Transform Calculus and its Applications in Differential Equations

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Lecture - 10
Bessel Function and its Laplace Transform

In the last lecture, we have defined the Dirac Delta function and discussed how to find out its Laplace transform. In this lecture, we will go through another function that is the Bessel function. Bessel function is also very well-known and is widely used in various engineering and science problems and therefore, we should know its Laplace transform as well.

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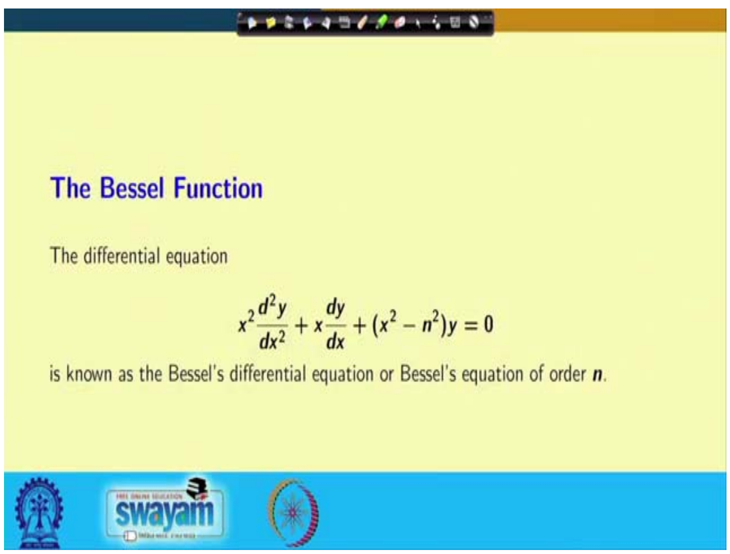

The differential equation given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

is known as the Bessel's differential equation or Bessel's equation of order n.

Now when n is not an integer or 0, the complete solution of Bessel's equation is

$$y=AJ_n(x)+BJ_{-n}(x)$$

where J_n and J_{-n} are totally independent and $J_n(x)$ is given by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

and we call it as Bessel function of first kind of order n. The entire expansion is presented in the attached lecture slide.

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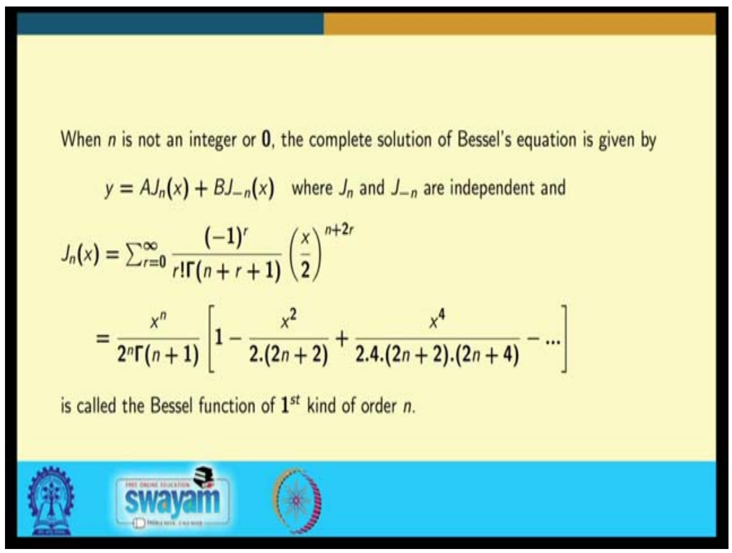

When n is an integer or 0, the complete solution of Bessel's equation is

$$y = AJ_n(x) + BY_n(x)$$

where J_n and J_{-n} are not independent and

$$J_{-n}(x) = (-1)^n J_n(x)$$

and $Y_n(x)$ is Bessel function of second kind of order n. It is also termed as the Neumann function.

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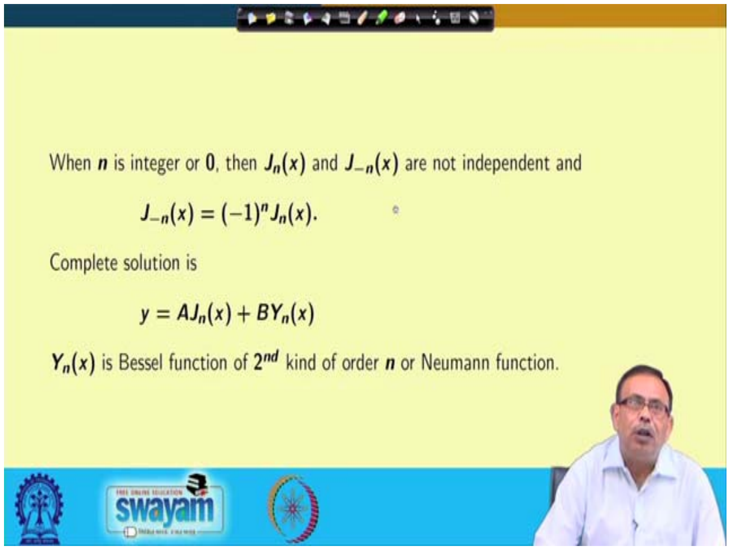

We will now discuss about the solution of Bessel's equation for $n = 0$.

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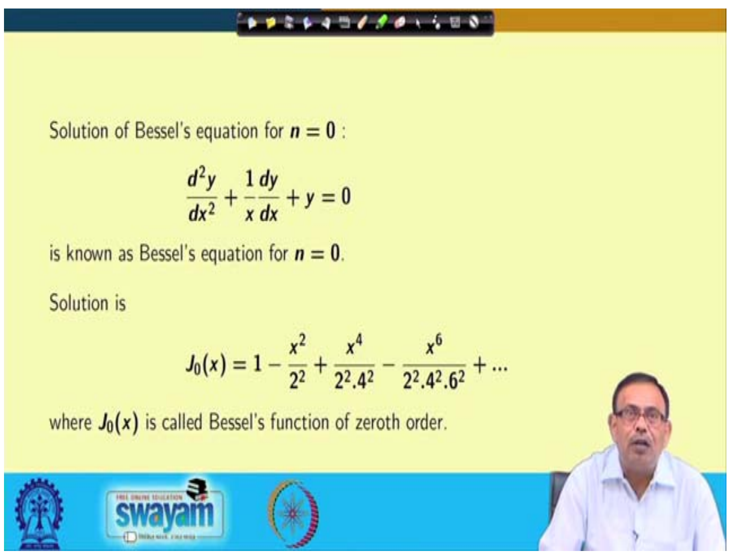

