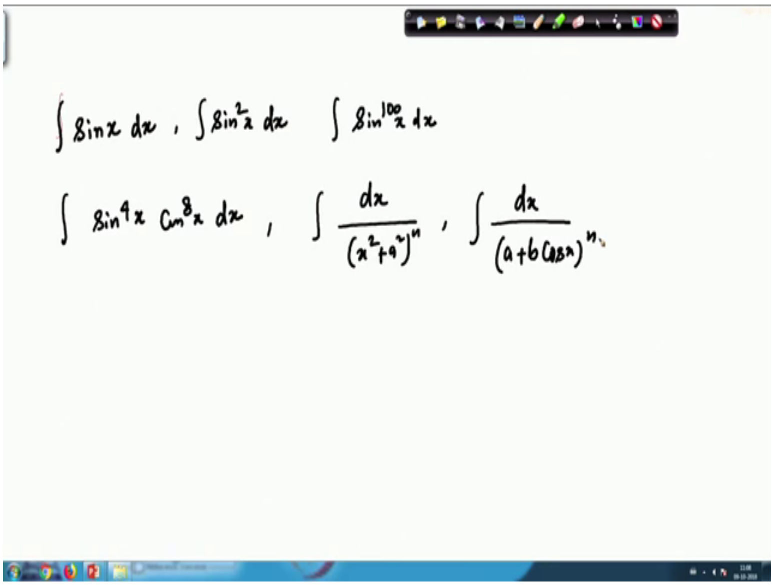
Integral and Vector Calculus
Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 07
Reduction formula

Hello students. So today we are going to start a new topic in our integral calculus section. So, up until last class we looked into Riemann integration, Riemann integrable functions as a fundamental theorem of integral calculus, mean value theorems, and we also worked out few examples. I will also try to include some problems in your assignment sheet, and I will also provide the solution, so that you will get to practice some problems from Riemann integration.

Now, today we will start with reduction formula, and we will derive some formula based on the based on different types of integrals, so where we use the reduction formula. So, what do we mean by reduction formula actually?

(Refer Slide Time: 01:21)

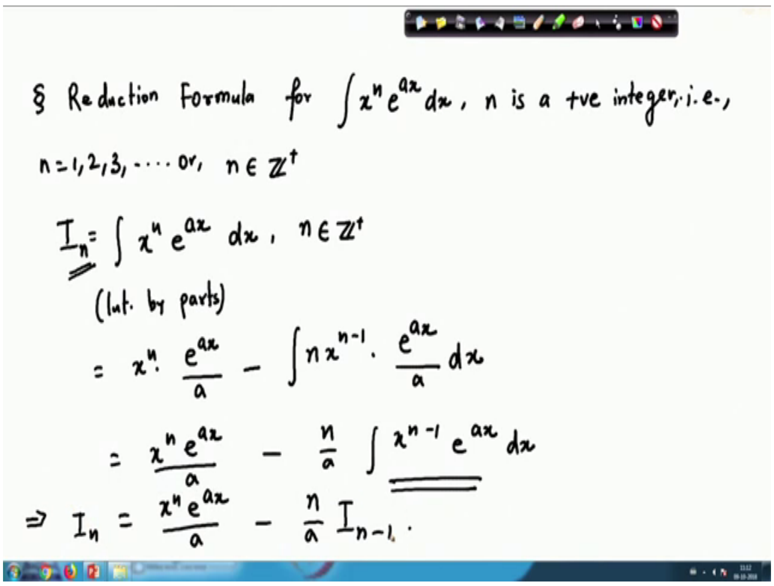


So, first of all in a integral calculus if I ask you to integrate a function of a type let us say a function of type let us say if it is sine x dx then its relatively simple to integrate. Even if it is a sine squared x dx, then it would also be quite simple to integrate you have to multiply by 2, and then you adjust the factor two and then you do the integration. But suppose if you have an integral of type sine to the power 10 x dx, or even worse if you have sine to the power 100 x dx, then in that case that it would not be that much straightforward to apply our traditional methods of integration.

