Transform Calculus and Its Applications in Differential Equations

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Lecture - 01
Introduction to Integral Transform and Laplace Transform

Welcome all of you to the course of Transform Calculus and Its Applications in Differential Equations. Transform calculus is basically a technique by which we transform the problem from its original domain to some other domain.

The advantage is that in the original domain, if we try to solve the original equation, it may become difficult for us to find the optimum solution. Therefore, using these transform techniques, we convert this original problem into some other problem in some other domain, where it becomes easier for us to find the solution of a particular equation. Now, among these particular topics, at first we will cover integral transform.

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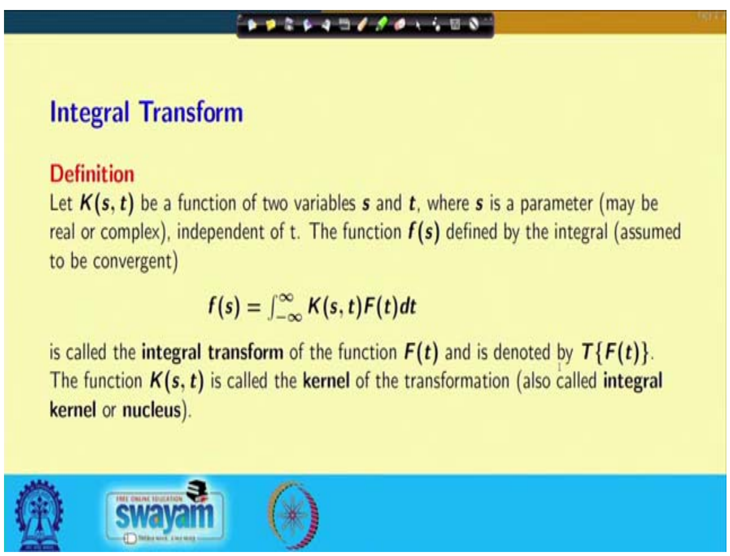

First let us see the definition of integral transform. Let, $K(s,t)$ be a function of two variables s and t where s is a parameter such that s may be real or complex and s is independent of t. The function $f(s)$ can be defined by the integral as

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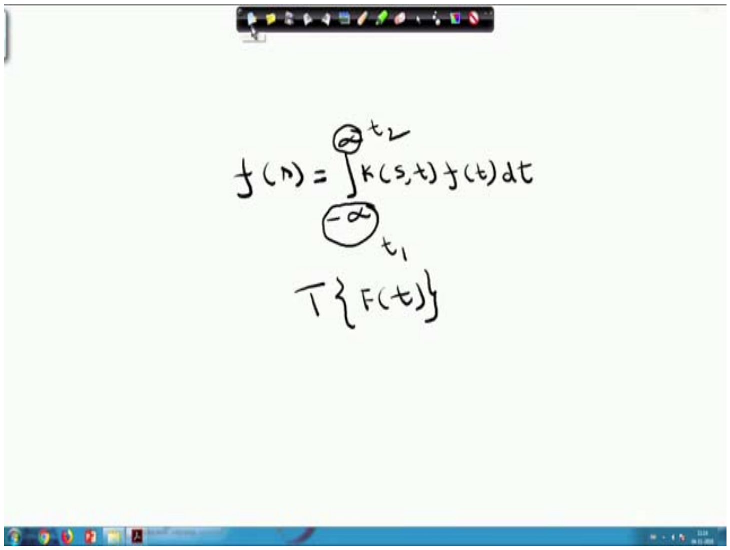

$$f(s) = \int_{-\infty}^{\infty} K(s,t)F(t)\,dt.$$

The limits of the integration $\int_{-\infty}^{\infty} K(s,t)F(t)\,dt$ sometimes, may be written as, t_1 to t_2 also when it is taken on some finite domain $[t_{1,}t_2]$.

The function $f(s)$ defined as $f(s) = \int_{-\infty}^{\infty} K(s,t)F(t)\,dt$ is known as the integral transform of the function $F(t)$ and is denoted by $T\{F(t)\}$. The function $K(s,t)$ is called basically the kernel of the transformation and also sometimes, we call it as integral kernel or nucleus.

So, effectively, if we change the value of $K(s,t)$, then we can find out various types of transforms like Laplace transform, Fourier transform etc.

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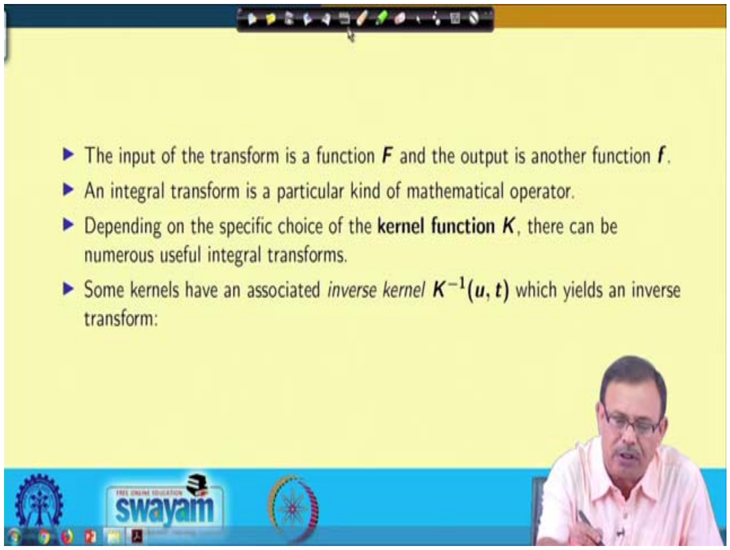

The input of the transform is a function $F(t)$ and the output is another function i.e., $f(s)$.

