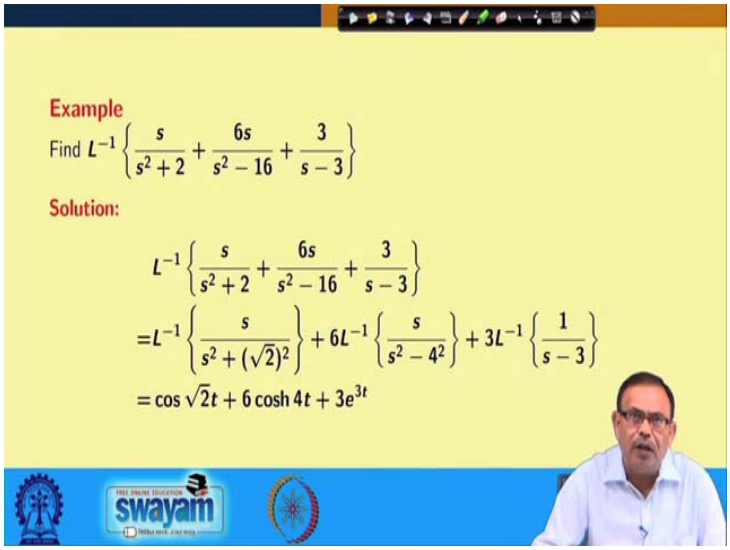

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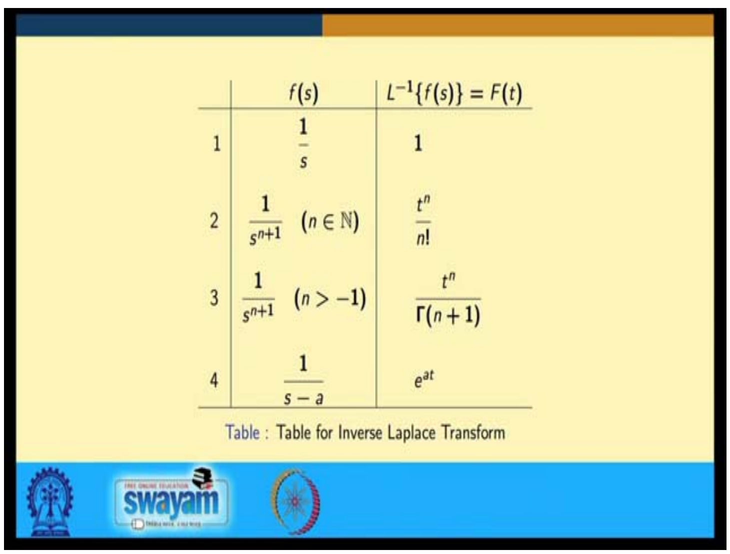

For our easy remembrance, we have presented here a set of Laplace transforms and the corresponding function in tabular form.

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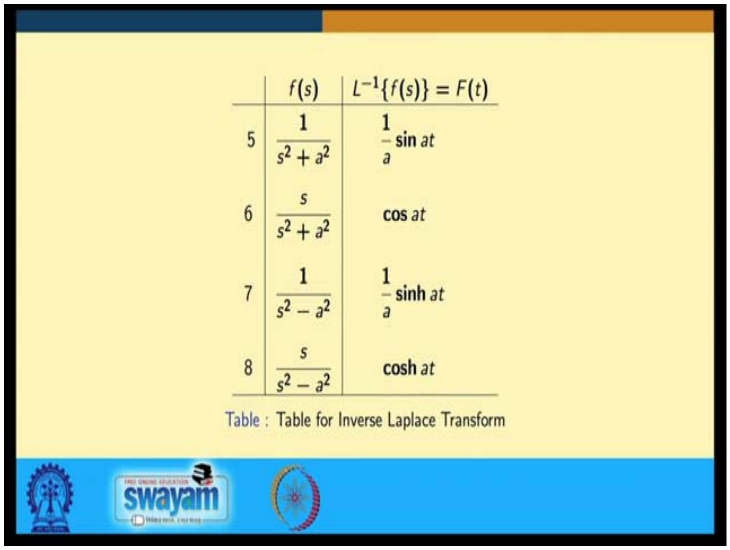

This table can be used for ready reference.