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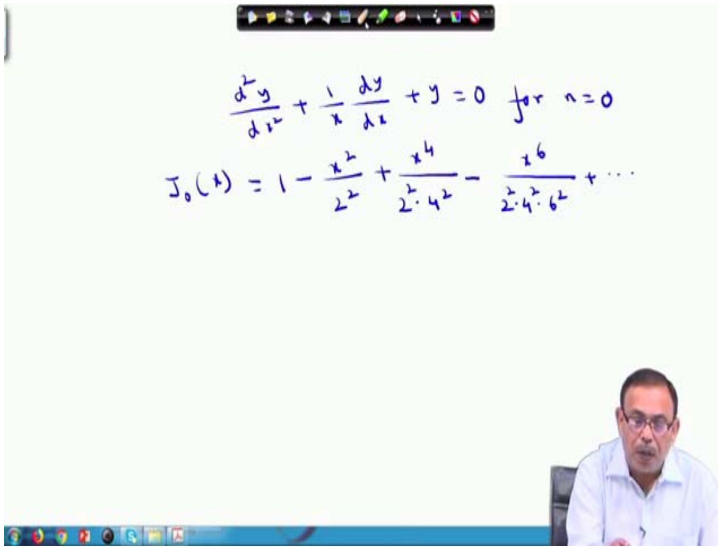

Whenever n is 0, Bessel's equation is transformed into

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0.$$

and the solution is

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\cdots$$

where $J_0(x)$ is known as the Bessel function of order 0. So, basically when n equals 0, we are getting the Bessel function of order 0 denoted by $J_0(x)$.

Now, let us solve some examples. First, we have to prove that Laplace transform of Bessel function of order 0 i.e., $J_0(t)$ is $\frac{1}{\sqrt{1+s^2}}$. From there, we will try to find out Laplace transform of $J_0(at)$, $tJ_0(at)$, $e^{-at}J_0(at)$ and we will also evaluate $\int_0^\infty J_0(t)\,dt$.

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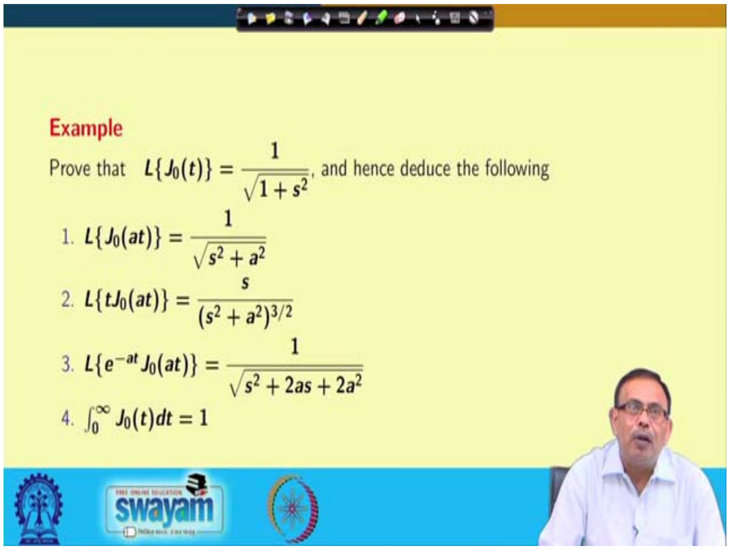

So we start with the Laplace transform of $J_0(t)$.

As we know, $J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots$ Therefore, we can easily calculate the Laplace transform of $J_0(t)$ using the linearity property and the formula $L\{t^n\} = \frac{n!}{s^{n+1}}$ as follows:

$$L\{J_0(t)\} = L\{1\} - \frac{1}{2^2}L\{t^2\} + \frac{1}{2^2.4^2}L\{t^4\} - \frac{1}{2^2.4^2.6^2}L\{t^6\} + \cdots$$

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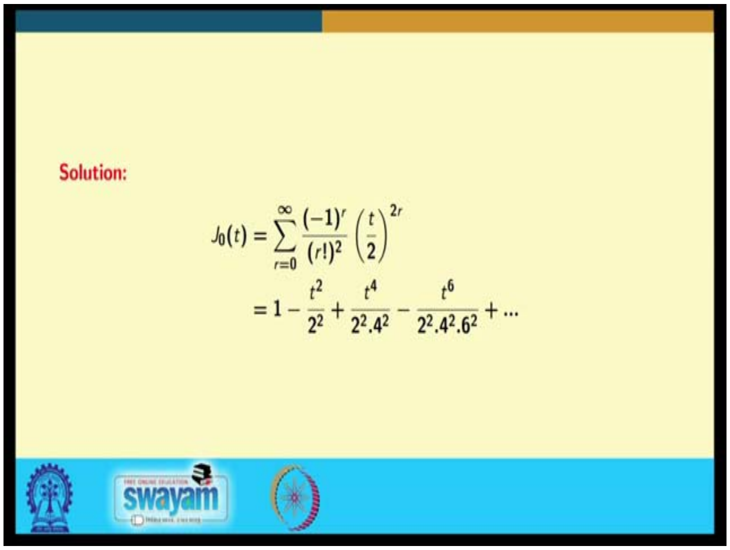

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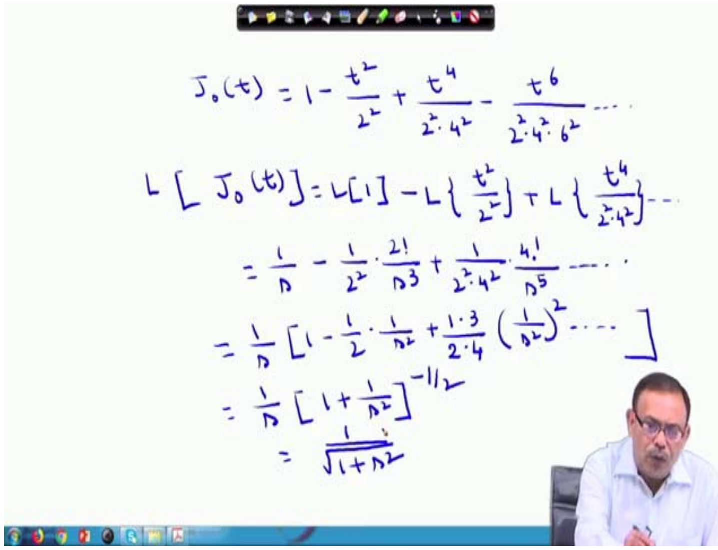

$$\therefore L\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2} . \frac{2!}{s^3} + \frac{1}{2^2 . 4^2} . \frac{4!}{s^5} - \frac{1}{2^2 . 4^2 . 6^2} . \frac{6!}{s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2} . \frac{1}{s^2} + \frac{1.3}{2.4}\left(\frac{1}{s^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2}$$

$$= \frac{1}{\sqrt{1 + s^2}}.$$

This proves the desired result.