An integral transform is a particular type of mathematical operator. Depending upon the specific choice of the kernel function K, there can be numerous useful integral transforms like Laplace transform, Fourier transform, Hankel transform etc.

Some kernels have an associated inverse which we call as $K^{-1}(u, t)$.

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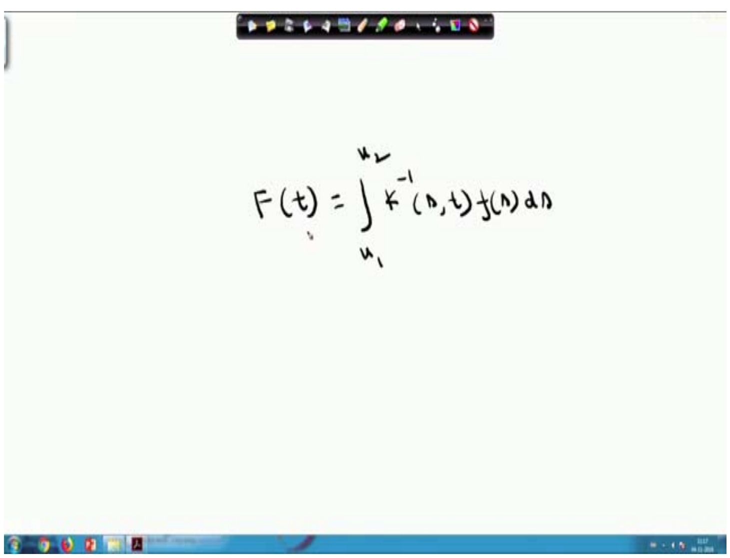

And this $K^{-1}(u, t)$ is defined as

$$F(t) = \int_{u_1}^{u_2} K^{-1}(s,t)f(s)ds$$

whose integrand will be a function of s, t and we are integrating it over s so that it will be a function of t.

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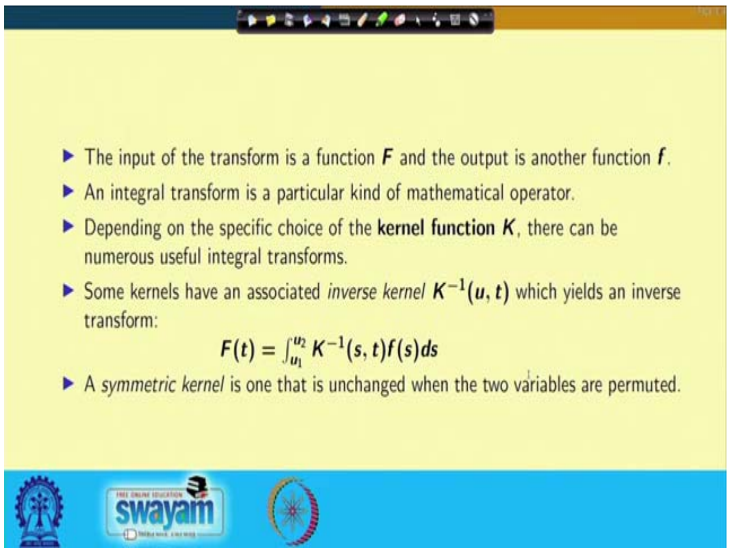

A symmetric kernel is one that remains unchanged if we interchange or permute the variables s and t.

Next we come to the Laplace transform. Laplace transform is a mathematical tool which is used to solve various kinds of engineering and mathematical problems. It is used mostly by the science departments, engineering departments to solve a particular type of problem. Laplace transform was first introduced by French mathematician Laplace.

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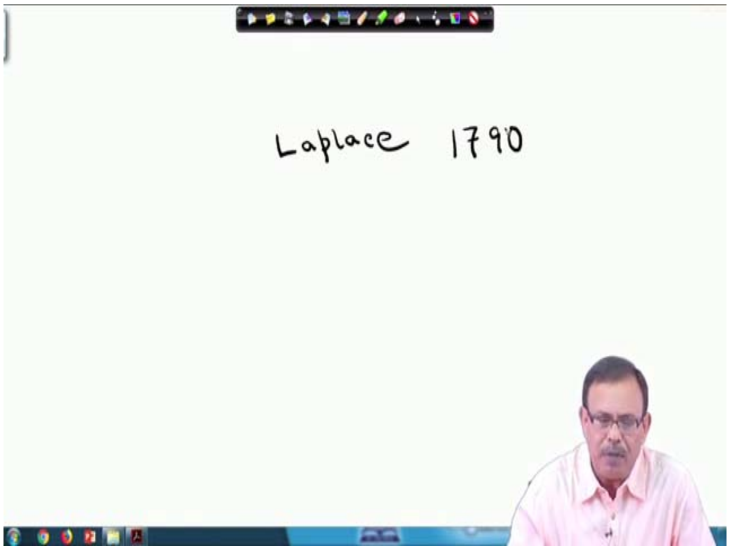

