Integral and Vector Calculus

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Lecture – 08
Reduction formula (Contd.)

Hello students, we will continue our this lecture with the Reduction formula and we will start from where we left off. So, we were trying to derive the direction formula for the product of 2 trigonometrical functions and we basically started with integral of type I m n equals 2 sin m x times cos n x d x

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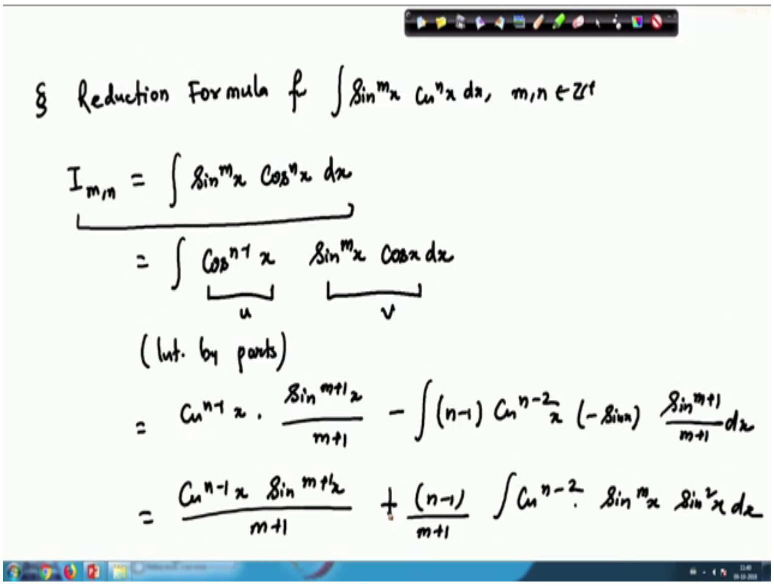

I believe here there was a small error, so it should be plus and then we ultimately obtained this reduction formula.

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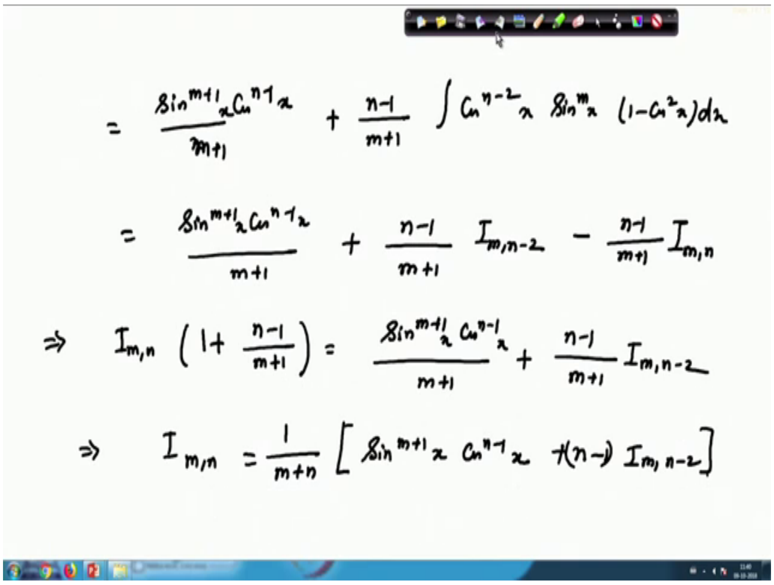

So now, we will see an example where we can apply this reduction formula, so an another example.

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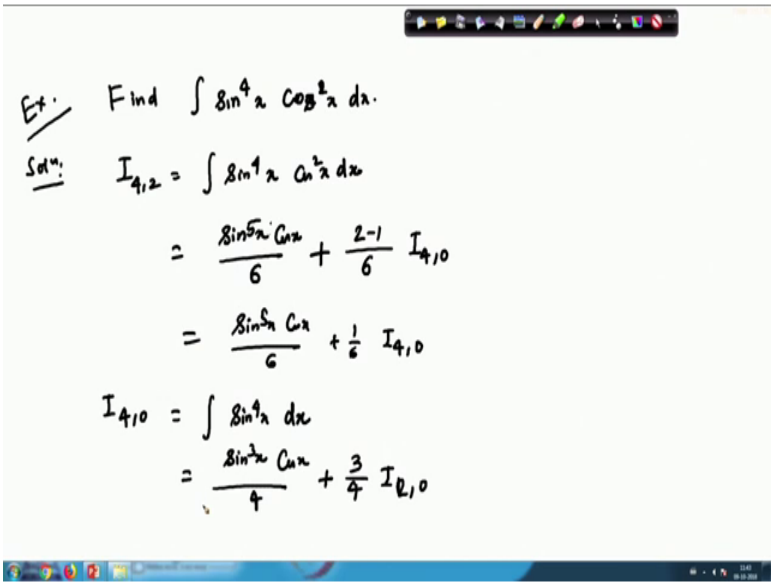