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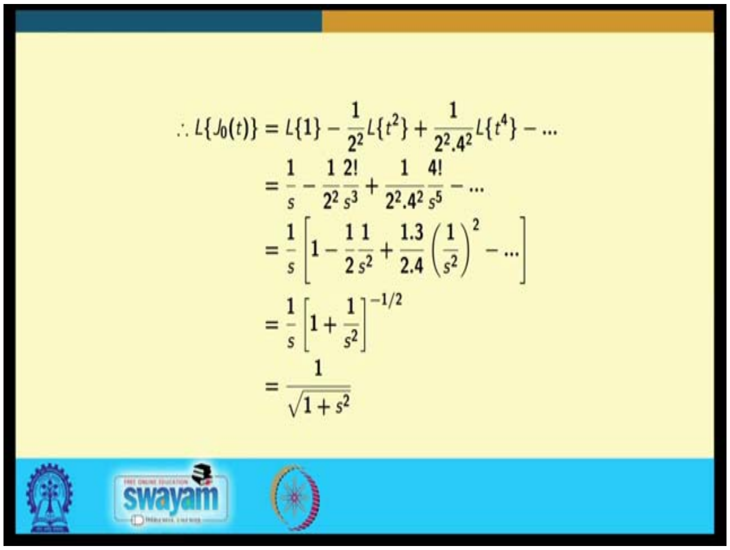

The next problem is to evaluate the Laplace transform of $J_0(at)$.

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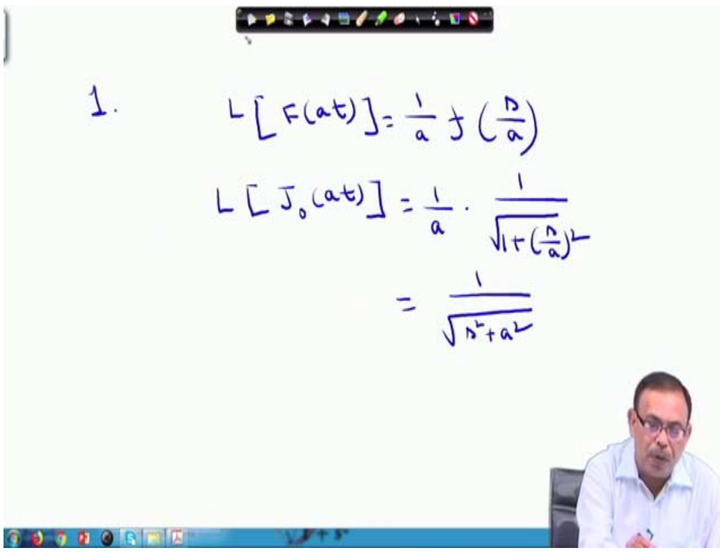

Let $F(t) = J_0(t)$ so that $F(at) = J_0(at)$. Again we know by the change of scale property,

$$L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

where $f(s) = L\{F(t)\} = L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$ as obtained in the previous problem. Therefore,

$$L\{J_0(at)\} = \frac{1}{a}\frac{1}{\sqrt{1+\left(\frac{s}{a}\right)^2}}$$
$$= \frac{1}{\sqrt{a^2+s^2}}.$$

So, once we know the Laplace transform of $J_0(t)$, very easily we can find out the Laplace transform of $J_0(at)$.

Now we move to $L\{tJ_0(at)\}$.

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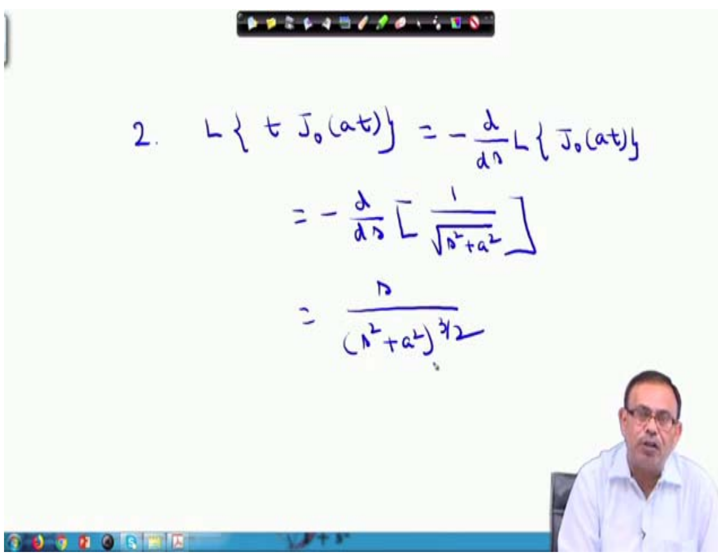

We know, by the property of multiplication by power of t, that $L\{tF(t)\} = -\frac{d}{ds}L\{F(t)\}$. Here, we assume, $F(t) = J_0(at)$ so that $L\{tF(t)\} = L\{tJ_0(at)\}$ which we need to evaluate. Thus we can write,

$$L\{tJ_0(at)\} = -\frac{d}{ds}L\{J_0(at)\}$$
$$= -\frac{d}{ds}\left(\frac{1}{\sqrt{a^2+s^2}}\right)$$

$$= \frac{s}{(s^2 + a^2)^{3/2}}.$$

Next, we want to evaluate $L\{e^{-at}J_0(at)\}$.

