Integral and Vector Calculus
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Lecture – 09
Improper Integral

Hello students. So, up until last class, we looked into reduction formulas. We also derived several types of reduction formulas involving trigonometrical functions, involving algebraic functions. And we also tried to work out few examples. And I hope I was I gave you enough how to say introduction, and also the flavour of the topic related to reduction formula. You might also be able to get some problems in your assignment sheet, and I will also provide the solution. So, hopefully the concept would be much clearer, when you work out few examples by your own.

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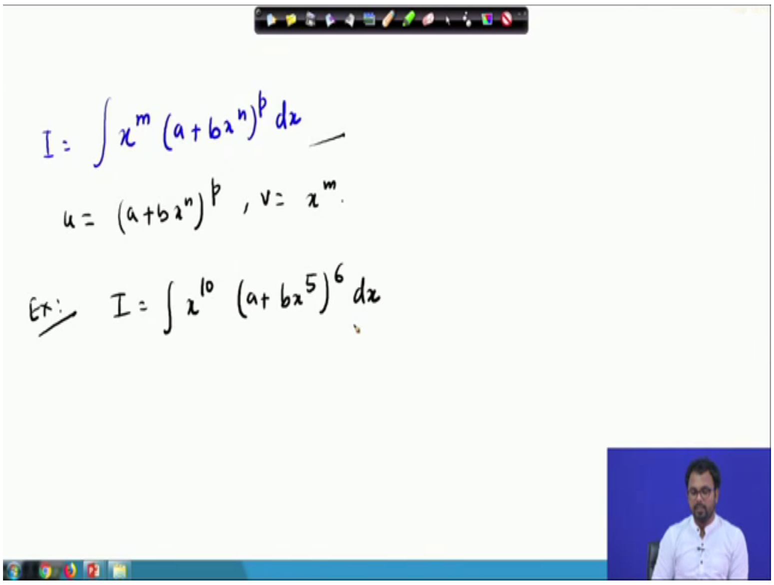

There might be one another type of reduction formula, which is of type integral x to the power m a plus b x to the power n whole to the power p d x. So, you may come across algebraic expressions or integrand of this type. So, let me change the colour first. So, in such cases what you can do, like we did earlier you can substitute u is equals to a plus b x to the power n whole to the power p. And then we can choose our function v as x to the power m, and then we just use the integration by parts formula.

And after doing some calculation at the end, we will be able to obtain our required reduction formula of this type. So, our required reduction formula can be obtained by doing the integration by parts on u and v. And basically, you need to find some kind of iterative reduction formula to calculate any such problem of this type. So, for example one possible example can be let us say, I equals to x to the power 10 a plus b x to the power 5 whole to the power I do not know 6. So, this is one such example, which falls into this category and we basically how to say find the reduction formula of this type of integral here to calculate an integral of this type.

So, I mean this is one such example, we can have lot of other types of integrals and based on which you can derive some kind of reduction formula. So, the bottom line is that in case of reduction formulas all you have to do is to find out some kind of iterative scheme or some kind of how to say an n type formula for every n, you can be able to calculate a certain integral. So, I leave this reduction formula topic here. And since we have very compressed time to cover a lot of things so, I will move on to our next topic, which is Improper Integral. And we will work out few examples on reduction formula in our assignment sheet.