Laplace transform was developed by French mathematician Laplace in 1790. It is used to solve ordinary differential equations (ODE). In general, whenever we try to find the solution of an ODE, we try to find out the general solution and from the general solution, using the initial conditions (IC), we try to find out the values of arbitrary constants. Whereas, if we use the Laplace transform, it simply converts the ODE into an algebraic equation.

So, in order to find the solution, we have to solve only one algebraic equation which is basically very easy to find. Similarly, if we try to find out the solution of a partial differential equation (PDE) using Laplace transform, it reduces the number of independent variables by one so that the solution process becomes easier.

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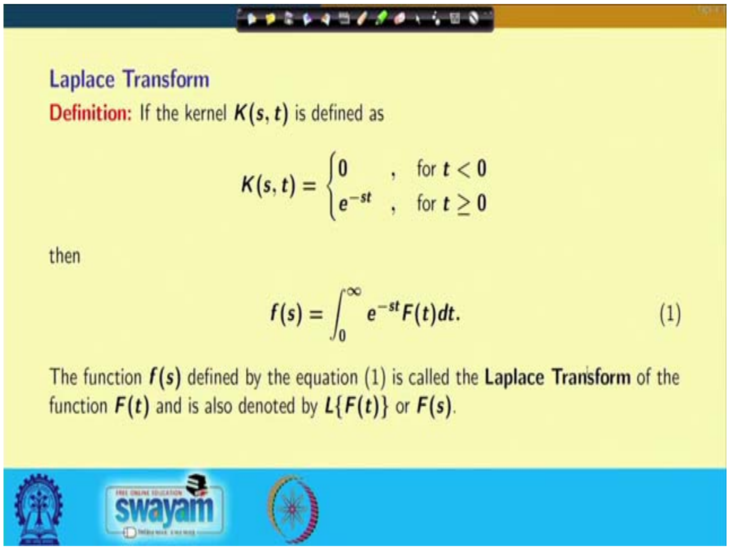

Now we define the Laplace transform. If the kernel $K(s,t)$ is defined as

$$K(s,t) = \begin{cases} 0, & t < 0 \\ e^{-st}, & t \geq 0 \end{cases}$$

then the Laplace transform of $F(t)$is defined as

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt. \tag{1}$$

So, we have a function $F(t)$ and the Laplace transform of the function $F(t)$ is denoted by $f(s)$ and is defined as $\int_0^\infty e^{-st} F(t)\, dt$.

Whenever we try to use the Laplace transform, the value of the independent variable t will always be lying between 0 to ∞. So, the function $f(s)$ defined by the equation (1) is called the Laplace transform of the function $F(t)$ and we denote it by $L\{F(t)\}$ or $f(s)$ or $F(s)$.

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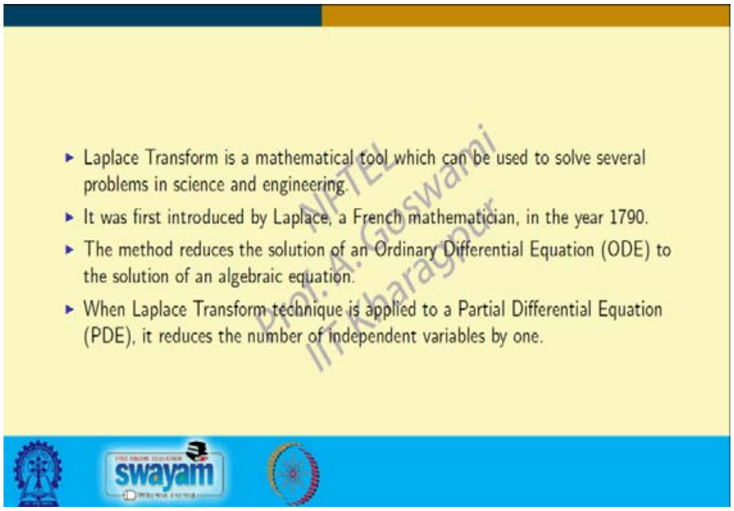

Laplace transform is a mathematical tool which can be used to solve several problems in science and engineering. It was first introduced by Laplace, a French mathematician in the year 1790. The method reduces the solution of an ODE to the solution of an algebraic equation.

So, basically if we try to solve an ODE and apply Laplace transform, then the ODE will be transformed into a problem in some other domain creating one algebraic equation. And to find the solution of the original problem, instead of solving the ODE, we have to simply solve the algebraic equation. When Laplace transform technique is applied to a PDE, it reduces the number of independent variables by one.