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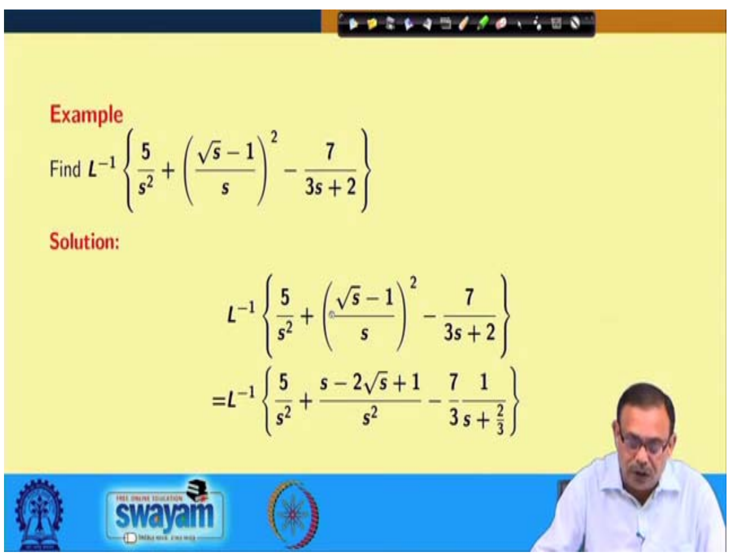

Now, let us try to find out the inverse Laplace transform of this particular function.

$$\frac{5}{s^2}+\left(\frac{\sqrt{s}-1}{s}\right)^2-\frac{7}{3s+2}.$$

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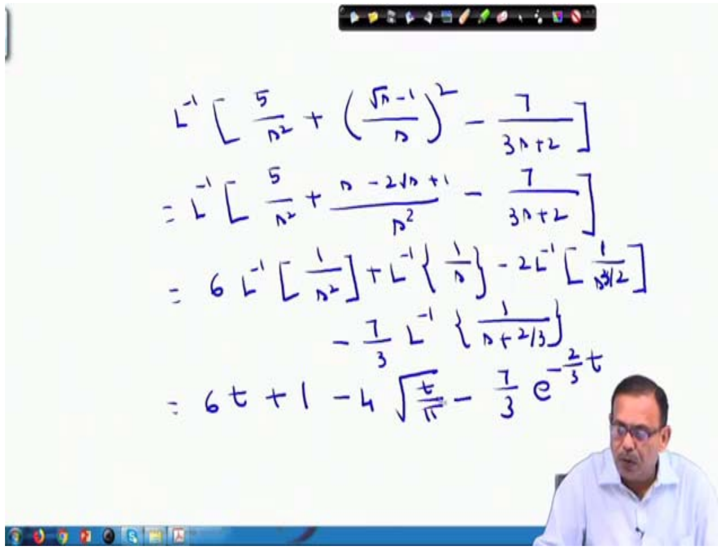

So, this we can write down

$$L^{-1}\left\{\frac{5}{s^2}+\frac{s-2\sqrt{s}+1}{s^2}-\frac{7}{3}\frac{1}{\left(s+\frac{2}{3}\right)}\right\}$$

$$=6L^{-1}\left\{\frac{1}{s^2}\right\}+L^{-1}\left\{\frac{1}{s}\right\}-2L^{-1}\left\{\frac{1}{s^{3/2}}\right\}-\frac{7}{3}L^{-1}\left\{\frac{1}{s+\frac{2}{3}}\right\}.$$

Actually we have written it in such a fashion, so that directly the corresponding function can be identified as

$$L^{-1}\left\{\frac{5}{s^2}+\left(\frac{\sqrt{s}-1}{s}\right)^2-\frac{7}{3s+2}\right\}=6t+1-4\sqrt{\frac{t}{\pi}}-\frac{7}{3}e^{-\frac{2t}{3}}.$$

So, like this way we try to find out the solution of these functions.

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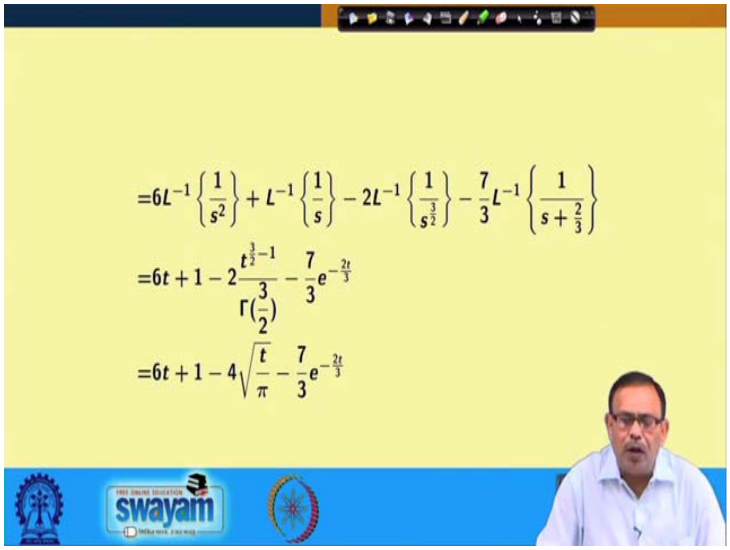

Now we come to certain important properties of Inverse Laplace transform.