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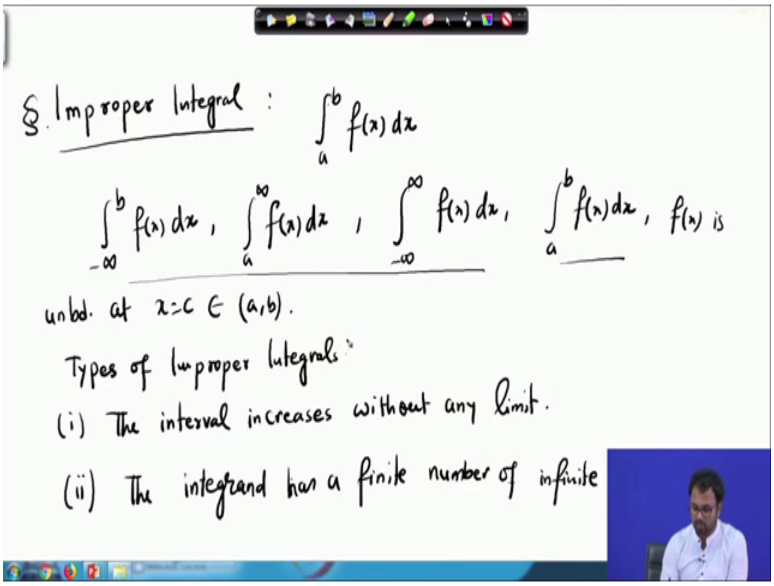

So, let us go to our next topic, which is improper integral, so that is our next topic in agenda.

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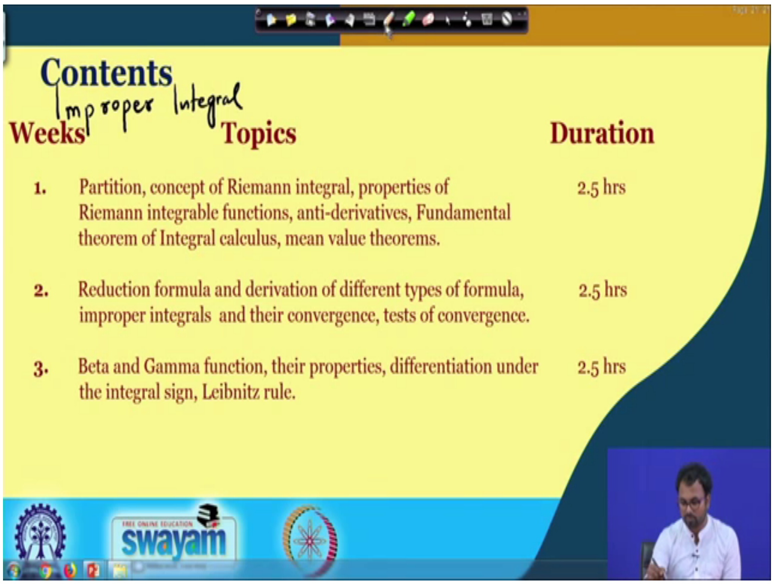

So, as you can see we were supposed to reduction formula and the derivation. And the next topic, we are supposed to cover improper integrals, their convergence and some tests that will confirm the convergence of an improper integral.

So, let us go back to our topic here. So, when we say improper integral, what do we mean by it? The term improper itself somehow signifies that something that is not good in a way or that is not proper in a way. And, something which is not good in case of a function or in case of an integral is that a function may be having a discontinuity or a function is not differentiable. So, those kind of properties are somehow related to something is not good with a function.

So, here in this case it means exactly the same. So, when we say an improper integral, so it could mean that the, we know that a definite integral can be written in this form integral a to b f x d x. And when we say that this integral is improper that means, something is not good with this integral, it means that either the lower limit is infinity or the upper limit is infinity or both the lower limit and upper limits are infinity or the lower limit and upper limits are fine, but the function is discontinuous or unbounded at a point in between these two these two points.

So, f x in this case f x is unbounded at x equals to c in the interval a to b. So, do you see that what do I mean, I mean it means that there is some kind of issue either with the limit points like here, like in the first three sets of definite integral or there is some kind of issue with the function itself in the given interval. And if such kind of things happen with an integral, then such type of integral are called as an improper integral or integral who which are not behaving nicely.

So, we can put this in a small statement. So, the small statement is types of improper integral. So, when we say improper integral, what are their types, so or what do we actually mean by them. So, the improper integral types of improper integrals are the 1st type is the interval increases, the interval increases without any limit. So, this could be one problem. Like here in this case or in this case the interval is increasing up to infinity. So, these type of integrals can be of type improper integral or the 2nd type is the integrand.