So, in such cases you need some kind of how to say iterative formula or some kind of an induction formula that will how to say give you an easy way to calculate these type of integrals not only with one trigonometrical function, you could also have sine to the power 4 x dx, and then cos to the power 8 x dx. And if you are asked to evaluate an integral of this type then even in this case it will be a little bit tedious to evaluate this integral using our traditional method. So, then in that case you need induction formula to evaluate these integrals.

You can also have integrals of type dx by x square plus a square whole to the power n or dx by a plus b cos x whole to the power n. So, all these type of all these type of integrals can be put into some kind of reduction formula. And we will see how with the help of reduction formula, we can actually calculate these integrals very easily. So, let us start with our first reduction formula. So, let us go into a new page.

(Refer Slide Time: 03:39)



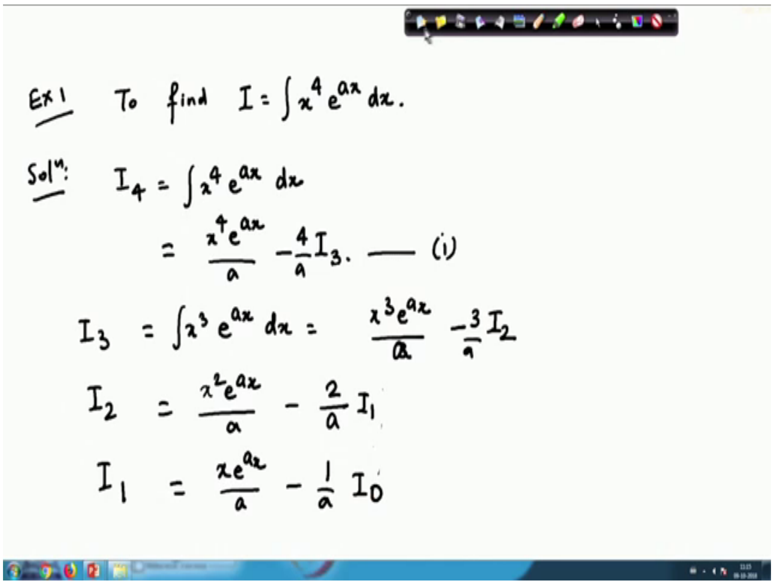
So, suppose we want to derive a reduction formula reduction formula for reduction formula for x to the power n e to the power a x dx, where n is a positive integer. So, when I say positive integer, it usually mean n is, so that is n is n is 1, 2, 3 dot dot and so on. And in short I will try to write n belongs to or n belongs to z positive. So, if I write z positive that basically mean that we are talking about set of all positive integers not 0, which does not include 0.

Now, you may have seen an integral of this type. So, instead of n it could be 1 or 2 or any positive integer. So, if you have any positive integer sitting at x to the power sitting at the place of n, then how do you evaluate actually. So, instead of doing the traditional integration by parts, we will derive an reduction formula that will help us evaluate for any n. So, let us start.

So, our given integral I equals to x to the power n e to the power x dx. Now, since we have n here I can put a small n here, and so where n is a positive integer. Let us write in short ok. So now, I will do in the integration by parts. So, I will do basically integration by parts. So, if I do integration by parts, then this will become x to the power n times e to the power a x divided by a minus integration differentiation of the first term, so n to the power x n minus 1 then integration of the second term. So, e to the power a x divided by a dx and then this will become x to the power n e to the power a x divided by a minus n by a x to the power n minus 1 times e to the power a x dx.

So, now if you see this term here; if you see this term here, this is nothing but I n minus 1. So, if I take here, if I take in place of n, if I take n minus 1 then this formula is basically x to the power n minus 1 times e to the power a x dx. So that means, this will be equal to x to the power n e to the power a x by a minus n by a I n minus 1. So, this is basically our reduction formula. Now, this implies I n equals to x to the power n e to the power x by a minus n by a times n minus 1. So, this is our required reduction formula all right.