So, find integral sin to the power 4 x times cos square cos square x d x. So, the solution would go something like this, so we can write I 4 comma 2 equals to sin to the power 4 x times cos square x d x.

Now, I 4 2 can be written as sin to the power, it can be written as sin to the power m plus 1, so, m plus 1 would be m plus 1 would be 5. So, where is that formula? So, here it will be m plus 1. So, 4 plus 1 divided by m plus n. So, m plus n is 6 and cos n minus 1 so, cos n minus 1 is our basically so m plus n is 5 x d x and cos n minus 1 is cos x and this will be minus plus n minus 1. So, n minus 1 is 2 minus 1. So, this is basically 2 minus 1 divided by m plus n. So, m plus n is again 6 and I m is 4 and n is n minus 2 is basically 2 minus 2, so that is basically 0.

So, here we will have sin 5 x times cos x by 6 plus 1 by 6 times I 4 0. So, I 4 0, next I 4 0 is basically sin to the power 4 x d x. Now, I 4 0 and here we have sin to the power 4 x d x will again fall into that sin n to the power x formula and sin n to the power x formula was sin n minus 1. So, this is basically sin cube x times cos x by 4 plus 4 minus 1. So, this is basically 3 by 4 I 4 minus 2.

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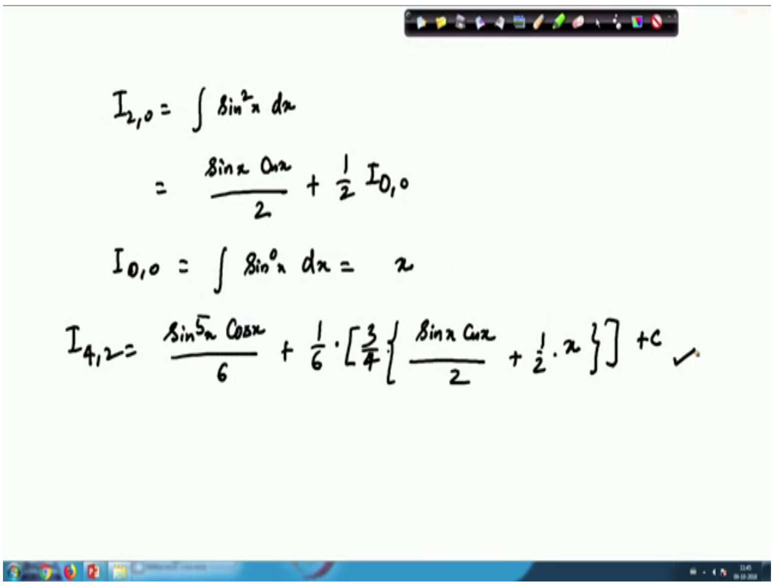

So, I 2 comma 0 and I 2 comma 0 can be written as, I 2 comma 0 can be written as integral sin square x d x and sin square x d x can again follow the same what to say integral reduction formula. So, this can be written as sin 2 minus 1, so basically sin x times cos x by 2 plus n minus 1. So, this is basically 1 by 2 I 0 and I 0 0 is nothing, but sin to the power 0 x d x and sin to the power 0 x is basically simply x.

So, I can write it this whole thing as I 4 0 equals to I 4 0 equals to sin to the power 5 x times cos x divided by 6 plus 1 by 6 times I 4 0, I 4 0 is 3 by 4 times I 2 0. So, this one is 3 by 4 times I 2 0, I 2 0 is sin x times cos x by 2 plus half I 0 I 0 0 is basically x plus a constant c. So, you will obtain whatever after the multiplication you will get from here.