So, the integrand is basically the function f x. The integrand is or the integrand has a finite number of infinite discontinuity discontinuities. So, any one of these two things can happen with our function or with our integral and then such type of integrals are called as the improper integral.

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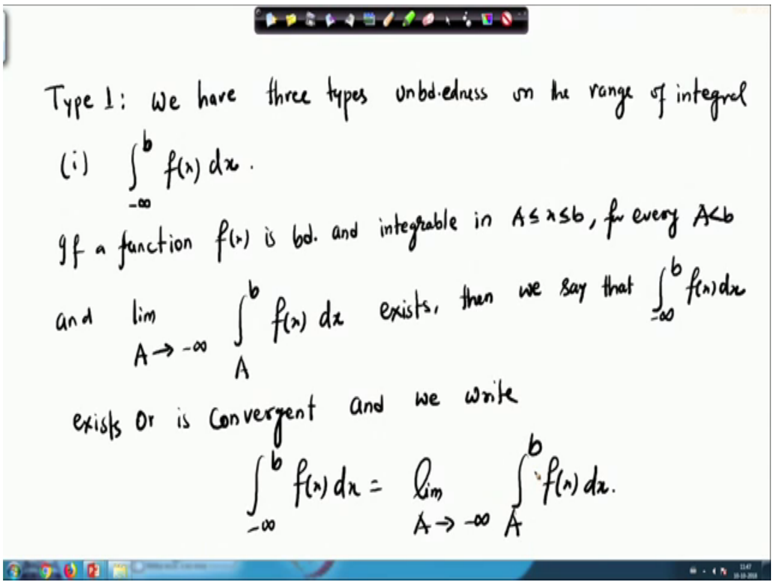

So, let us look into the type-1 so type-1 type of improper integral. So, type-1 type of integral is so in this case we have three types of unboundedness on the range of integral.

So, the first one is limit let us say our lower integral is infinity. So, we have first type of unboundedness of this type. So, here if a function f x is bounded and integrable in A less or equal to x less or equal to b, where for every A capital A less than b. And limit A goes to minus infinity integral from A to b f x d x exists, then we say that integral from minus infinity to b f x d x exists or is convergent or it is convergent. And we write integral from minus infinity to b f x d x is basically the limit A goes to minus infinity integral from A to b f x d x.

So, the thing is the if the interval is unbounded means if we have the range of integral from minus infinity to b, then in that case what we do is basically, we get rid of that minus infinity part by putting it in terms of limit. And then we basically evaluate this integral here from A to b, and afterwards we pass on the limit. And that will be exactly the same the value will be the same as the value of this integral. So, for this type of integral, where the lower limit is minus infinity, we basically calculate this limit and this limit is actually however the value of the integral.

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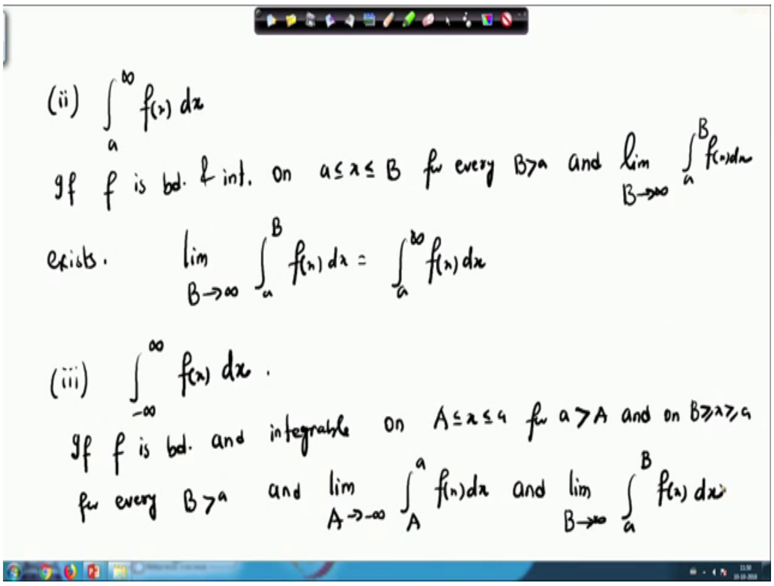

In type-1 category another type of integral could be of this type, let us say we have a to infinity f x d x. Then the definition pretty much goes the same that if f x is a bounded function, let us go to the previous slide. So, if f x is a bounded function and integrable from A to capital from A to capital B for every B greater than A.