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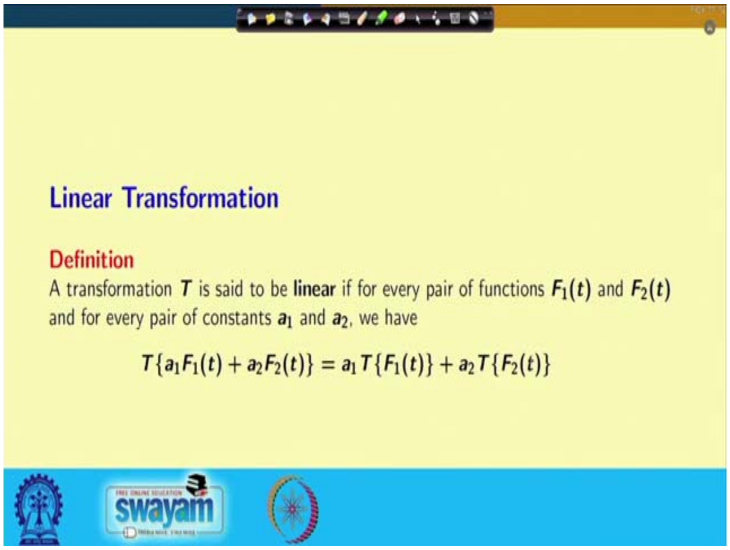

Next we come to linear transformation. Any transformation T is said to be linear, if for every pair of function $F_1(t)$ and $F_2(t)$ and for every pair of constants a_1 and a_2, we have,

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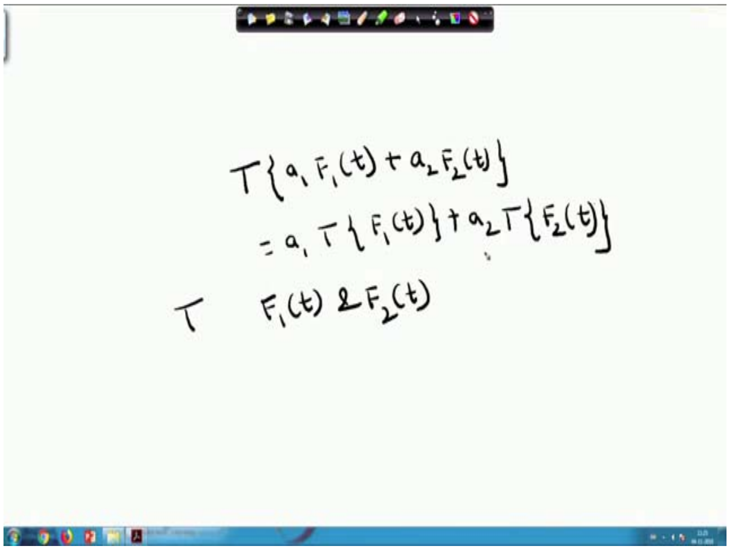

$$T\{a_1F_1(t) + a_2F_2(t)\} = a_1T\{F_1(t)\} + a_2T\{F_2(t)\}$$

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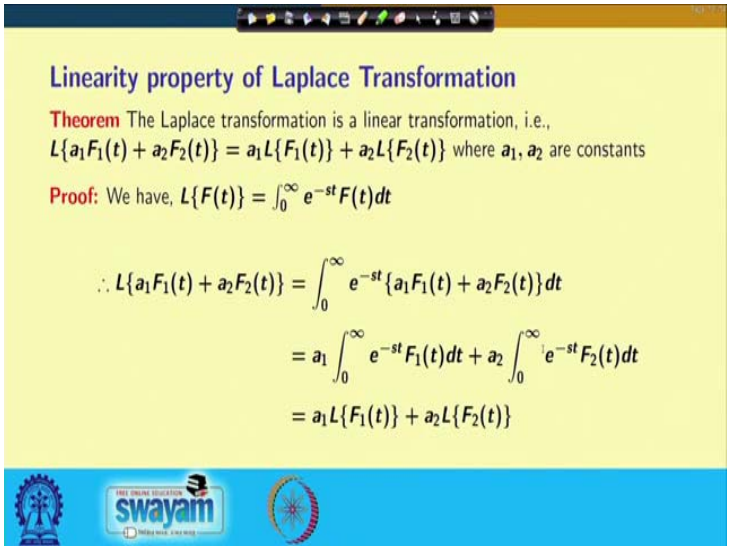

Now, we come to the linearity property of Laplace transform. Laplace transform is a linear transformation, i.e.,

$$L\{a_1F_1(t) + a_2F_2(t)\} = a_1L\{F_1(t)\} + a_2L\{F_2(t)\}$$

where a_1 and a_2 are constants. Now, let us look at the proof.