So, this is one another way to calculate the product of 2 trigonometrical functions and you can play with any power here. So, you can play with any power here you please sorry, here you can play with any power of power of sin x and cos x here. So, and then you just apply this formula, this one here and that will give you the required answer for that particular integral and this is one such iterative formula.

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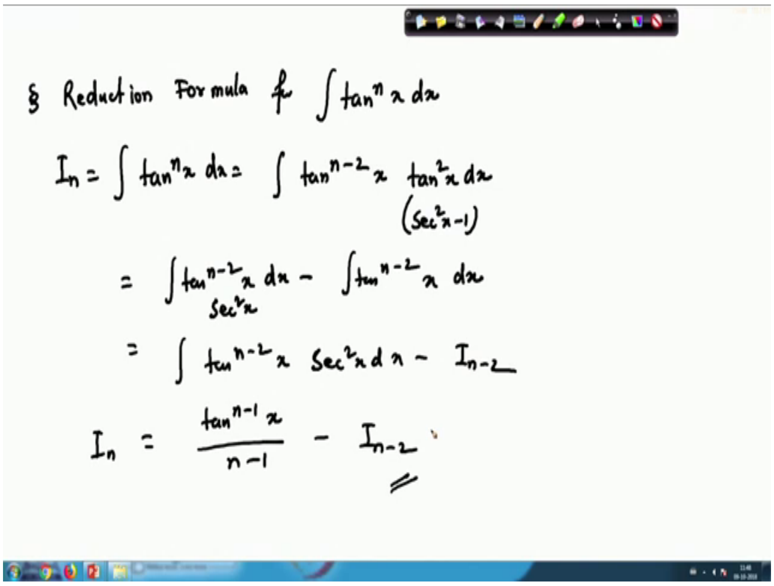

Next we will derive the reduction formula of type, let us say tan to the power n x d x where n is again a positive integer. So, to do that I can again write I n because of that n here and then I will write tan to the power n x d x and I can split this tan n to the tan to the power n x d x as tan n minus 2 x times tan square x d x. And, then you just write tan square then you just write tan square x d x as a sec square x minus x like we know from the trigonometrical formula and then we just proceed as before.

So, we write this one as sec square x minus 1 here and then you will basically end up with an iterative formula of type tan n minus 2 x d x minus so, tan square x. Then this will become sec square x minus 1 and then this one becomes tan n minus 2 x d x and this is again our I n minus 2. So, we can write tan n minus 2 times sec square x d x minus tan n minus 2. So, this can be written as tan n minus 2 x times sec square x d x minus I n minus 2 and here if I substitute tan x equals to z then in that case that d z will be sec square x d x. And, we integrate and then this will become tan to the power n minus one x by n minus 1 minus I n minus 2. So, this is basically the reduction formula for tan to the power n x d x I mean all you have to do is play with some trigonometrical formula it is not that tricky

And similarly if you have some range for the definite integral that is if it is 0 to pi by 4 tan n x d x then you just have to integrate this part and keep on calculating depending on the power of n here. So, if you have n equals to 4 then this will become 4 minus 2 and then again in the next step it will become 2 minus 0 and if it is odd then you will end up with I 1 at the end. So, it is pretty much on the similar footsteps of sin n x and cos n x formula. So, I leave the examples for this direction formula up to the students you can look into any book.

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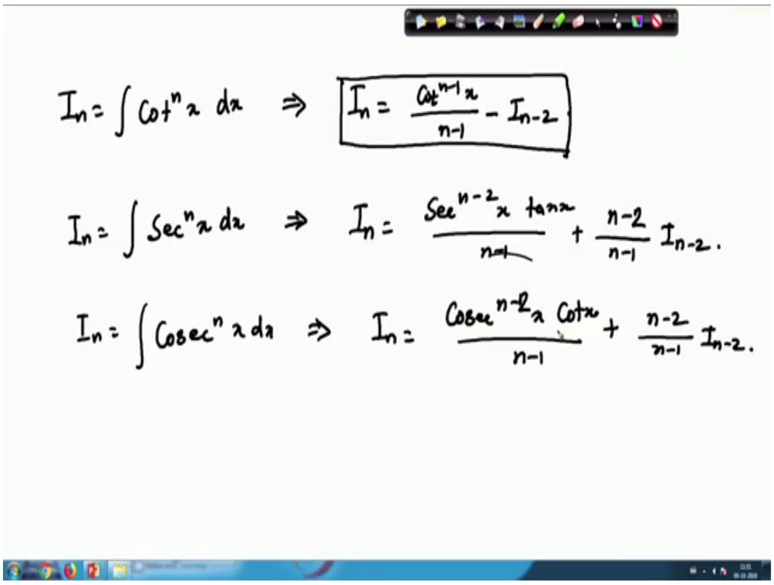