So, here if f is bounded and integrable on a less or equal to x less or equal to capital B for every B greater than a and this limit B goes to infinity integral from a to B f x d x exists, then we say that this integral is convergent. And we can write limit B goes to infinity a to

B f x d x is equal to the value of that integral right. So, it follows pretty much the same definition like the definition 1.

And another type of improper integral could be from minus infinity to plus infinity f x d x. So, here both the upper limits and lower limits are infinity. So, then in that case, what we will do? We will choose point, so we will write if f is bounded and integrable and integrable on capital A less or equal to x less or equal to a for capital for small a greater than capital A and on capital B on capital B greater or equal to x greater or equal to a for every B for every B greater than a. And the limits A goes to minus infinity A to small a f x d x. And limit B goes to infinity integral from a to capital B f x d x.

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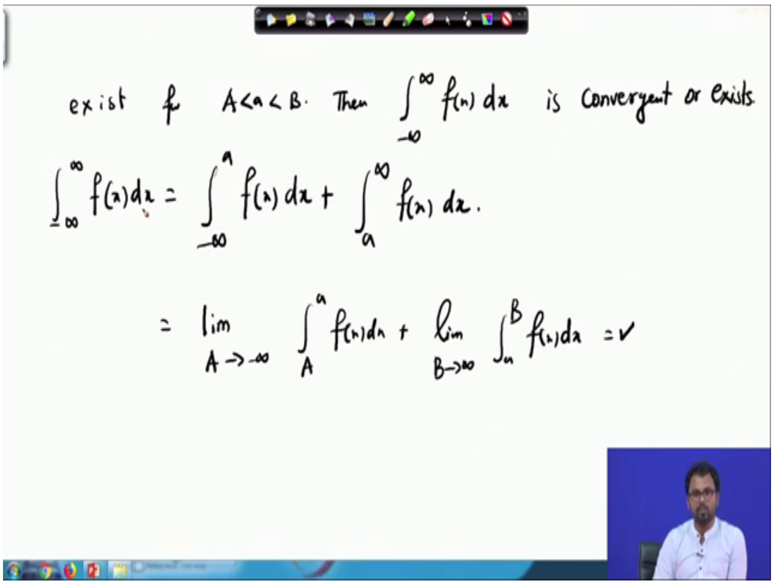

They both exist for capital A less than small a less than capital B. Then we say that this integral minus infinity to plus infinity f x d x is convergent or exist, so this into this integral will exist or convergent. And the value of this integral f x d x, value of this integral can be split into two sub integrals, so addition of two sub integrals.

So, you basically see what we are doing? So, we had minus infinity and plus infinity at both the end points, so what we have done? We have chosen a point in between minus infinity to plus infinity, so that now we have infinity at only one of the endpoints. And then it falls into the definition of one definition of one or the definition of two, so that is that is what we are basically doing?

So, we are dividing the whole interval into two parts. And then we are using the definition one and two to show that these two limits these two individual limits. So, here I can write limit A goes to minus infinity, so these two individual limits A to capital A to small a f x d x plus limit B goes to infinity a to capital B f x d x and if this limit exists if this limit exists, then the value of the original integral would exist.

So, with the limit points these are the three types of how to say improper integral, we can we can come across. Now, let us evaluate an example. So, we have given the how to say a few definitions, and now we are going to as work out few examples. So, let us go to a new page.