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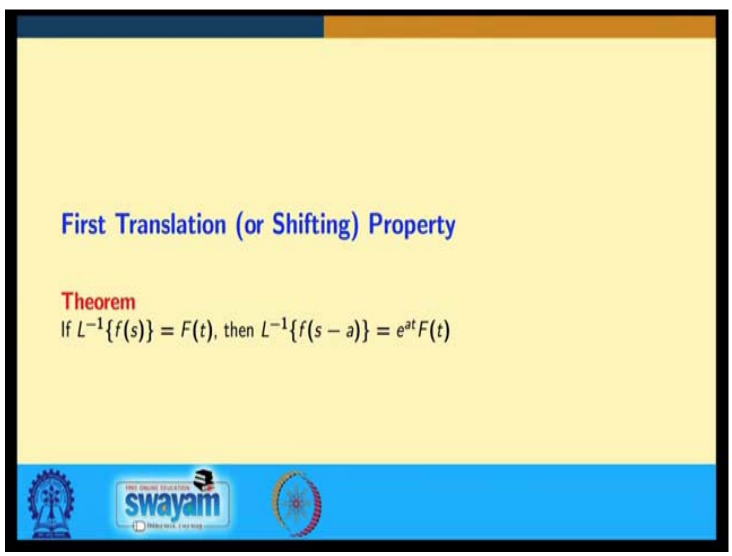

The first property is First Translation or Shifting property which states that if $L^{-1}\{f(s)\}=F(t)$ then $L^{-1}\{f(s-a)\}=e^{at}F(t)$.

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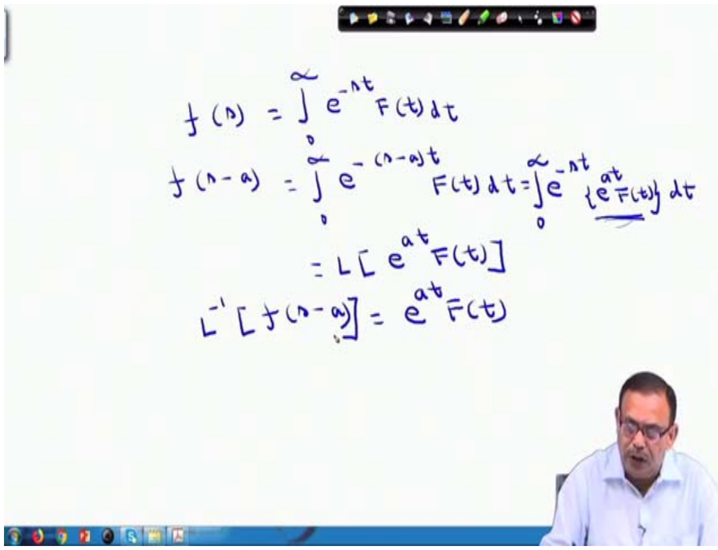

So, for the proof, we have,

$$f(s) = \int_0^\infty e^{-st} F(t) dt$$

$$\Rightarrow f(s-a) = \int_0^\infty e^{-(s-a)t} F(t) dt$$

$$= \int_0^\infty e^{-st} \left(e^{at} F(t)\right) dt$$

$$= L\{e^{at} F(t)\}$$

Therefore, Laplace inverse of $f(s-a)$ is equals to

$$L^{-1}\{f(s-a)\} = e^{at} F(t).$$

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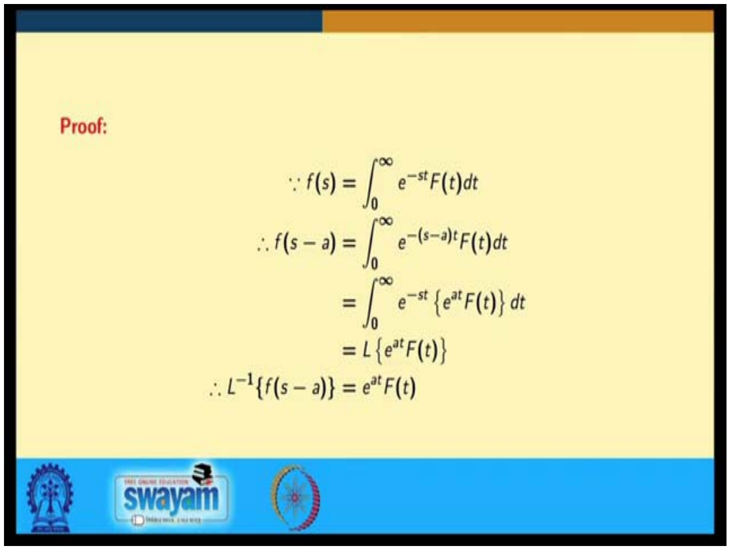