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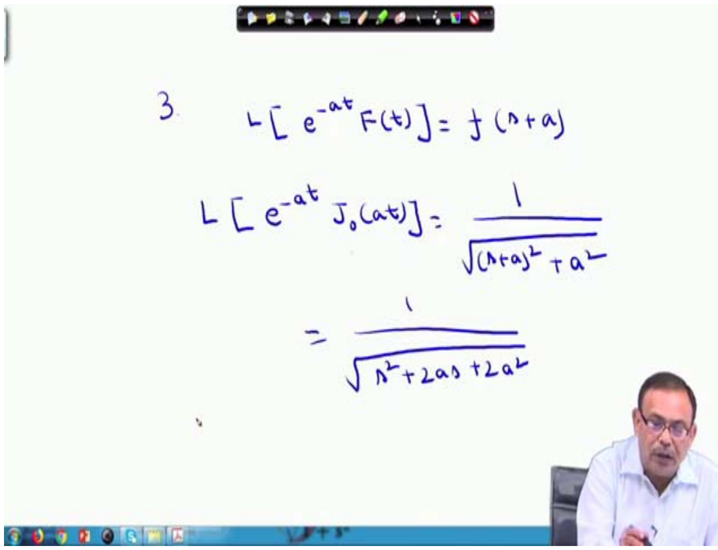

We already know by the first shifting property that $L\{e^{-at}F(t)\} = f(s+a)$ where $f(s) = L\{F(t)\}$. We assume $F(t) = J_0(at)$ and it is known to us that $L\{J_0(at)\} = \frac{1}{\sqrt{a^2+s^2}}$.

$$\therefore L\{e^{-at}J_0(at)\} = \frac{1}{\sqrt{(s+a)^2 + a^2}}$$

$$= \frac{1}{\sqrt{s^2 + 2as + 2a^2}}.$$

Lastly, we need to evaluate the integral $\int_0^{\infty} J_0(t)\,dt$ where $J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots$

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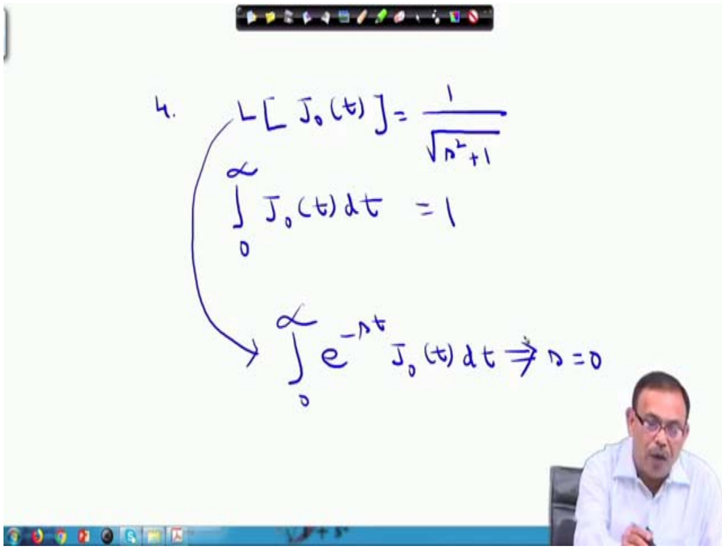

We know already the Laplace transform of $J_0(t)$ i.e.,

$$L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}.$$

Using the definition of Laplace transform, we can write

$$\int_0^\infty e^{-st} J_0(t)\,dt = \frac{1}{\sqrt{1+s^2}}. \qquad (1)$$

So, in order to evaluate $\int_0^\infty J_0(t)\,dt$, we can simply put $s = 0$ in (1). Hence we obtain

$$\int_0^\infty J_0(t)\,dt = 1.$$

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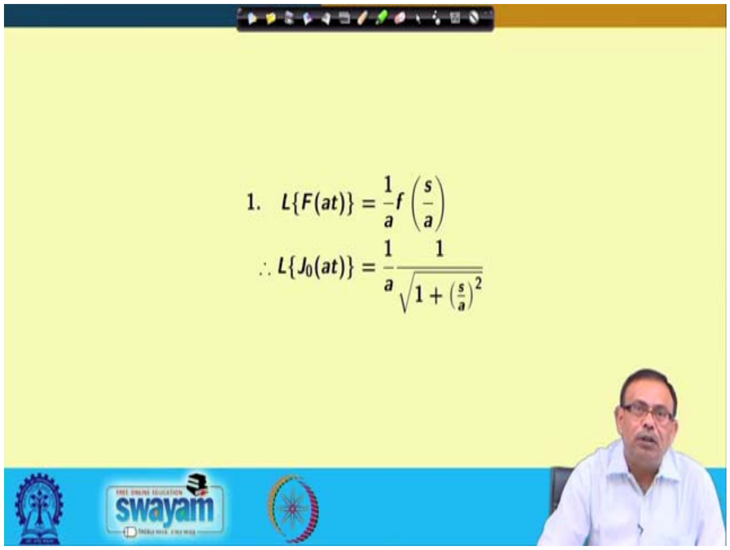

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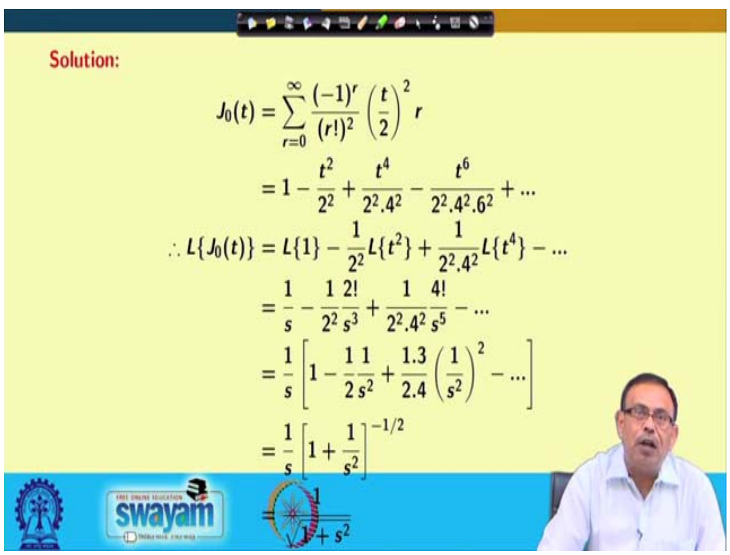