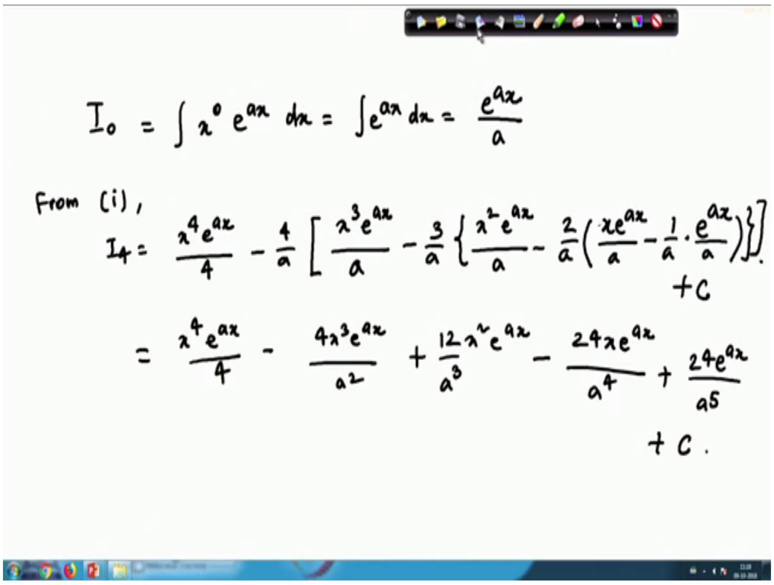
(Refer Slide Time: 07:31)



Now, let us say we want to apply this reduction formula to find, so example 1. So, let us say we want to use that reduction formula to find integral of x to the power 4 e to the power a x dx. So, as you can see if I go with the traditional integration by parts formula, then this will be a little bit how to say how to extensive in a way I mean it will require some time to calculate this integral, instead we will use the reduction formula. So, how we can write solution?

So, first of all n equals to 4, so we write I 4 equals to I 4 equals to x to the power 4 e to the power a x dx. And this can be written as x to the power 4 e to the power a x dx by a minus I 4 minus 1, so I 3 and so I 3. So, this is basically I 3. Next, we have to calculate I 3. So, let us calculate I 3. I can name this equation as equation 1. Now, I 3 will be x to the power 3 a x, and then this can be written as x to the power 3 e to the power a x by 3 by a minus I 2. Then again I 2 can be written as x to the power e to the power a x by a minus 2 by ok. So, here it is n by a. So, n is 4 here its 3 by a, then I 2, and then this one is 2 by a, then I 1. And again I 1 equals to x e to the power a x by a minus 1 by a I 0.

(Refer Slide Time: 09:57)

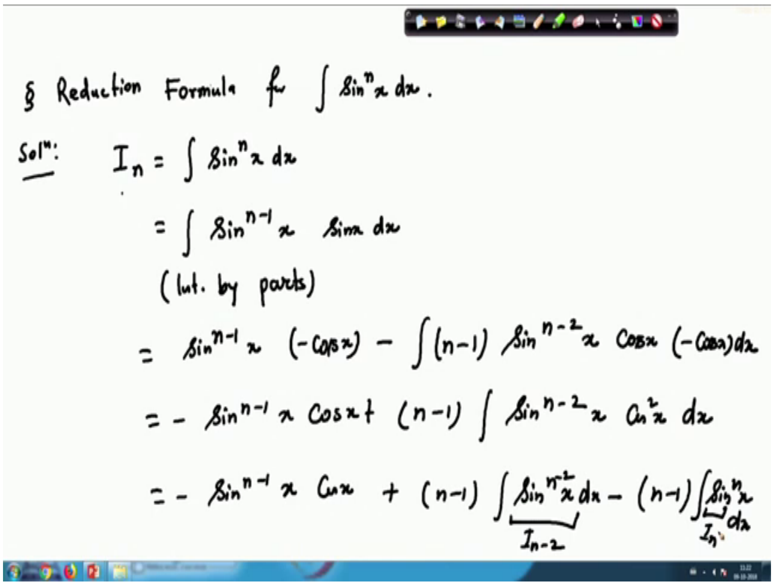


Now, I can calculate I 0. What is I 0? I 0 is x to the power 0 e to the power a x dx. So, x to the power 0 is 1. Then we are left with e to the power a x dx and e to the power a x dx is basically e to the power a x by a. So now, we will go backwards. So, we will go backwards. And we can write we can write from 1, we have I 4 equals to x to the power 4 e to the power a x by 4 minus 4 by a I 3. What is our I 3? I 3 is x to the power 3 e to the power a x by a minus 3 by a times I 2. What is our I 2 x e to the power a x by a minus minus 2 by a times I 1.