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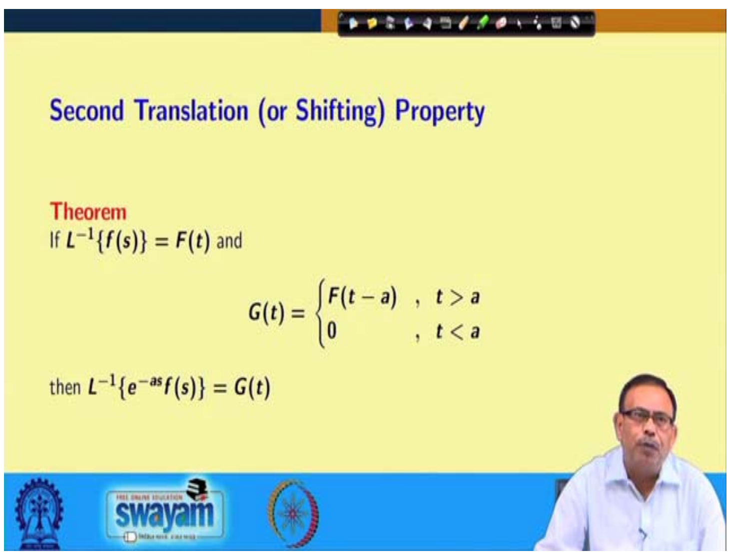

Next property is Second Translation (Shifting) property. If we have $L^{-1}\{f(s)\} = F(t)$ and we have a function $G(t)$ such that

$$G(t) = \begin{cases} F(t-a), & t > a \\ 0 \quad , & t < a \end{cases}$$

then $L^{-1}\{e^{-as}f(s)\} = G(t)$.

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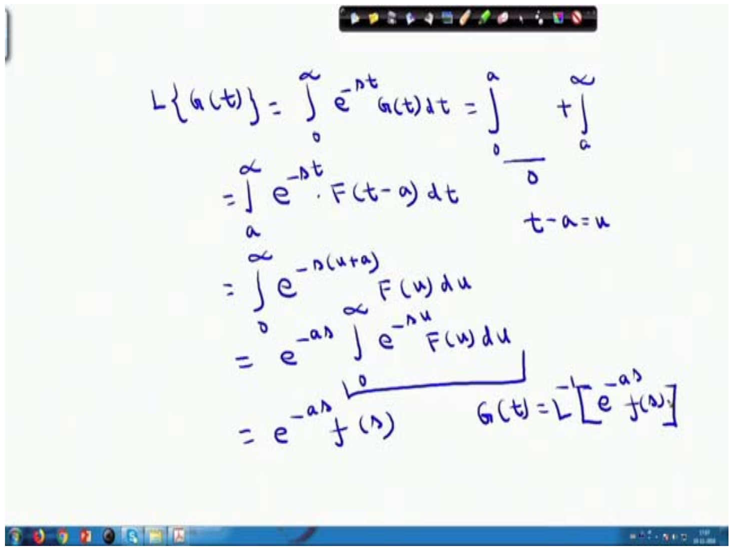

For the proof of this, we have to start with Laplace transform of the function $G(t)$, that is

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t) dt$$

This we can break it into two parts that is

$$L\{G(t)\} = \int_0^a e^{-st} G(t) dt + \int_a^\infty e^{-st} G(t) dt$$
$$= \int_a^\infty e^{-st} F(t-a) dt$$

by the definition of the function $G(t)$. If we substitute $t - a = u$ so that $dt = du$ and the lower limit at $t = a$, u will be 0, upper limit will remain ∞. Therefore,

$$L\{G(t)\} = e^{-as} \int_0^\infty e^{-su} F(u) du = e^{-as} \int_0^\infty e^{-st} F(t) dt = e^{-as} f(s)$$

So, we can write down

$$L^{-1}\{e^{-as} f(s)\} = G(t).$$

This completes the proof of this particular theorem.