Similarly, if you have let us say if you have other trigonometrical functions for example if you have I n equals to integral cot n x d x then you can again write cot n x d x as got n minus 2 x times cot square x and cot square x can be written as cosec square x minus 1 and then you follow the similar steps like before. So, ultimately here the reduction formula would be I n minus 1 equals I n equals to cot to the power n minus 1 x divided by n minus 1 minus I n minus 2. So, this will be the reduction formula for cot n minus cos cot n x d x.

Similarly, one can have I n equals to sec n to the power x d x. So, here also you write sec n x d x as a sec n minus 2 x times sec square x d x and we substitute how to say sec as sec square x as how to say 1 plus tan square x and then we use some how to say basic method of substitution. So, this will again give you I n equals to at the end sec to the power n minus 2 x times tan x divided by n minus 1. I am just writing this because these are very straightforward all you have to do is do some basic integration and this will be n minus 2 by n minus 1 times I n minus 2 and use some basic algebraic operations.

So, similarly we have the reduction formula for sec n to the power x d x you can also have in the similar fashion you can also have cosec n x d x and you can write cosec x cosec n x d x as cosec n minus 2 x times cosec square x d x. And, then you use again some trigonometrical formula and you will basically end up with cosec n minus 2 x times cot x by n minus 1 plus n minus 2 by n minus 1 I n minus 2. Here you can see a pattern that say sin to the power n x and cos to the power n x follows the same formula reduction formula, similarly tan x and cot x follow the same trigonometrical formula and also sec sec n x and cosec n x also follow pretty much the same trigonometrical formula.

So, instead, so when you have when instead of sec x if you have cosec n x then we just replace this sec x by cosec x and we replace tan x by cot x and then you will pretty much end up with a similar type of formula. And as I was talking the derivation of these formulas are not difficult, so I will leave those task to the students for practice. Next we have reduction formula of type let us go to a new page.

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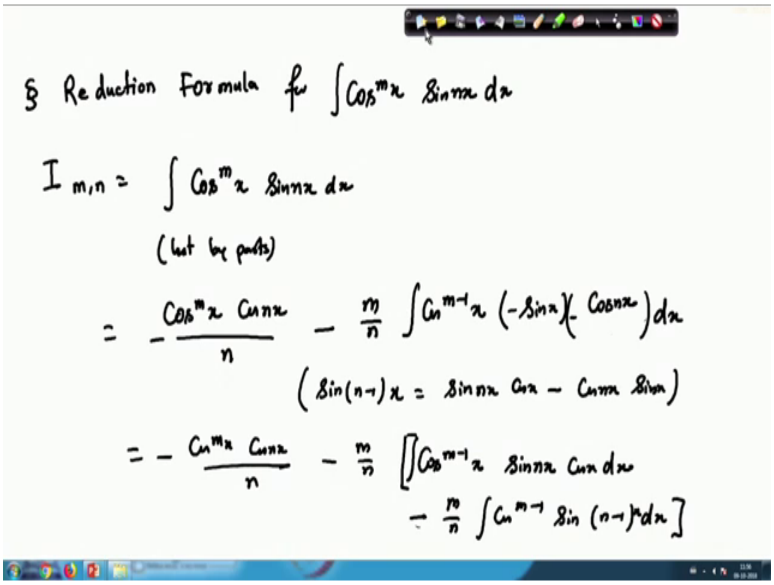

So, next we have a reduction formula of type let us say, reduction formula. So, as you are seeing that by the help of these reduction formulas we are deriving some kind of iterative schemes that will help us calculate very complicated powers of functions involved in your integral and you just have to put in those reduction formula you just I mean and you just calculate if it is either known I to the I 10 and then you may have to calculate I 8 I 6 I 4 I 2. And, then I 0 and ultimately you just put those in that formula and that will give you the value of that I 10 for example, or if it is an odd one then you can just calculate how to say let us say if it is I 11, then you calculate I 9, I 7, I 5 I 3 and then ultimately I 1 and that will give you the answer.