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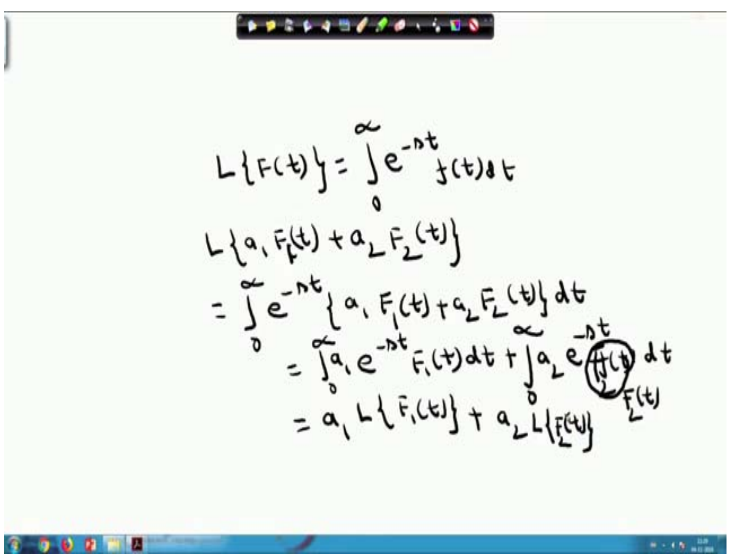

We know that

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

using the definition of Laplace transform.

Substituting $F(t)$ by $a_1 F_1(t) + a_2 F_2(t)$, we have,

$$L\{a_1 F_1(t) + a_2 F_2(t)\} = \int_0^\infty e^{-st} [a_1 F_1(t) + a_2 F_2(t)] dt.$$

We can break this into two integrals as

$$L\{a_1 F_1(t) + a_2 F_2(t)\} = \int_0^\infty a_1 e^{-st} F_1(t)\, dt + \int_0^\infty a_2 e^{-st} F_2(t)\, dt$$

$$= a_1 \int_0^\infty e^{-st} F_1(t)\, dt + a_2 \int_0^\infty e^{-st} F_2(t)\, dt$$

Again by the definition of Laplace Transform, we can write this as:

$$L\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$$

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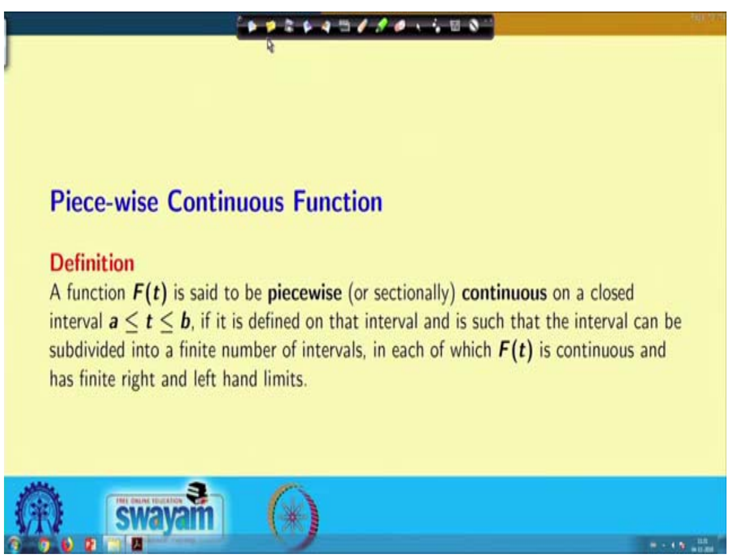

Now, we come to piecewise continuous function. A function $F(t)$ is said to be piecewise continuous or sectionally continuous on a closed interval $[a, b]$ if it is defined on that interval and is such that if the interval is divided into a finite number of subintervals, then in each of these subintervals, $F(t)$ is continuous and has finite right and left hand limits. The meaning is that a function $F(t)$ is defined in an interval $[a, b]$ and if the entire domain is subdivided into n sub-domains, then in each of the sub-domains, the function $F(t)$ will be continuous. Then we call the function $F(t)$ as a piecewise continuous function.

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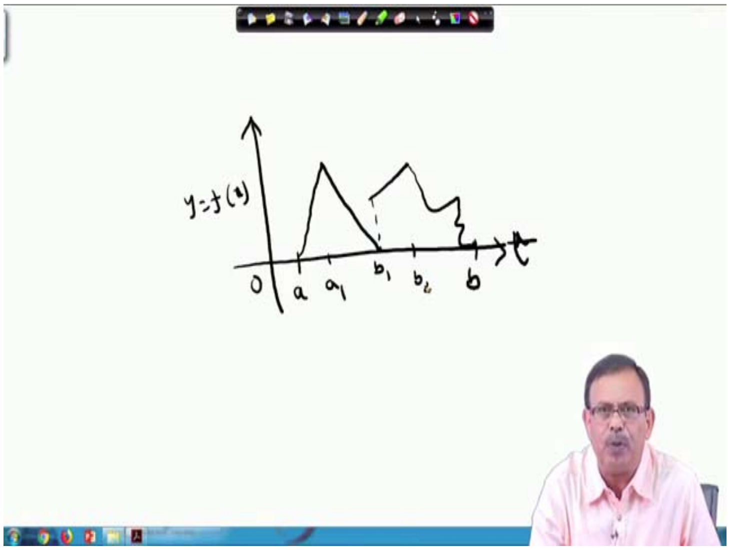

We draw a function $y = f(x)$ in $[a, b]$ as shown in the slide. So, if we consider the entire domain $[a, b]$, then it is observed that the function $f(x)$ is not continuous, but if we subdivide $[a, b]$ into some smaller subdomains, say, $[a, a_1]$, $[a_1, b_1]$, $[b_1, b_2]$, $[b_2, b]$, then in each of these sub-domains, the function is found to be continuous. Then we can tell that the function $y = f(x)$ is piecewise continuous.