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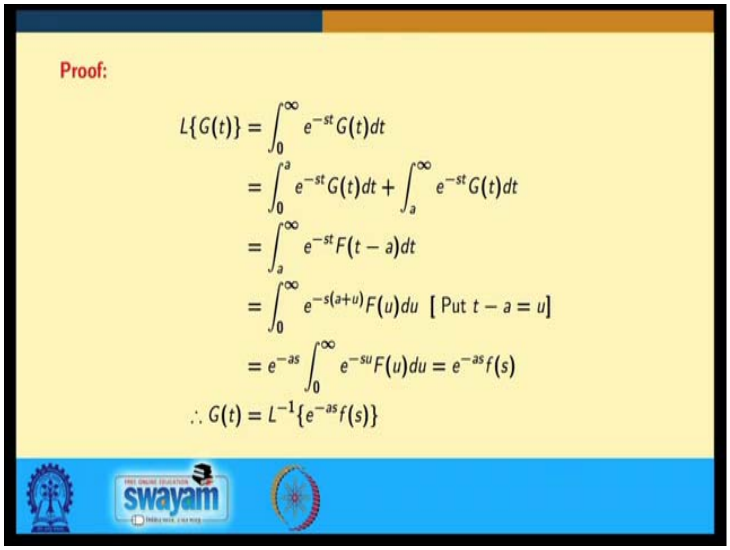

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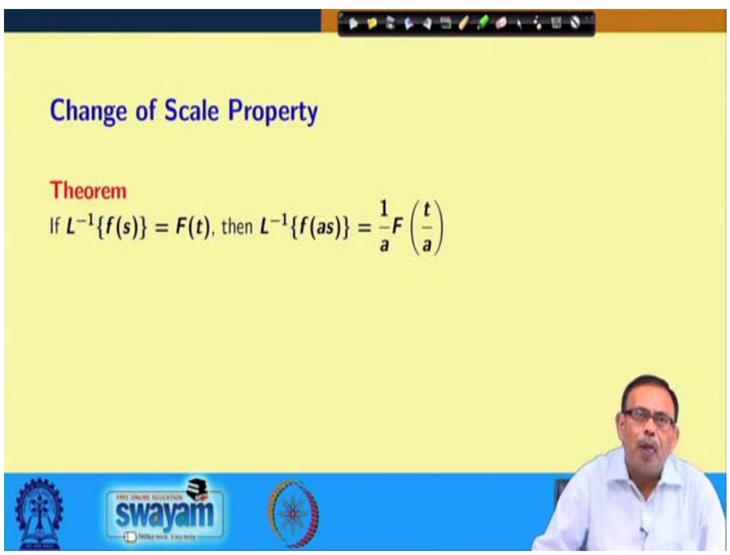

Now, next theorem is Change of Scale property. It says that if $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right)$.

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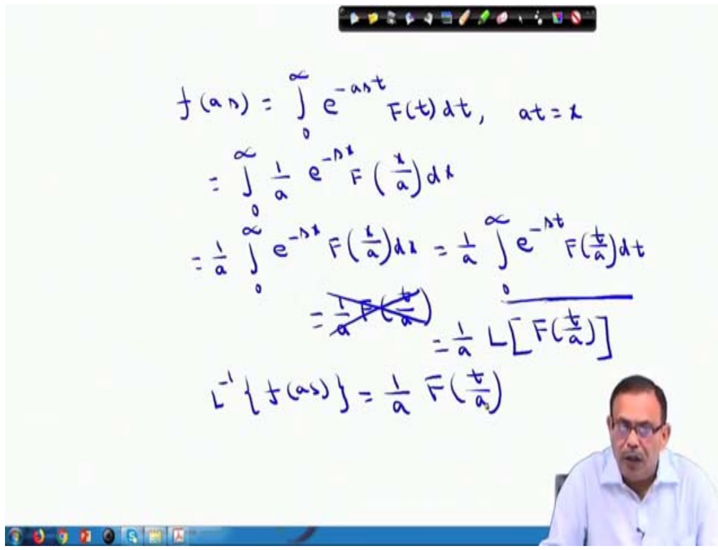

From definition, we have,

$$f(as) = \int_0^\infty e^{-ast} F(t) dt.$$

On this if you substitute $at = x$ so that $dt = \frac{dx}{a}$ and the limits of integration remain unchanged, then

$$f(as) = \frac{1}{a} \int_0^\infty e^{-sx} F\left(\frac{x}{a}\right) dx.$$

So, x can be replaced by t to obtain

$$f(as) = \frac{1}{a} \int_0^\infty e^{-st} F\left(\frac{t}{a}\right) dt = \frac{1}{a} L\left\{F\left(\frac{t}{a}\right)\right\}.$$