And what is our I 1? I 1 is I 1 is x to the power I 1 is x to the power e to the power a x by a minus 1 by a times I 0; and that I 0 is e to the power a x by a, I will close all these brackets. So, basically if I multiply plus a constant c of course, so at every step we will get a constant. So, since it is a definite integral at every step, we will get a constant. And we can finally, write a big constant at the end. And then we can write this as x to the power 4 e to the power a x by 4 minus 4 x e to the power a x by a square minus minus plus 12 a square 12. So, this one is 4 12 x square e to the power a x whole to the power 3 minus 24 x e x a to the power 4 plus this will be 24 again e to the power a x by a to the power 5 plus the constant c.

So, this is our how to say this answer or the solution of this problem. So, as you can see that instead of doing integration by parts again and again. We basically use the reduction formula to calculate the value of this integral. So, this is our required induction formula for any form of x to the power n times e to the power a x dx type. So, you can play with any power here. So, instead of x to the power 4, we can even take x to the power I do not know 20. And then you just calculate I 20, I 19, I 18, I 17 dot dot up to I 0, and then you just put them together and that will give you the value of I 20 actually. So, this one was how to say pretty straightforward.

(Refer Slide Time: 13:49)



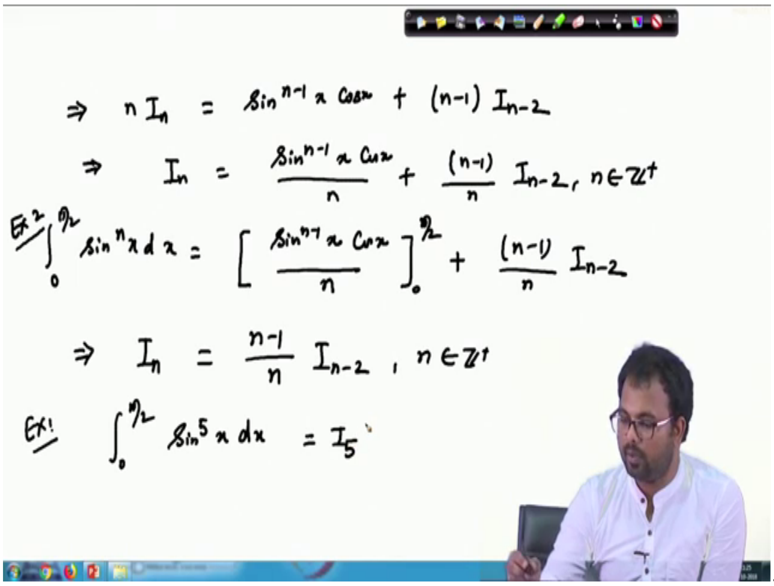
Next let us say you have an integral; next let us say you have you have an integral of time; so reduction formula for integral of sine x whole to the power n x dx. So, in order to find the reduction formula for this, we can write again I n integral from sine x whole to the power n dx. Now, we can do the integration by parts. So, if I do the integration by parts, I can first write this as sine to the power n minus 1 x dx times sine x dx. And then I will use integration by parts.

So, if I use integration by parts here, this will reduce to sine to the power n minus 1 x times minus of cos x minus n minus 1 sine to the power n minus 2 x times cos x times minus of cos x dx. So, it is like differentiation of the first function which was sine n minus 1 x times integration of the second function which was sine x. So, hence we are getting minus of cos x. So, this can be written as minus of sine n minus 1 x times cos x minus of n minus 1. So, minus and minus will turn into plus. And then this will be sine n minus 2 x times cos square x dx. And we can write cos square x dx as 1 minus sine square x dx. And therefore, this will reduce to sine n minus 1 x times cos x plus n minus

1, and this will be sine whole to the power n x dx. So, this will be 1 minus sine square x. So, 1 minus, so this is n minus 2 and minus n minus 1 sine whole to the power n x d x.