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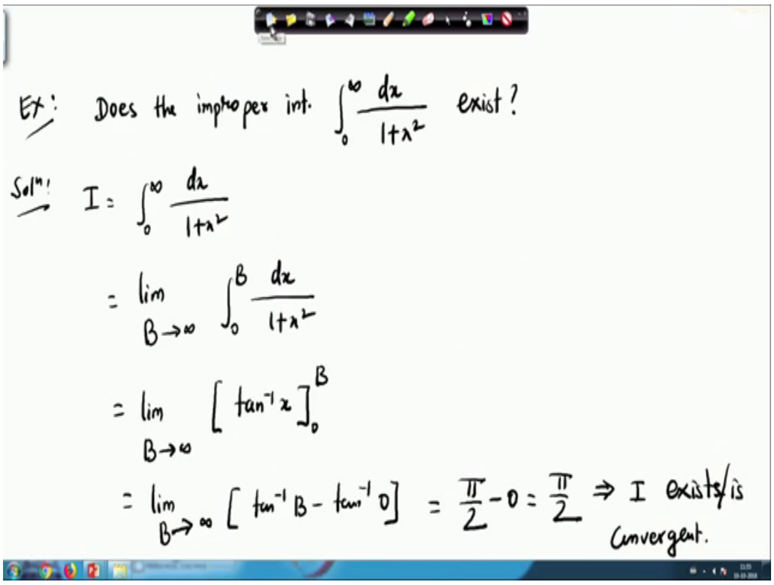

Example, sometimes I am not good at keeping track of the examples, so I will just write E X ok. So, the very first question is does the improper integral 0 to plus infinity d x by 1 plus x square exist. So, of course it is an improper integral, because you have the upper limit as infinity.

Now, if the upper limit is infinity, then it falls into the definition 2 criteria. And therefore, we know that the value of this integral let us write it as I the value of this integral would actually be equal to this limit B goes to infinity, integral from 0 to B d x by 1 plus x square.

Now, integral of 1 plus d x by 1 plus x square is tan inverse x, so we will write tan inverse x here, then integral from 0 to B. This will turn into limit B goes to 0 B goes to infinity I beg a pardon, and then this one is tan inverse B minus tan inverse 0, now tan inverse 0 is 0. And tan inverse B, when B goes to infinity will go to pi by 2, so the value of the integral is pi by 2. And therefore, this improper integral, it converges or it exists. So, from here we can say that I exist or it I is convergent or it is I is convergent all right. So, this was a fairly simple example to verify.

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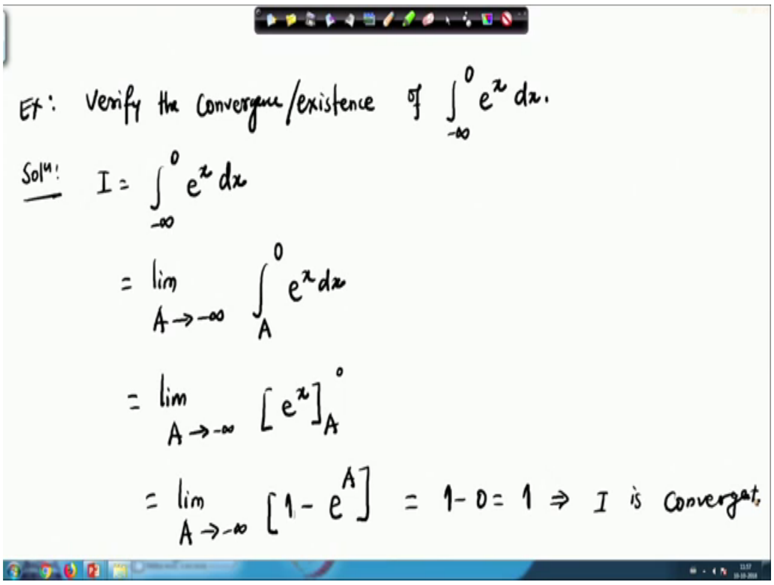

Next let us consider an example of type of type integral let us say verify the convergence or existence convergence or existence, we can use either one of these terms, existence of integral from minus infinity to 0 e to the power x d x.