So, even to you initially have a very complicated or a difficult integral or a product of 2 functions to you just have to use these formula and that will give you the answer. So, in a way a reduction formula is a nice shortcut or a nice tool that will help you calculate those difficult integral. Next we have a function of type this. So, let us write I m n equals to cos m to the power x sin n x d x. So, basically I will integrate by parts. So, I will integrate by parts and if I integrate by parts then this will be cos m x integration of sin x will be cos n x divided by n and then there will be a minus sign minus m by n this will be cos m minus one x differentiation of cos x is minus sin x and then how to say integral of sin x is cos n x with a minus sign divided by n d x. So, basically minus minus will turn into plus and this n we have taken here. So, we no need to write this n here. So, we do not write n here.

Now, if you remember the formula for sin n minus 1 x, so the formula for sin n minus 1 x is equals to sin n x cos x minus cos n x sin x. So, I will use this formula here and then this whole thing will ultimately reduce to minus cos m x times cos n x divided by n minus m by n this whole thing will turn into cos m minus 1 times x sin n x cos x d x plus so minus sorry this will be minus minus m by n cos m minus 1 sin n minus 1 x d x.

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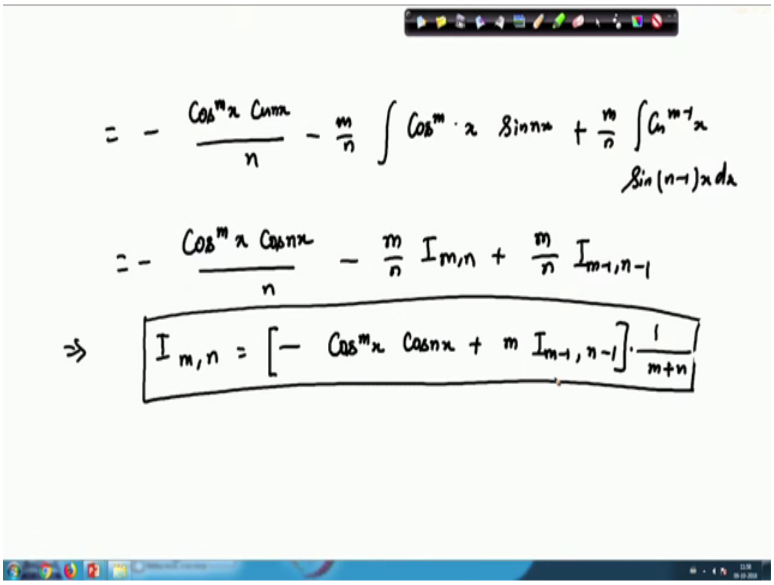

So this will reduce to minus of cos m x times cos n x by n plus minus m by n this will reduce to cos m minus 1 x sin n x and time cos x. So, this will be basically cos m x and then plus m by n this will reduce to cos m minus 1 x times sin n minus 1 x d x. So, this is basically our I m n and this 1 is basically I m minus 1 n minus 1.

So, I can write it as cos m x times cos n x by n minus m by n this is nothing, but I m n and this one is again I m minus 1 n minus 1. So, if I take I m n on the on the left hand side and if I do multiply both sides by n then ultimately we will end up with I m n equals to minus of cos m x times cos n x plus m times I m minus 1 n minus 1 divided by, so the whole thing is divided or multiplied by 1 by m plus n and this is our required reduction formula for I m n.

So, if you have the product of cos to the power, so if we have a product of cos to the power let us say m x times sin n x our required reduction formula would be of this type. So, again instead of m if you have 10 and instead of n if you have 20, then you just have to put here and then do the calculation and ultimately you will be able to up you will be you will be able to end up with either I 0 0 or I 1 1 or I 1 0, I mean a relatively simple integral to evaluate basically and just how to say reverse engineers, so that engineer all those things; that means, you just have to put everything backwards and then that will give you the answer.