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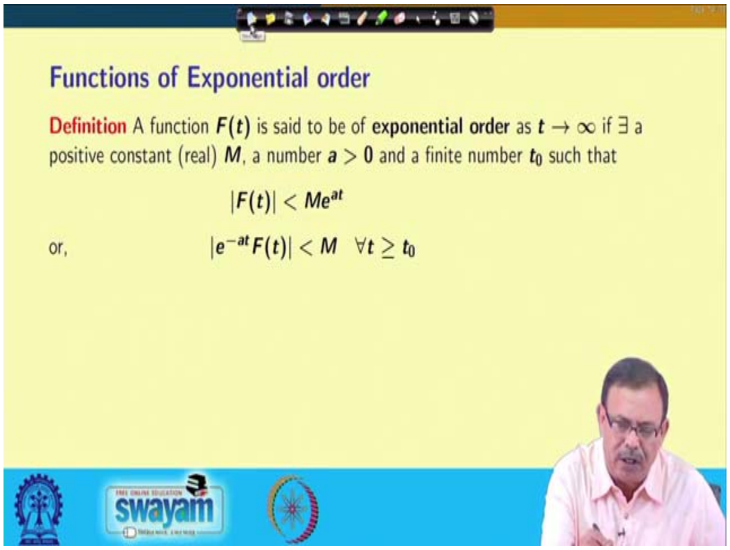

So, now we come to the function of exponential order. A function $F(t)$ is said to be of exponential order as t approaches ∞, if there exists a positive real number M and a number $a > 0$ and a finite number t_0 such that absolute value of $F(t)$ i.e., $|F(t)|$ is less than Me^{at} i.e.,

$$|F(t)| < Me^{at}$$

$$\Rightarrow |e^{-at}F(t)| < M$$

which holds for all $t \geq t_0$. Then we say that the function is of exponential order.

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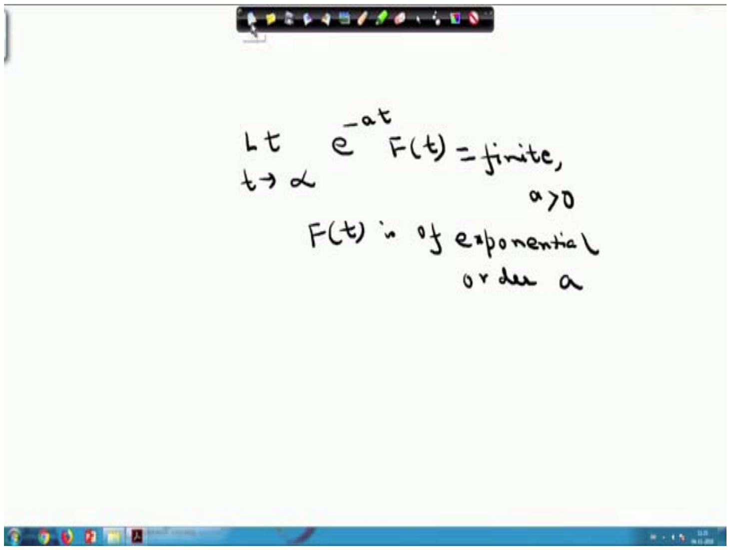

Therefore, if $\lim_{t\to\infty} e^{-at}F(t)$ is a finite constant, say, $a > 0$, then we can say that $F(t)$ is of exponential order a. And we sometimes write it as $F(t) = o(e^{at})$ as $t \to \infty$.

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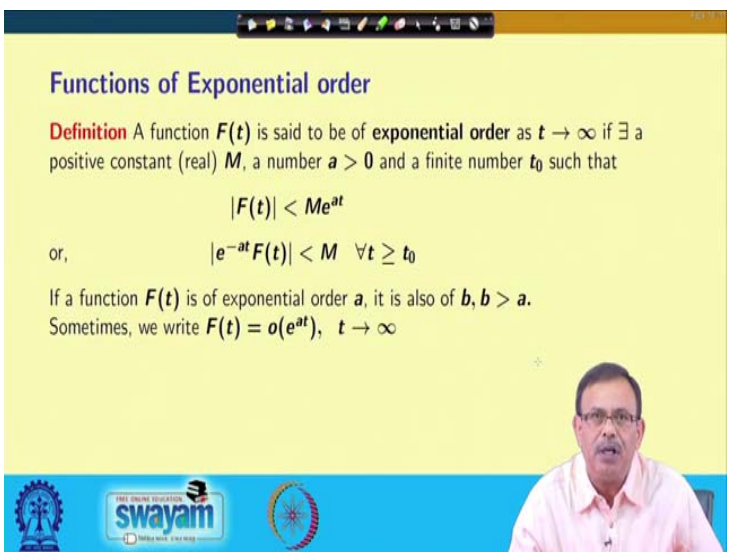

Now, if a function $F(t)$ is of exponential order a, then for any $b > a$, $F(t)$ is also of exponential order b.