So, let us start. Now, this one falls into the category of type-1 one. So, we can write I as integral from minus infinity to 0 e to the power x d x. Now, we know that from definition 1, where is that so here. So, we know that if we have an integral of this type, then the value of that integral will be the value of this limit here. So, we will do exactly the same thing. So, I can write limit A goes to minus infinity integral from A to 0 e to the power x d x.

Now, if you integrate e to the power x, then will obtain basically e to the power x integral from A to 0. Now, this can be written as limit a goes to minus infinity e to the power 0 is 1. And e to the power x at x equals to A would be e to the power A all right.

So, now when A goes to minus infinity, the 1st term is constant, so it will remain unaffected. And the 2nd term, when e goes to minus infinity basically this whole thing will go to 0, because e to the power minus infinity will be 1 by e to the power infinity, and that will be basically 0. So, this will be 1 minus 0, so 1. And from here, we can say that I is convergent or I exist convergent all right. So, this is one such example, where you can use the definition of type-1. Now, let us play with this example this integral here a little bit. And then we can say that the integral is actually divergent.

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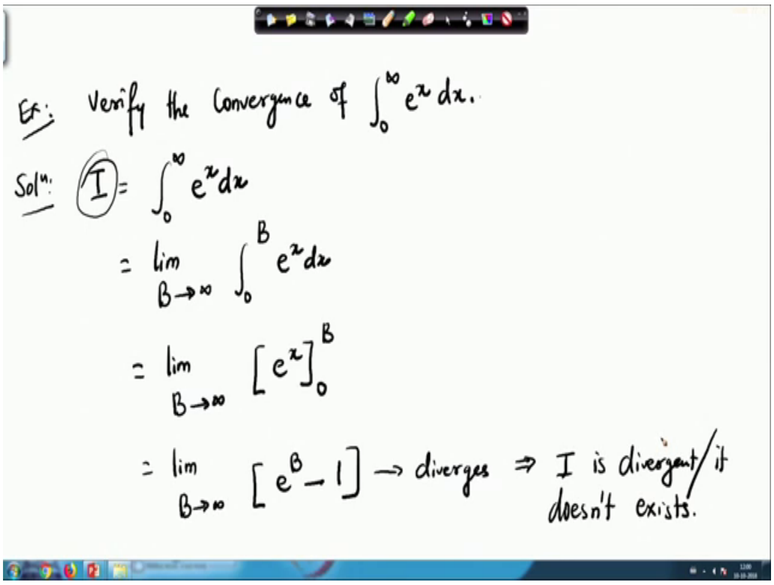

So, let us consider an another example, verify the convergence the convergence of integral from 0 to infinity e to the power x d x. So, we have to verify the convergence of this integral here. So, this one is again of type-2 type-1 2, so that means, where the upper limit is infinity.

So, I can write our integral as I, which is equals to 0 to infinity e to the power x d x. And then I can use the limit definition, so I can write it from 0 to B e to the power x d x. And then this can be written as limit B goes to infinity, the integral of e to the power x would remain e to the power x, and this one will be from 0 to B.

Now, we have limit B goes to infinity e to the power B minus e to the power 0, e to the power 0 is again 1. And here when B goes to infinity e to the power B will also diverge. So, this whole thing will diverge. So, it diverges that means, this increases, how to say beyond all bounds actually as B goes to infinity that means, this integral I here, this integral I here, it does not converge to a finite limit. So, it does not converge into any finite number or the limit does not exist. And therefore, from here our integral I is divergent or it does not exist or it does not exist. So, you see just by playing with the limits same integrand can be shown to be divergent all right.