So, this is one such reduction formula. Similarly instead of instead of cos to the power m x you can have sin to the power m x times cos n x and then you proceed in the similar fashion and then you will probably end up with a similar formula of this type. Next we will look into and next we will look into formula of type let us say formula of type.

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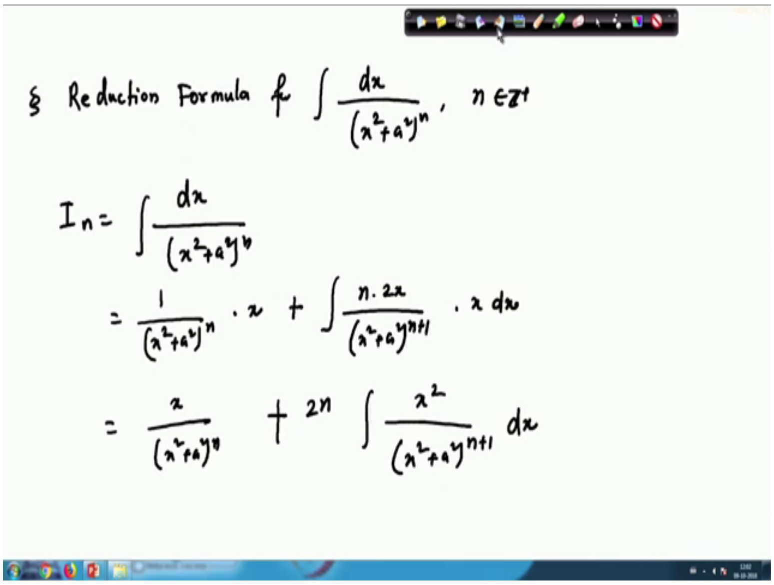

So, reduction formula d x by x s square plus a square whole to the power n where n is a positive integer sir. Next what we will do we will write I n equals to d x by x square plus a square whole to the power n and what we will do is basically we will write this function. So, we will write this function as integration by parts we will do integration by parts. So, our first function is x square plus a square times n and the second function is 1 So, this will be x square plus a square whole to the power n and the second function is 1, so integration of the second function would be simply x minus differentiation of the first function.

So, this will be n times 2 x divided by x square plus a square to the power n plus 1 times x d x all right. So, this can be written as x by x square plus a square whole to the power n minus n 2 2 n basically this will be x square divided by x square plus a square whole to the power n plus 1 d x right.

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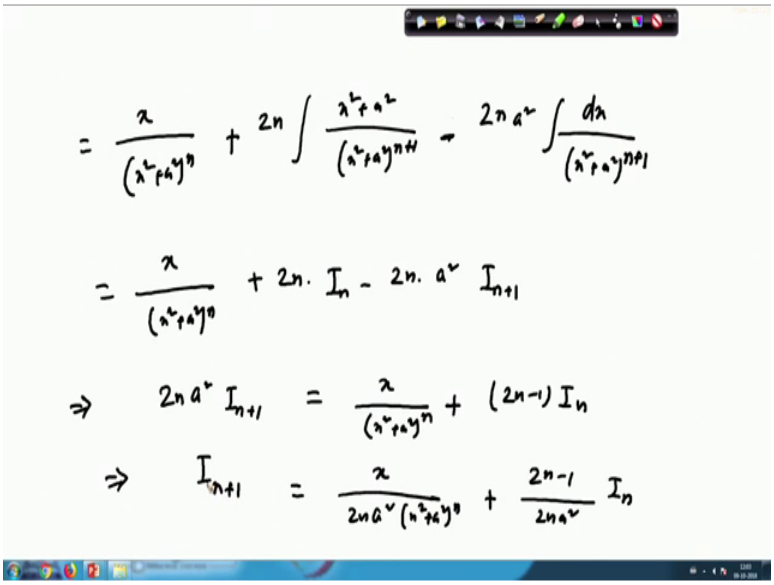

Now, I can write x x square plus a square whole to the power n minus 2 n and I can write x square as x square plus a square divided by x square plus a square whole to the power n plus 1 minus 2 n minus minus this will be plus. So, here we have minus and then this integral, so this will basically turn this whole thing into a plus, so this is plus and if I go back then this will be plus and then this will be a minus. So, this is a minus 2 n times a square d x by x square plus a square whole to the power n plus 1.