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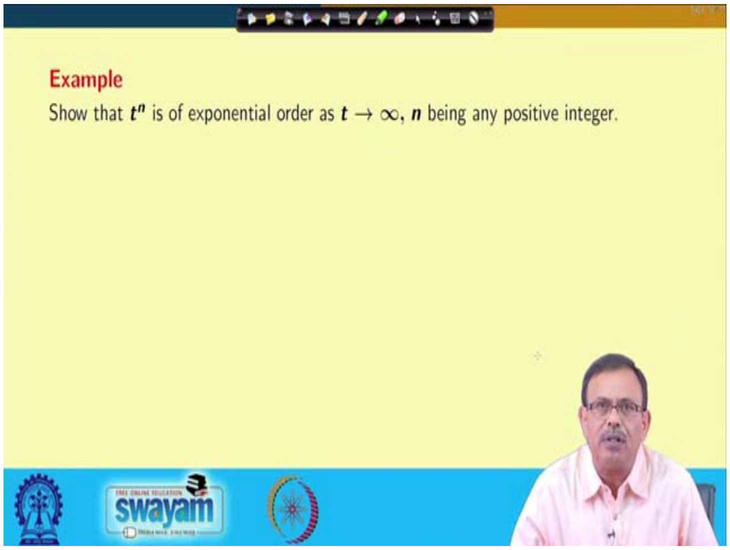

Let us now solve an example. Say, we have to show that t^n is of exponential order as $t \to \infty$, n being any positive integer.

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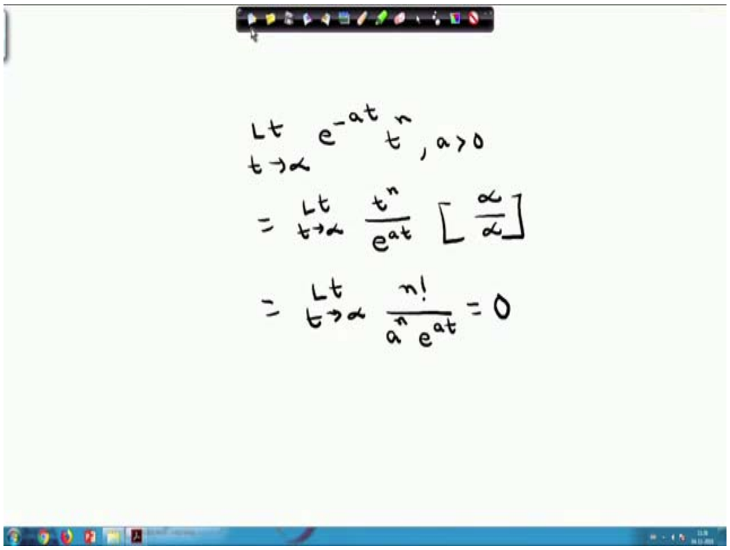

So, we have to evaluate

$$\lim_{t\to\infty} e^{-at} t^n$$

because we want to find out whether t^n is of exponential order a or not where $a > 0$. So, we write it as

$$\lim_{t\to\infty}\frac{t^n}{e^{at}}$$

which is of the form ∞/∞ so that, in order to evaluate this limit, we can use L' hospital's rule n times. As a result, we can obtain this as

$$\lim_{t\to\infty}\frac{n!}{a^n e^{at}}$$

whose value becomes 0. Therefore, the limiting value of $e^{-at}t^n$ as $t\to\infty$ always will be 0. Since the limit has a finite value 0, so we can say that t^n is of exponential order as $t\to\infty$.

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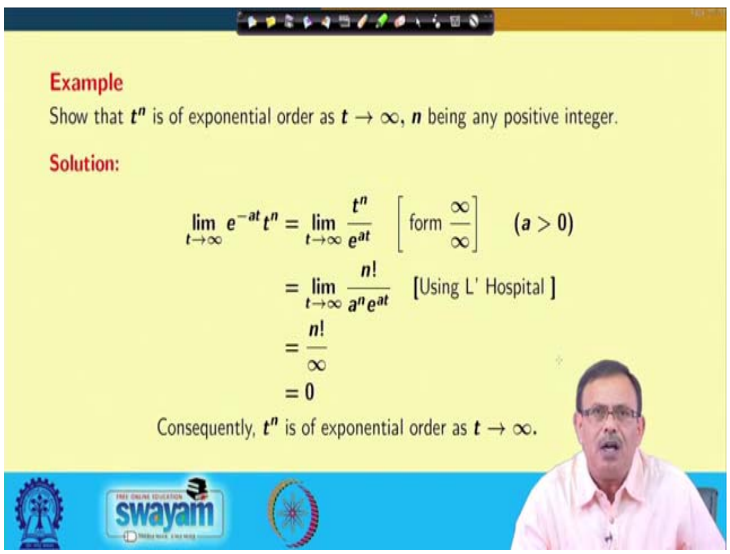

So, in order to check whether a function $F(t)$ is of exponential order or not, we have to just find out the limiting value of $e^{-at}F(t)$ as $t\to\infty$. And if it is a finite value, we say that the function $F(t)$ is of exponential order.