So, now I can write this as let us go to a new page. So, here this is sine n minus 2 x dx. So, this is basically my I n minus 2. So, if I take instead of I, if I take I n minus 2, then this will be basically sine n minus 2 x dx. So, this is our I n minus 2, and this is again our I n. So, I will bring I n on the left hand side.

(Refer Slide Time: 17:07)

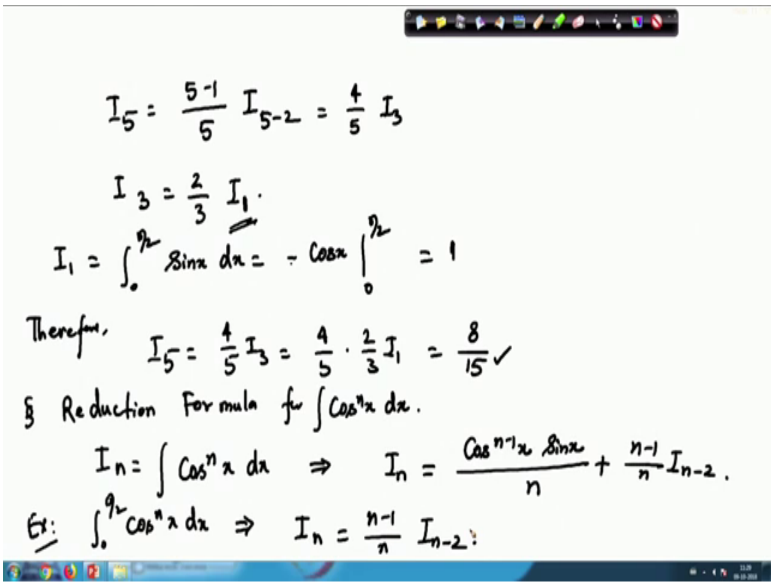


So, this will become n times I n sine n minus x cos x the plus n minus 1 times I n minus 2. So, if I divide by n on both sides, since n is a positive integer, we can do that cos x divided by n plus n minus 1 divided by n times I n minus 2. So, this is our required reduction formula for sine n minus sine to the power n x dx. So, if you have instead of n, if you have let us say 20 or even 10. Then in that case we just have to put n equals to 10 or 20 whatever it is and then we keep on calculating I n minus I n then I n minus 1 and 2 dot dot so until we get here I 0. And then we can easily calculate I 0 and that will basically give us the answer of the required reduction formula.

Now, suppose we have suppose we need to calculate, suppose we need to calculate integral 0 to pi by 2. So, this is our required integral formula, we can write n belongs to positive integer. Suppose, we are asked to calculate integral from so this is basically an example sine n x dx. So, what, what this would be this would be basically sine n minus 1 times x cos x divided by n integral from 0 to pi by 2. And this will be a definite integrals of course, I am writing n minus 2, but this is basically my definite integral, and at x equals to at x equals to pi by 2 cos pi is cos pi by 2 is 0 and at x equals to 0 sine 0.

So, basically that one is also 0. So, the first term will actually vanishes and then we are left with left with I n equals to n minus 1 divided by n times n minus 2, where n is a positive integer. So, if you are given this range from 0 to pi by 2, then the required reduction formula would be I n equals to n minus 1 by n times I n minus 2, some people also denote it by j. So, instead of using I not to confuse with confuse, with this I they introduce j, but that does not matter here. So, for example, let us say if we are asked to calculate a integral from 0 to pi by 2, sine to the power 5 x dx, then this will be basically our, so suppose if we are asked to calculate this. So, let us call it as I 5.

(Refer Slide Time: 20:31)



And I 5 would be basically. So, I 5 would be basically 5 minus 1 by 5 I 5 minus 2. So, this is basically 4 by 5 I 3 and I 3 is 3 minus 1. So, 2 by 3 and then 3 minus 2 is 1. So, what is our I 1. So, our I 1 is integral from 0 to pi by 2 sine x dx. So, integral of sine x is minus cos x 0 to pi by 2. So, cos pi by 2 is 0 and cos 0 is 1. So, this is basically plus 1, because we have a minus here. So, thus or our I 5 is 4 by 5 times I 3 which is 4 by 5 times 2 by 3 times I 1, and this will be 8 by 15, because I 1 is 1.