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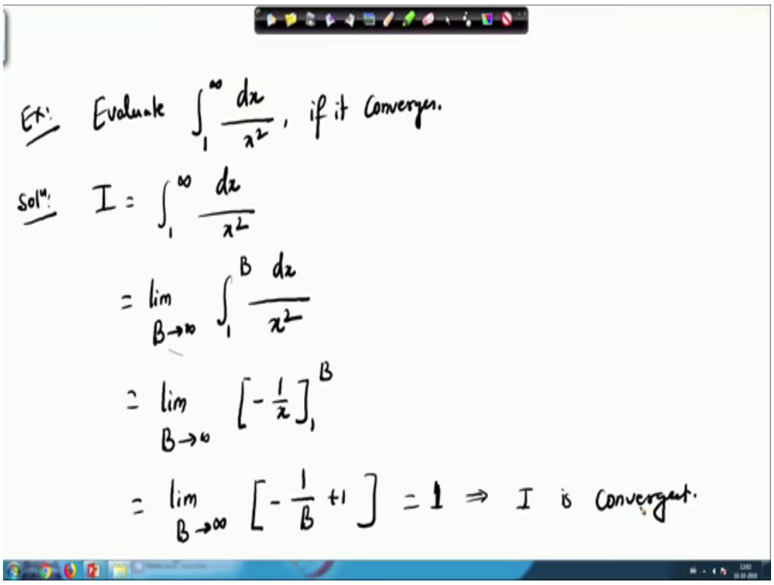

Now, let us consider an another example so let us consider an example. Let us say evaluate integral from 1 to infinity d x by x square, if it converges. So, here I can write I as integral from 1 to infinity d x by x square. And now I can write it as limit B goes to infinity integral from 1 to B d x by x square. And if I integrate the integrand, then I will end up with 1 by x integral from 1 to B.

And this one can ultimately be written as integral from B to infinity minus 1 by B plus 1. So, when B goes to infinity 1 by B will go to 0, and therefore will end up with 1. And this implies that our integral I is again convergent. So, any one of these examples if we consider, then we can see that if either one of the limits is infinity.

Then all we have to do is to consider the limit form limit form limit form of that of that integral, and then evaluate the integral like the way we do, and then finally pass on the limit, and the value of that limit will be the value of the integral. So, if it is finite, then in that case the integral is convergent; and if it is not, then it is divergent. And an integral is can never oscillate, so a integral is not oscillatory, so either it is divergent or it is convergent. And in order to do that we are we can verify in this fashion or we will also look into some tests that will assure the convergence.

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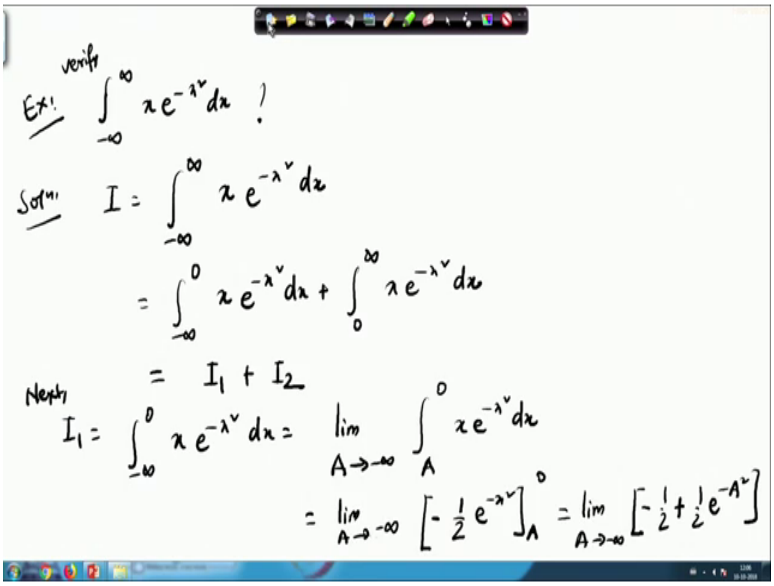

Now, and another example could be of type of this type, where both of our limits. So, let us write verify the convergence. So, verify the convergence verify the convergence of this integral here. So, I hope you got the statement, what I am trying to say. So, verify the convergence of this integral.