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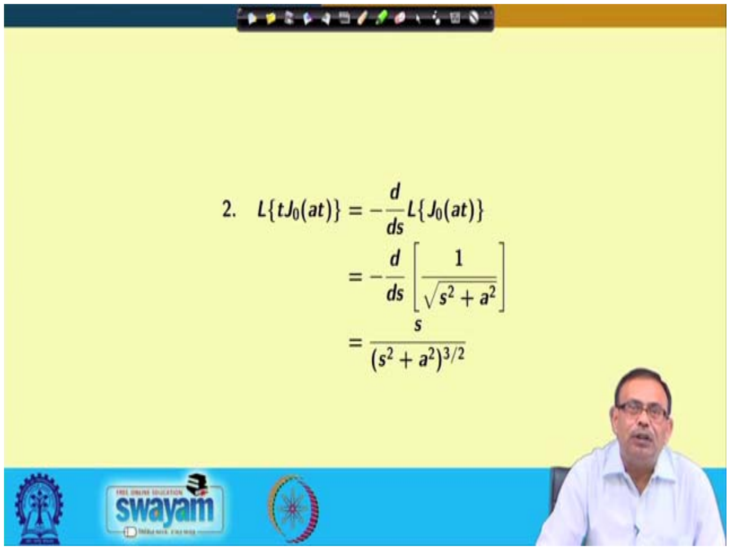

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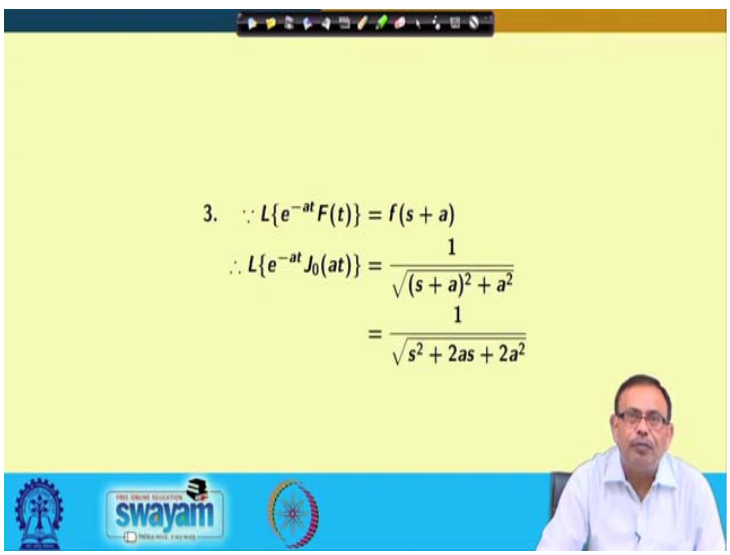

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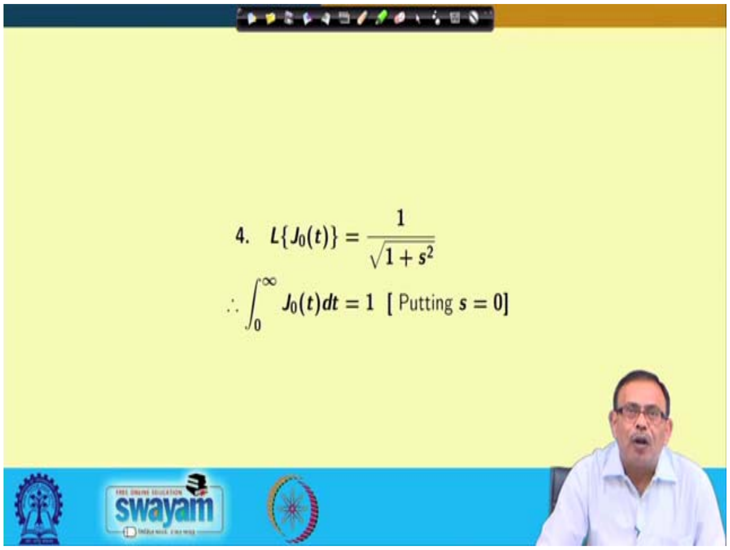

So, we have discussed the Laplace transform of $J_0(t)$. Now we will consider the function $J_1(t)$ where $J_1(t)$ denotes the Bessel function of order 1.

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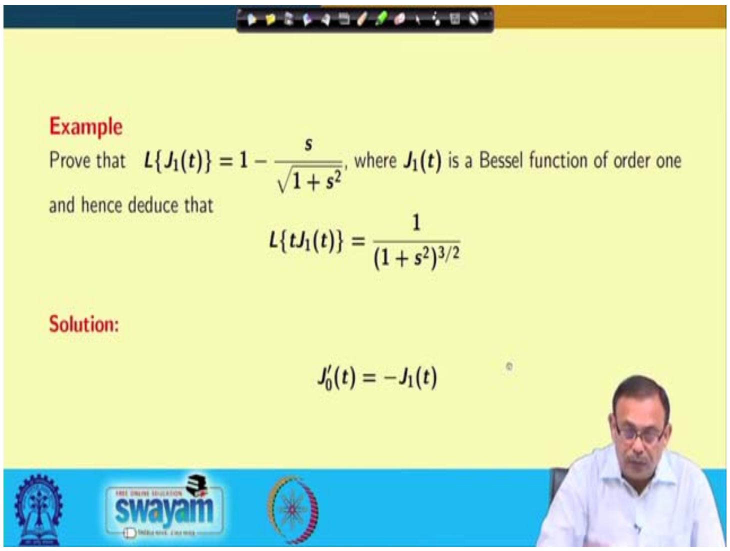

We have to prove that Laplace transform of $J_1(t)$ is $\left[1 - \frac{s}{\sqrt{1+s^2}}\right]$ and once we know that, from there we can find out the Laplace transform of $tJ_1(t)$.