So, this is our required answer to these reduction formulas instead of n equals to 5, you can have n equals to 15. And you just proceed in the similar manner. You may have noticed a trick here that when n is odd that means if you have n to the power 5, 7, 13 or

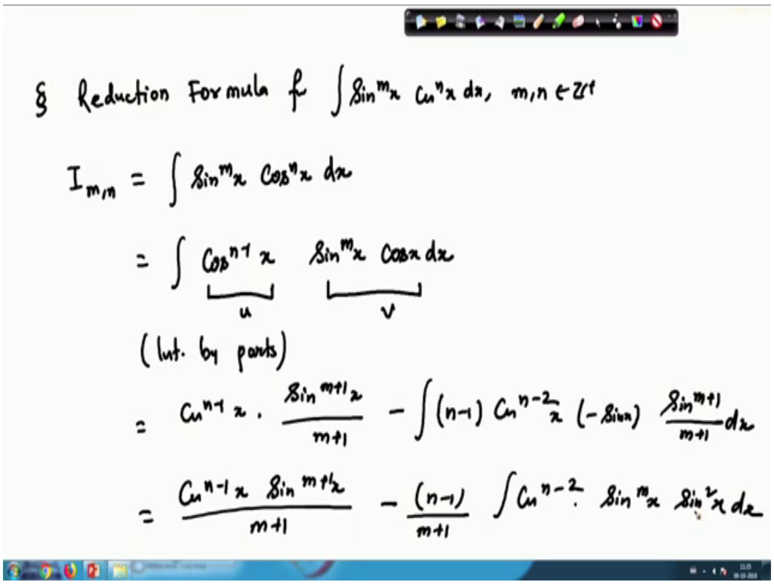
15, then you will basically end up with calculating I 1. And if n is even, that means, if you have sine to the power 6, 10, 14, then in that case you will end up with I 0. And sine to the power 0 is basically 1. And then you just calculate integral from 0 to pi by 2 dx, and that will give you the answer. So, if n is odd, then you end up with calculating I 1; if n is even, then you are end up calculating I 0 at the end, so that is the trick here.

Now, that we have the reduction formula for sine x, the reduction formula for cos x; so reduction formula reduction formula for cos to the power n x. So, this one is cos to the power n x dx. So, this follows on the similar footsteps basically. So, the way we have calculated the induction formula for sine to the power n x dx, we can also calculate the induction formula for sine to the power cos n x dx. You just have to write it as cos n minus 1 times cos like the way we did for the sine. So, I leave this task to the students. And I am pretty sure you can be able to do it.

So, I am just writing the formula. So, the formula would be from here it would imply our reduction formula would be I n equals to cos n minus 1 x times sine x divided by n plus n minus 1 by n I n minus 2. And if we have if we have example, if we have let us say integral from 0 to pi by 2 cos n x dx, like we saw in the sine sine n x dx, then from here our I n would be just n minus 1 by n 1 I n minus 2. So, it is pretty much the same actually like the formula we looked for in sign n sine to the power n x dx.

So, we can also solve the similar examples like we did for the sine x part. So, I am going to I am going to leave those examples for cos n to the power x dx. You can look into any book which I put into the references and there you will be able to find these kinds of examples. Next is how to say a little bit interesting. So, instead of having just one trigonometrical function, you can have the product of trigonometrical functions, and then how you deal with calculating their integral.

(Refer Slide Time: 25:01)



So, let us say you have a reduction, you have an integral of the following type. So, we are calculating the reduction formula reduction formula for sine to the power m x d m x cos to the power n x dx, where m and n are both positive integers. So, the reason we are taking the positive integers is that I mean this is a very vital point. So, the reason we are taking a positive integers is that here we are always ending up with either I 0 or I 1. But if we take fractions, then we might end up with some kind of fraction here or the how to say this whole reduction formula will turn into a totally different; it may not follow how to say this type of nice integral representation.