So, let us write this integral as I equals to minus infinity to plus infinity x e to the power minus x square d x; so, again this integrand here. We basically choose any point between minus infinity to plus infinity, so that we can split this integral into 2 halves. So, I choose minus infinity to 0 x e to the power minus x squared d x plus integral from 0 to infinity x e to the power minus x square d x. We have to make sure that in this interval the function is not unbounded.

So, whatever the point, I choose now in between those two points the function should not be unbounded here also. So, if you see between minus infinity to 0, it does not matter whatever we substitute, it will always remain finite and similarly from 0 to infinity. So, we have to choose this point in such a way that the function or the integrand remains bounded.

Now, I can call this one as I 1, and I can call this one as I 2. And then I evaluate I 1, so next we can evaluate I 1, which is minus infinity to 0 x e to the power minus x squared d x. And if I follow the previous definition, then I can write it as minus A goes to minus infinity A to 0 x e to the power minus x squared d x. This is a traditional method of substitution example.

So, here if I substitute x square equals to z, then basically I will obtain x d x, and then you do basically the integral integration here. And this will reduce to limit A goes to minus infinity A goes to minus infinity half e to the power minus x square A to 0, and e to the power minus z yeah.

And if I substitute x equals to 0, so this will be limit A goes to minus infinity, this will be minus half plus half e to the power minus A square. Now, when A goes to minus infinity, then this will be actually A to the power this will be actually a to the power so this will be again A to the power minus infinity.

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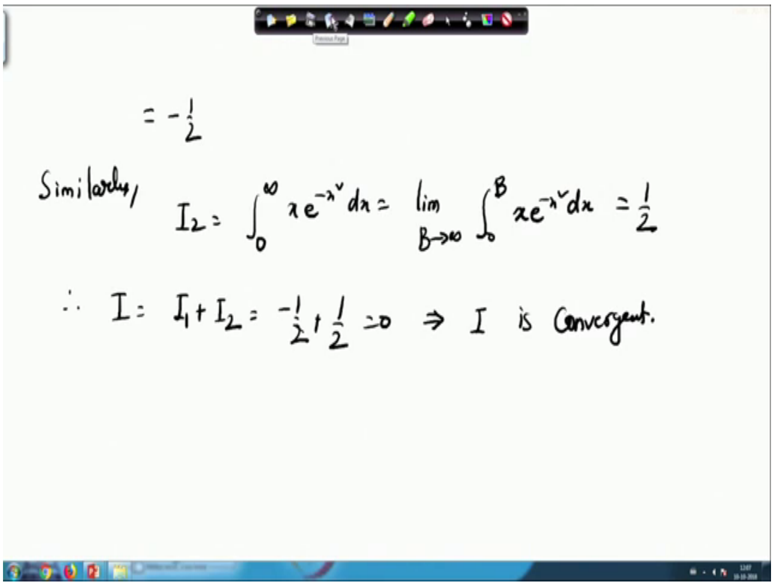

So, ultimately this will be our the value will be half so this will be our minus half. And similarly, we can evaluate similarly we can evaluate I 2 equals to integral from integral from 0 to infinity x e to the power minus x square, we can write it as limit B goes to infinity integral from 0 to B x e to the power minus x squared d x. And if we follow the similar steps like I 1, then this will give us the value as half.

So, therefore our capital I is basically I 1 plus I 2. Now, I 1 is minus half, and I 2 is half, so ultimately the value is 0, so that means, the value of this integral the value of this integral is nothing but 0. And since 0 is finite, we can say that I is convergent or the integral I exist.

So, we have seen different types of examples in this case, and where the both lower limit upper limit or both the limits can be 0, and we also worked out for example. So, in the next lecture, we will look into some integrals of type-2, I will give you an explanation what do we mean by type-2. So, we will stop here for today, and see you in next lecture.

Thank you.