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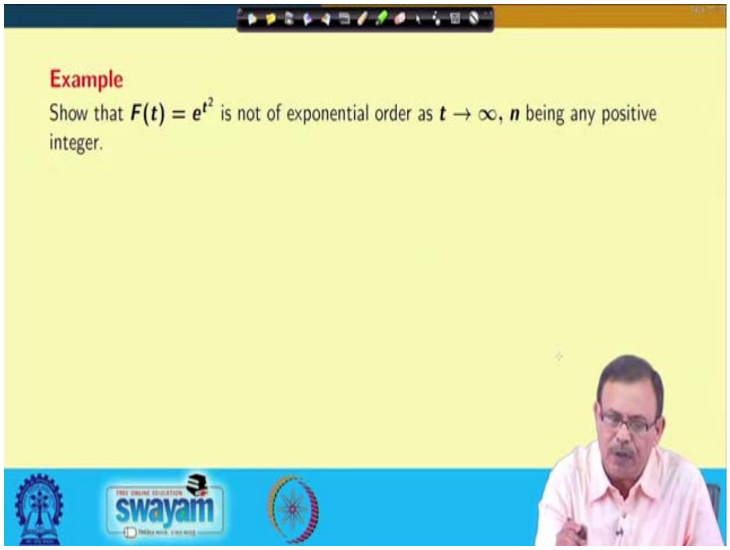

Let us come to the next example. We need to show that the function e^{t^2} is not of exponential order as $t \to \infty$, n being a positive integer.

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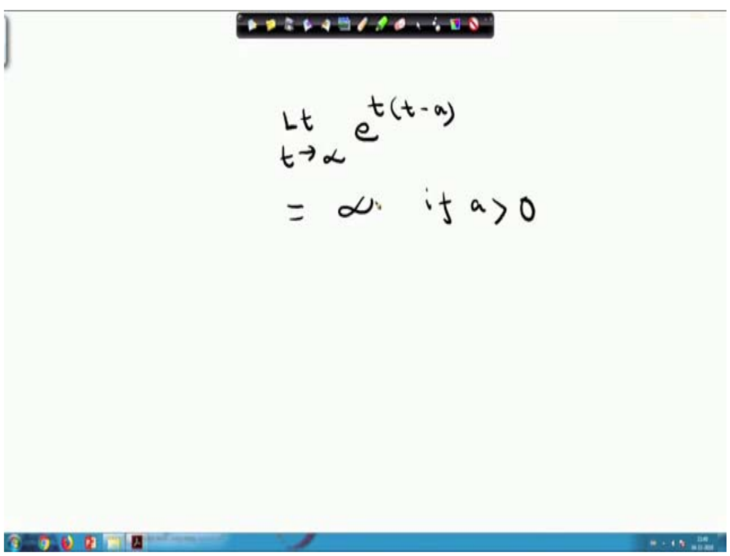

So, for this problem again, we have to find out the limiting value of $e^{-at}e^{t^2}$ as $t \to \infty$. This value clearly is equal to ∞ for $a > 0$ i.e.,

$$\lim_{t\to\infty} e^{-at}e^{t^2} = \lim_{t\to\infty} e^{t(t-a)} = \infty \text{ if } a > 0.$$

Since the limiting value is not a finite quantity, so the function e^{t^2} is not of exponential order.

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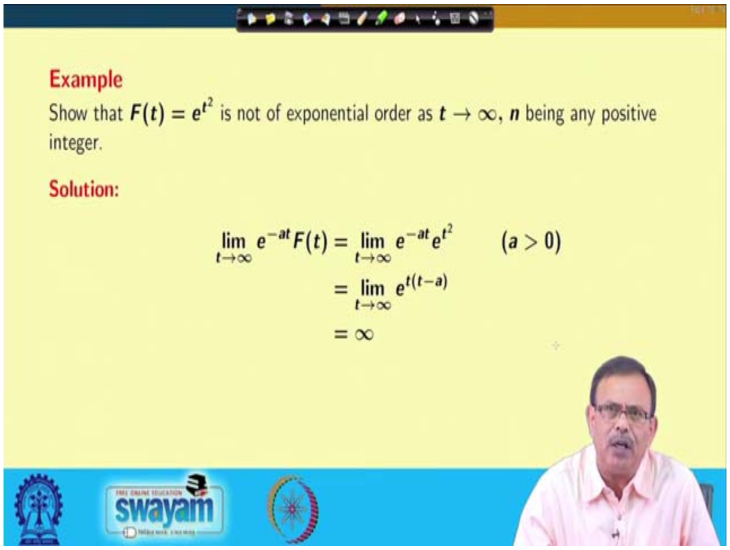

So, using this technique, we will be able to find out whether a given function is of exponential order or not. In the next lecture we will be studying the use of this exponential order i.e., how it can be related to transform calculus and what will be the corresponding effect.