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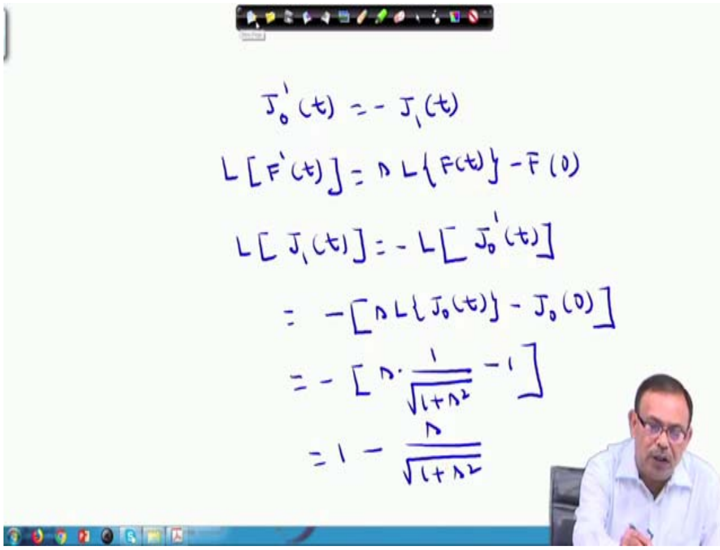

It is known to us that

$$J_0'(t) = -J_1(t).$$

Again by the Laplace transform of derivative of a function, we have,

$$L\{F'(t)\} = sL\{F(t)\} - F(0).$$

Using the above two relations, we get,

$$\begin{aligned} L\{J_0'(t)\} = L\{-J_1(t)\} &= sL\{J_0(t)\} - J_0(0) \\ \Rightarrow L\{J_1(t)\} &= -s\frac{1}{\sqrt{1+s^2}} + 1 \\ &= 1 - \frac{s}{\sqrt{1+s^2}}. \end{aligned}$$

This proves our result. Now, by using the theorem on multiplication by power of t, we have,

$$\begin{aligned} L\{tJ_1(t)\} &= -\frac{d}{ds}(L\{J_1(t)\}) \\ &= -\frac{d}{ds}\left(1 - \frac{s}{\sqrt{1+s^2}}\right) \\ &= \frac{1}{(1+s^2)^{3/2}}. \end{aligned}$$

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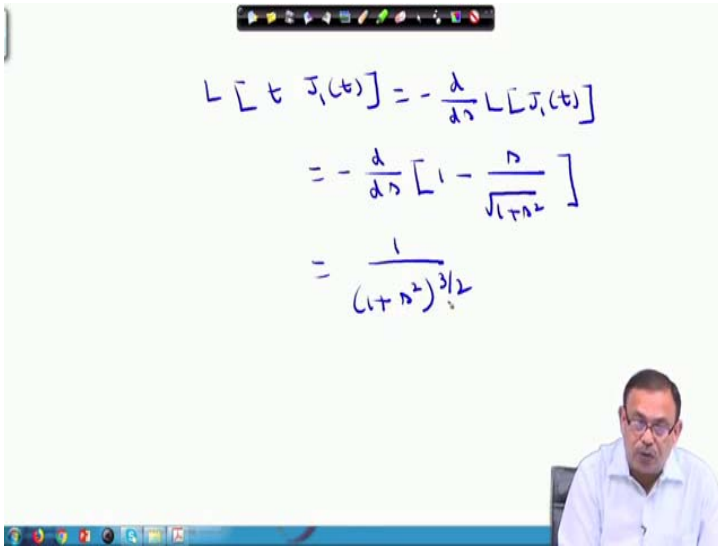

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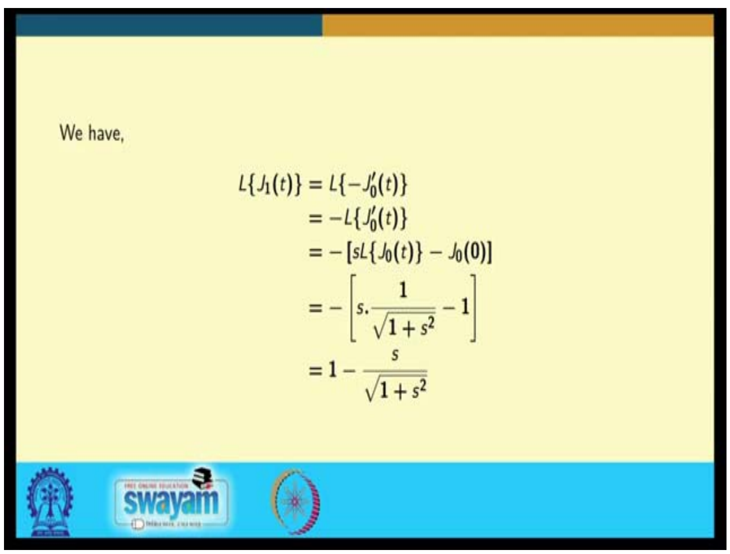

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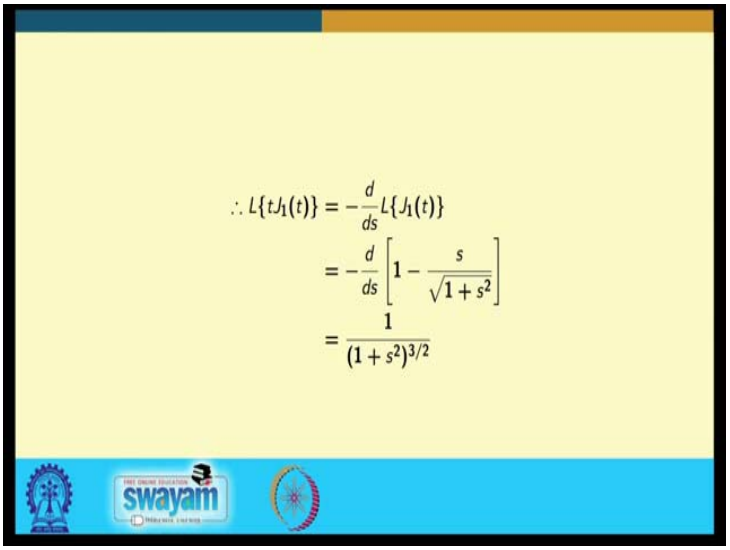

We now come to a new function i.e., the Null Function.