So, this is basically my I n. So, this is basically my I n minus 2 n times a square times I n plus 1 right. So, I can write 2 n times a square I n plus 1 equals to x divided by x square plus a square whole to the power n plus 2 n minus 1 times I n because this is this is this whole expression is equals to I n. So, I am bringing this on the left hand side and I am bringing I n on the right hand side, so we will end up with this.

Next we will divide the both sides by 2 n times a square I n and since we need the reduction formula for I n, so basically I will substitute n equals to n minus 1 on both sides.

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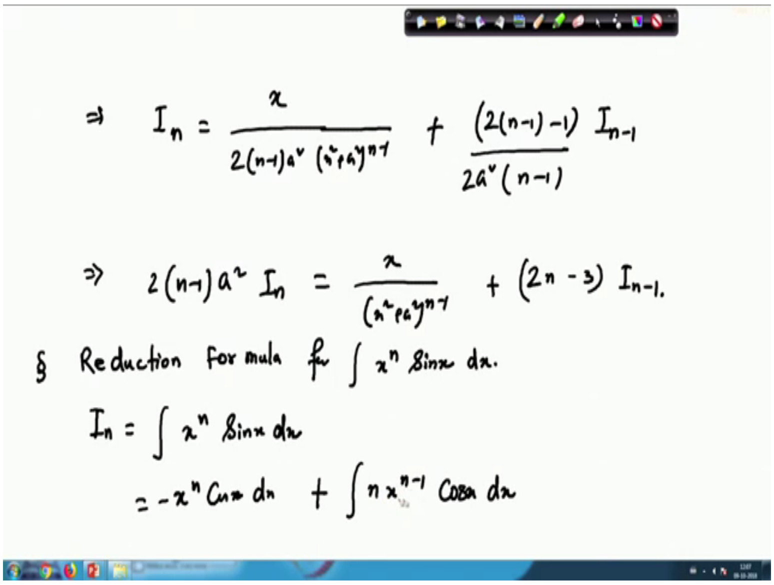

So, this will yield I n equals to I n equals to x by 2 n minus 1 a square x square plus a square whole to the power n minus 1 plus 2 n minus 1 minus 1 times I n minus 1 right. So, I substitute 2 n minus 1 minus 1 by, so 2 n minus 1 minus 1 by a square times n minus 1.

So, this will be our required reduction formula, let me verify. So, n minus one yes, so this will be our required reduction formula for x square d x by x square plus a square whole to the power n, so a function of this type all right. So, if you are given n equals to let us say n equals to 10 then you just have to substitute n equal to 10 here and then you keep on calculating I 9, I 8, I 7, up to I 0 and you just substitute the value of I 9, I 8, I 7 up to I 0 and that will give you the value of I 10 basically So, this is one such reduction formula.

Some people right prefer to write it in a rather how to say simpler form. So, you just keep this thing on the left hand side and you just write x square plus a square whole to the power n minus 1 plus 2 n minus 3 I n minus 1. So, this will be your required reduction formula, yes, from I n. Next you can have functions of type algebraic multiplied by trigonometrical functions.

So, our next reduction formula would be of type let us say reduction formula for x to the power n times sin x d x. So, to find the reduction formula for this function I can write I n equals to x to the power n sin x d x and here basically we have to do an integration by parts. So, we do integration by parts for, so the first function will remain unchanged integration of second function would be x to the power n cos x d x minus derivative of the first function say x to the power n minus 1 and integration of second function would be cos m cos sin x is minus of cos x. So, this will become plus and then we do the integration by parts again for this part.