Taking Laplace inverse of $f(as)$, we get,

$$L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right).$$

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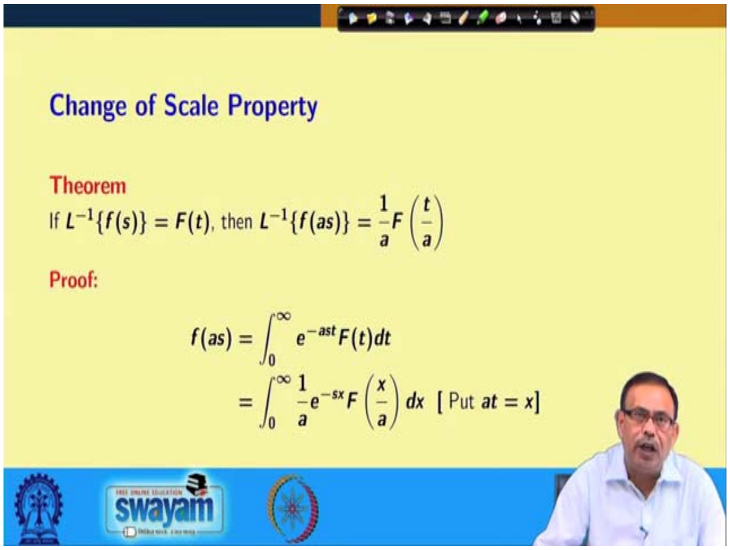

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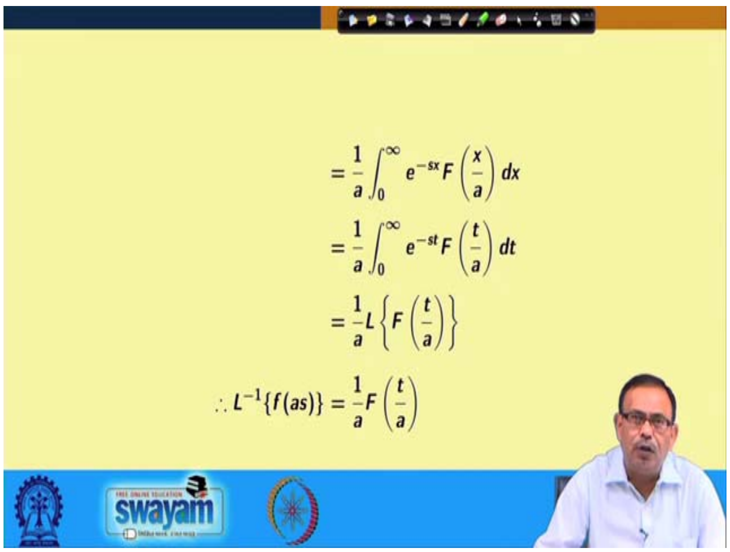

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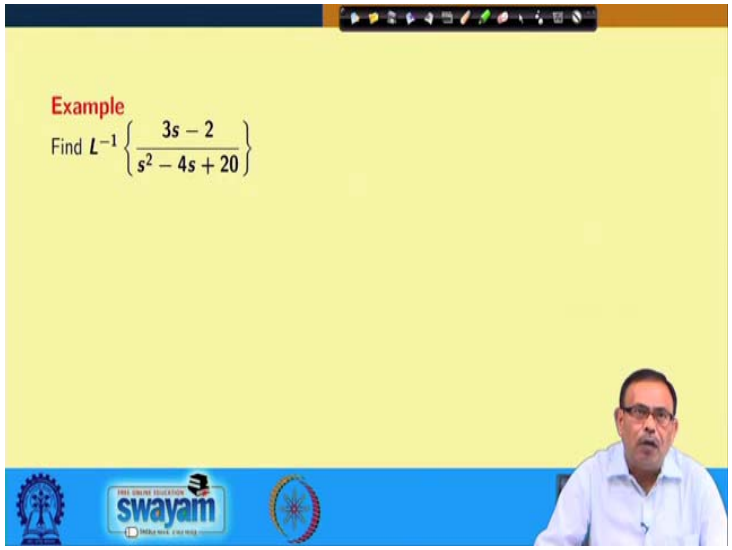