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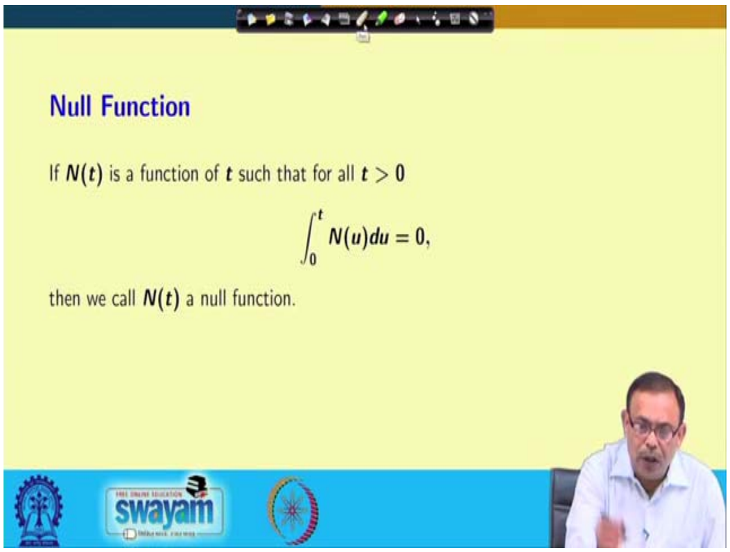

The Null function $N(t)$ is a function of t such that for all $t > 0$, we have

$$\int_0^t N(u)\,du = 0.$$

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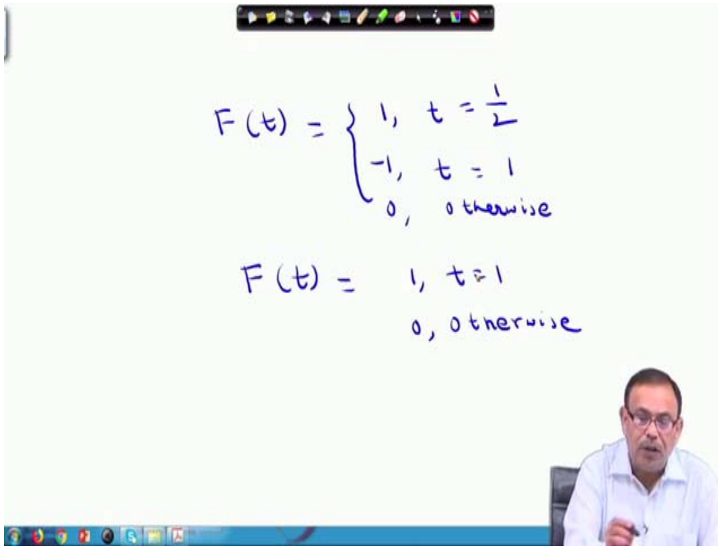

For example, consider a function given by

$$F(t) = \begin{cases} 1, & t = \dfrac{1}{2} \\ -1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$$

or a function defined by

$$F(t) = \begin{cases} 1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$$

Each of these two functions is a null function because if we take their integral within the limits 0 to t, the value always will be 0.

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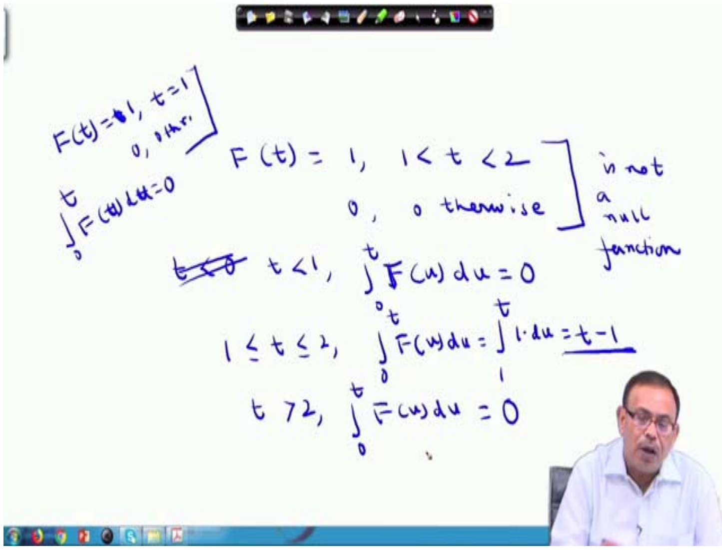

Whereas, if we consider a function $F(t)$ given by,

$$F(t) = \begin{cases} 1, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

So, if we integrate this between the limits 0 to t, we get

$$\int_0^t F(u)\,du = \int_1^t 1.\,du = t - 1 \neq 0 \text{ always.}$$

Therefore, this function is not a null function.

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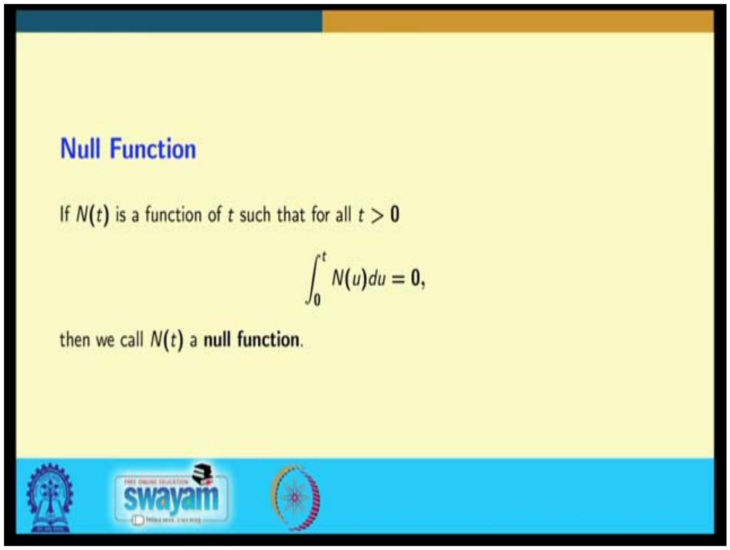

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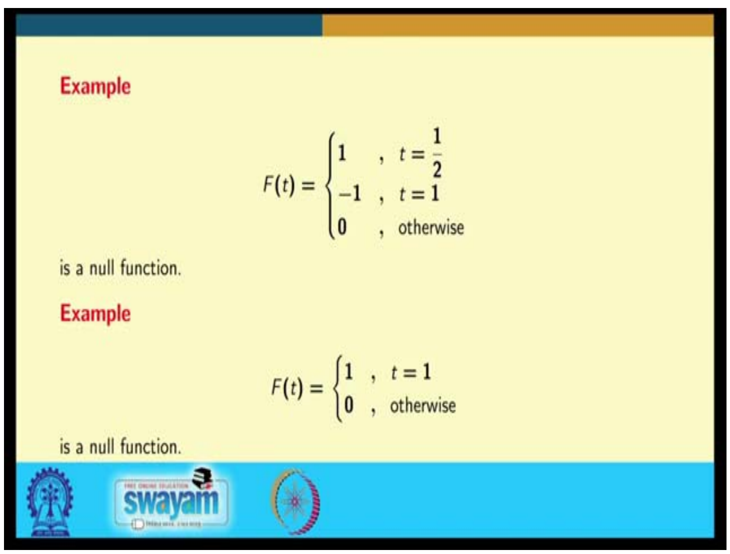