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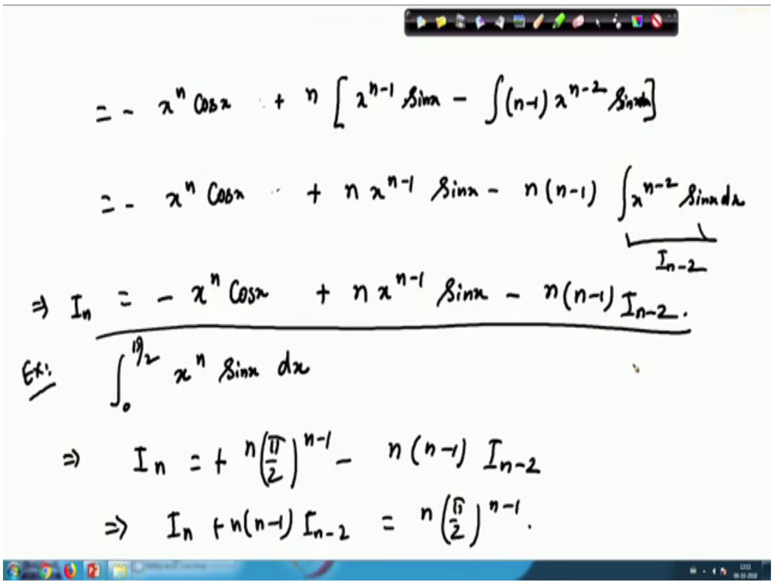

And if we do the integration by parts for this one, then you will basically obtain so minus x to the power n cos x d x plus n times I am keeping x to the power n minus 1 as my first function. So, x to the power n minus 1 we will remain unchanged integration of cos x would be sin x minus this one will be n minus 1, x to the power n minus 2 times integration of cos x is again sin x d x.

So, this will be minus of x to the power n cos x d x plus n times x to the power n minus 1 sin x minus n times n minus 1 integral of x to the power n minus 2 sin x d x. So, this one is our I n minus 2. So, this is our I n minus 2 So, I can write minus x to the power n cos x d x plus n times x to the power n minus 1 sin x minus n n minus 1 I n minus 2. So, this is our so this is our required reduction formula for I n of this type of this type.

You can have a integral from 0 to pi by 2 let us say an example So, you can have integral from 0 to pi by 2 x to the power n sin x d x. So, you can see that will basically be left with integral of x to the power n minus 1. So, we will be left with n times. So, if I substitute in there. So, we will have I n minus sorry, so there should not be any d x here, there should not be any d x here.

Now, if I substitute so there should not be any d x here. Now if I substitute x equals to 0 then the cos 0 is 1 and if I substitute x equals to pi by 2, so cos pi by 2 is x x equals to 0, so the cos 0 is 1, but x equals to 0. So, the first so at 0 this will vanish and at pi by 2 cos pi over 2 is 0; so this one will again vanish and therefore, and as far as this one is concerned, so at pi by 2 this will remain. So, this will be pi by 2 whole to the power n minus 1 and at x equals to 0 sin vanishes. So, this at x equals to 0 term will not exist and then this one will be n minus 1 I n minus 2.

So, I can bring and this one is plus and so, I can bring this whole thing on the left hand side. So, I n plus n minus 1 times n I n minus 2 and this will be n times pi by 2 whole to the power n minus 1. So, this will be basically the answer to this reduction formula for the integral from 0 to pi by 2. So, this is how we apply this induction formula instead of n you can have n equals to 50 and then you just have to calculate I 48, 46 and so on and then just put it back into this formula and that will give you the answer.

So, instead of sin x you can have any trigonometrical function and you just have to find in some way to up to write that trigonometrical part as I n minus 2 to get the formula of type I n minus 2 or I n minus 1 and then we will basically obtain a nice representation of this type. So, it is all about playing with the trigonometrical functions and some formulas and you sort of obtain the required reduction formula. So, it is not that difficult.

And as far as the examples are concerned just substitute the how to say range of integration range of this definite integral and then you just calculate or play with these algebraic formulas. And as far as this induction formulas are concerned at the end of it you can refer to if you prefer to write a constant just write the constant otherwise it is understood that there will a constant be here when you calculate let us say for n equals to 10 when you calculate I 0 or I 2 ultimately would obtain a constant.

So, it is up to you whether you write this constant here or not because even if you write the constant here and at the end of it when you are calculating I 2 or I 0 then you will obtain another constants, a constant plus constant is a new constant. So, we prefer to write it as c. So, it is totally up to you whether you want to write the c or not it is under understood that it will have a c here.

So, we will stop today’s lecture here and in the next lecture I will try to derive 1 or 2 more reduction formulas and afterwards we will move to the improper integral. And in your assignment we will I will include some problems from reduction formula just to make the concept clear. So, I hope this lecture was fruitful to you and I look forward to your next lecture.

Thank you.