Now, let us see one example. We want to find out the Laplace inverse of

$$\frac{3s-2}{s^2-4s+20}.$$

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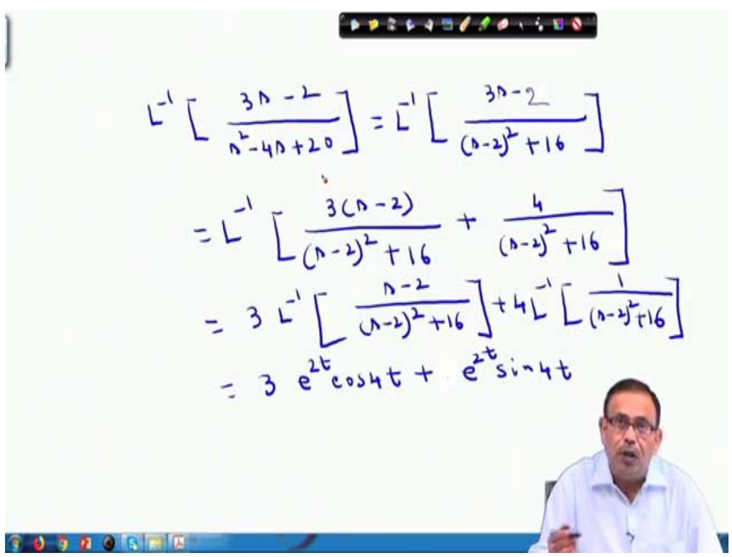

To find out the solution of the given function, we have to break it into parts, so that we can write it in some other way. Please note that we have some known forms of $\frac{1}{s+a}$, $\frac{s}{s^2 \pm a^2}$ or $\frac{a}{s^2 \pm a^2}$ whose inverse Laplace transform are known to us. So, basically whenever we try to find the solution using inverse Laplace transform, we should bring it in some form whose inverse Laplace transform is known to us, that should be the basic aim whenever we try to solve such problem.

For that reason, we are breaking the function into the following form:

$$L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\} = 3L^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + L^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

This directly we cannot solve because, here we have $s-2$ instead of s, therefore, firstly using first shifting theorem, we get,

$$L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\} = 3e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + e^{2t}L^{-1}\left\{\frac{4}{s^2+4^2}\right\}$$
$$= 3e^{2t}\cos 4t + e^{2t}\sin 4t.$$

So, effectively if we have a function, we will try to solve it in such a fashion or we will try to put it in such a form whose inverse Laplace transform is known to us.

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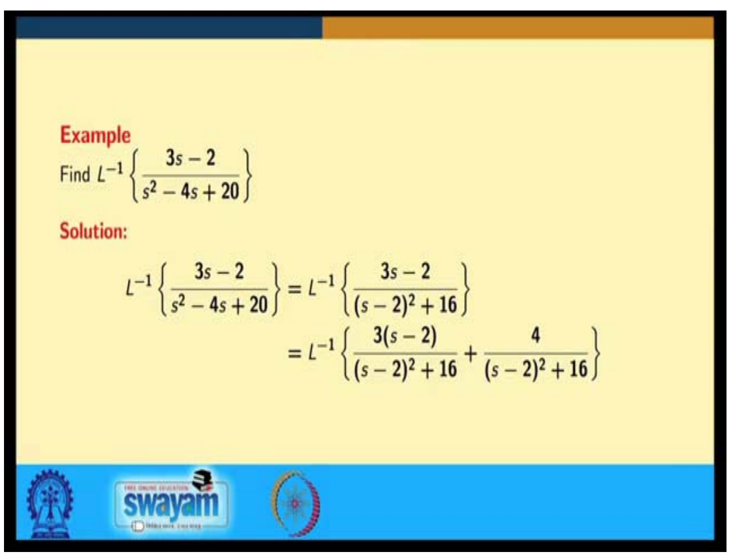

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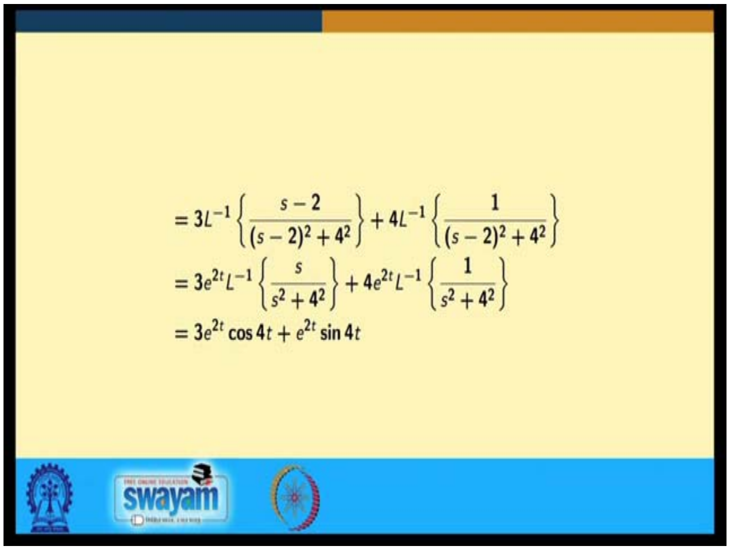