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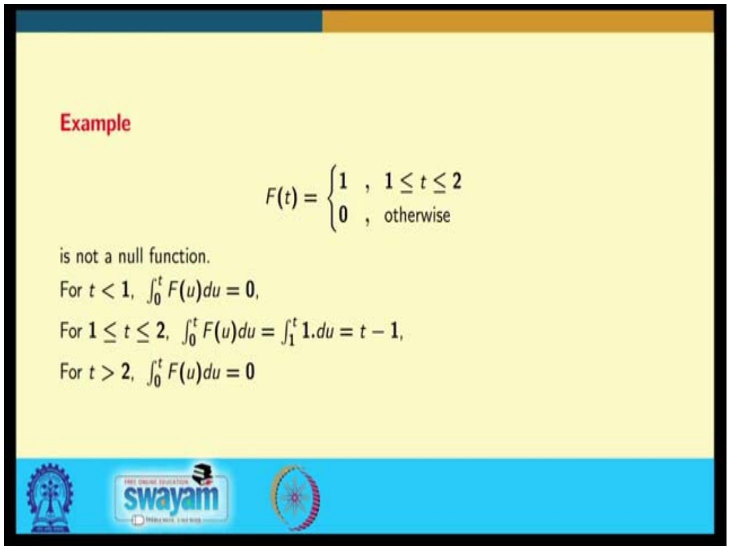

There is a remark in this regard.

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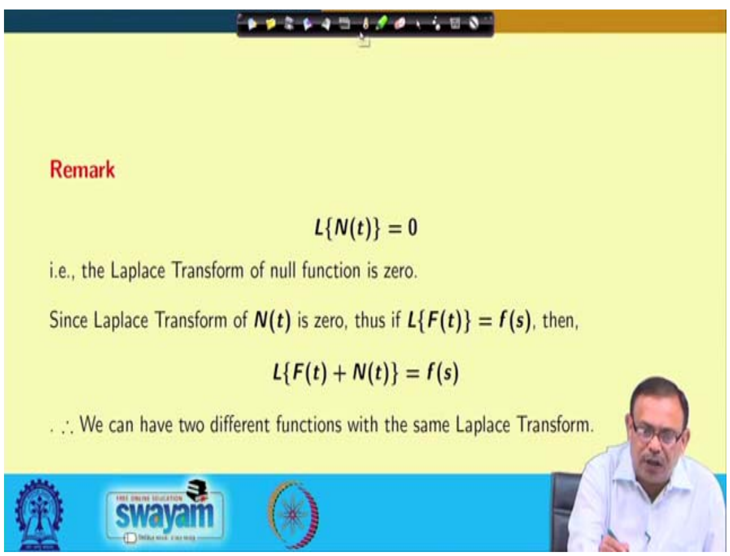

Laplace transform of a null function is always 0.

If the Laplace transform of $F(t)$ equals $f(s)$ (say), then in that case, we can write down Laplace transform of $[F(t) + N(t)]$ is equal to $f(s)$ (since the Laplace transform of null function is 0).

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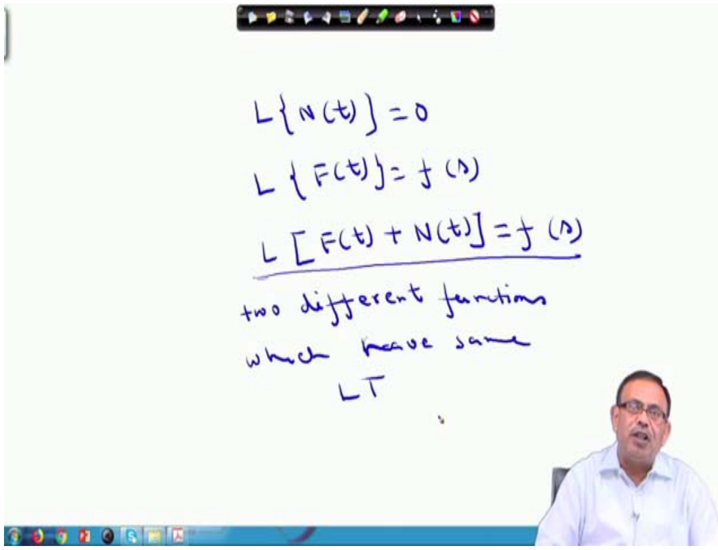

So, from the basic property, we can always write down $L\{F(t) + \text{a null function}\}$ is equal to $L\{F(t)\}$. This means we may have two different functions $[F(t)$ and $F(t) + N(t)]$ which have the same Laplace transform.

Consider 2 functions $F_1(t) = e^{-3t}$ and $F_2(t) = \begin{cases} 0, & t = 1 \\ e^{-3t}, & \text{otherwise} \end{cases}$

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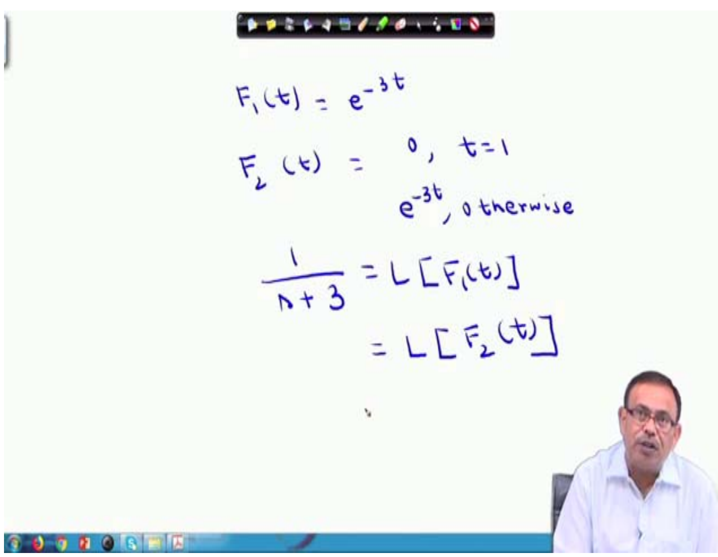

If we take the Laplace transform of these two functions, we will see

$$L\{F_1(t)\} = L\{F_2(t)\} = \frac{1}{s+3}.$$

So, we have two different functions but the Laplace transform of both are same in this case. In the next lecture, we will see what is the implication of this.

Thank you.