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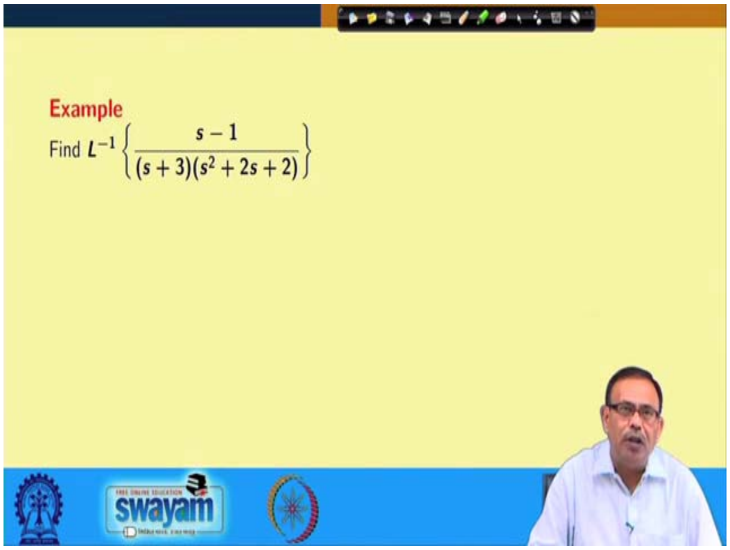

Our next example is to evaluate the Laplace inverse of

$$\frac{s-1}{(s+3)(s^2+2s+2)}.$$

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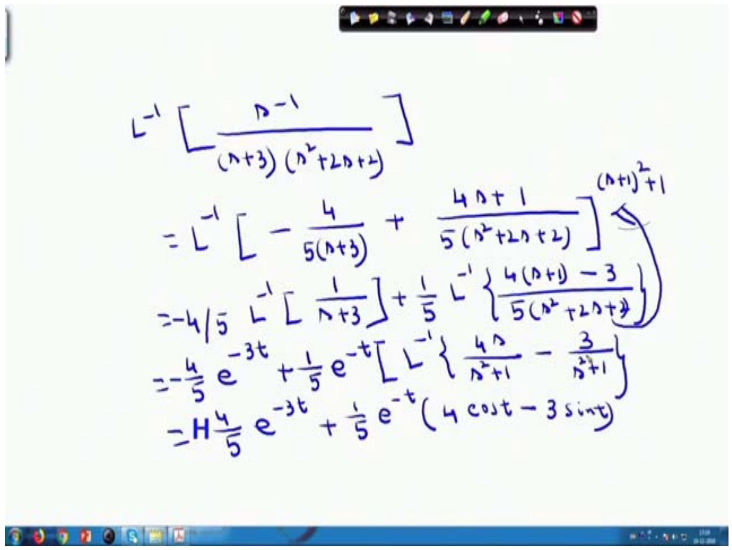

In a similar fashion as the earlier one, we have to break it into functions such that in each denominator there will be only one factor, not more than one factor. So using that trick, directly we are writing

$$L^{-1}\left\{\frac{s-1}{(s+3)(s^2+2s+2)}\right\} = -\frac{4}{5}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{5}L^{-1}\left\{\frac{4(s+1)-3}{(s+1)^2+1}\right\}$$

$$= -\frac{4}{5}e^{-3t} + \frac{1}{5}e^{-t}\left[L^{-1}\left\{\frac{4s}{s^2+1}\right\} - L^{-1}\left\{\frac{3}{s^2+1}\right\}\right]$$

$$= -\frac{4}{5}e^{-3t} + \frac{1}{5}e^{-t}(4\cos t - 3\sin t).$$

So, the trick is something like this, whenever we have a product of two or more functions in the denominator, then we have to break it in such a way that the denominator always has one factor in each term. And, in the denominator if we have something like $(s+a)^2$ or like that, then using the first shifting property, we can solve it.

In the next lecture, we will go through some more examples. Thank you.