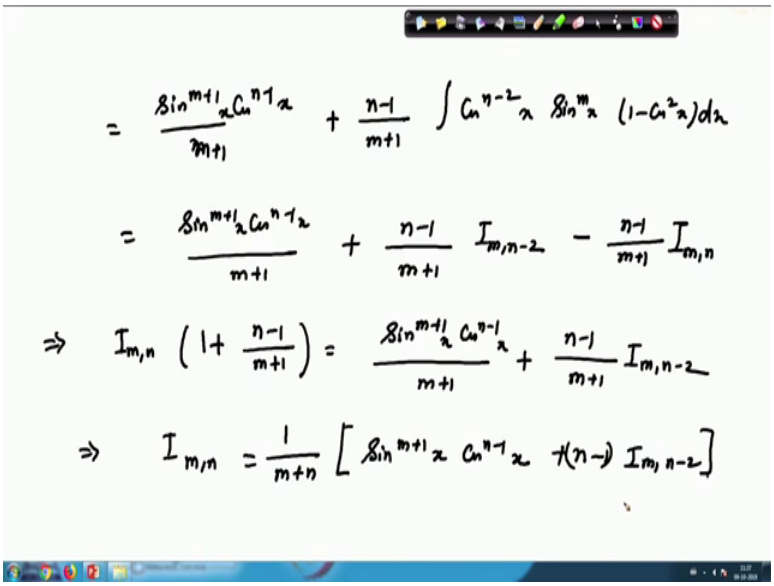
So, in order to in order to put it in a nice representation form like we have here or like we have here. So, these nice representation would not be how to say straightforward and that is one of the reason we are picking we are choosing basically we are choosing that is one of the reason we are choosing m n n as the positive integer. So, as I was saying a reduction formula reduction formula, for sine m x cos n to the power x dx, where m and n are both positive integers.

So, to derive the formula let us write since now we have two integers. So, I can write I m and n sine m x cos n x dx all right. And then we take this, this whole thing as the product of cos n minus 1 x times sine m x cos x dx. And then I choose this function as u and this entire function as v. And based on this, now I can apply the induction of the how to say a it is the integration by parts. So, now, I can apply integration by parts, integration by parts.

So, to do that if you calculate the integration by parts part, then this is basically cos n minus 1 x times integration of the first function unchanged integration of the second function. So, if I integrate the second function, then you can see that cos x dx is basically how to say it derivative of sine x, so the cos x is derivative of sine x. So, if I substitute sine x equals to z, then this will turn into cos x dx d z. And ultimately we will be able to obtain z to the power m plus 1 by m plus 1. And since our z is sine x it will reduce to sign m plus 1 by m plus 1. So, basically to calculate the integral of this thing we use the method of substitution.

Now, minus the derivative of the first function, so n minus 1 times cos n minus 2 x minus sine x; and then again if I integrate, then this will reduce to sine m plus 1 divided by m plus 1 dx. So, if I if I play with all these how to say functions, and if I use those how to say formulas, then in that case this will basically reduce to cos n minus 1 x times sine m plus 1 x divided by m plus 1 and minus n minus 1 divided by m plus 1. So, this will be sine m plus 2 x sine m plus 2 x and sine m plus 2 x can be written as cos n minus 2 times sine m x times sine square x. And now I can write sine square x as 1 minus cos square x and then this will become.

(Refer Slide Time: 29:59)



So, I can write sine square x as 1 minus cos square x, and then this will become sine m plus 1 by x times cos n minus divided by m plus 1. And this will become plus n minus 1 divided by m plus 1 cos n minus 2 x times sine m x times 1 minus cos square x. Now, if I see if here, so this is basically if I take the first term, then this is basically I n minus I m n minus 2 and this will be again minus I m n. So, I can write this as sine m plus 1 times cause n minus 1 x dx divided by m plus 1 plus n minus 1 divided by m plus 1 I m n minus 2 minus n minus 1 divided by m plus 2 divided by m plus 1 sorry this will be I m n.

And if I take everything on the, if I take everything on the on the right hand left hand side, then this will become I m n 1 plus n minus 1 m plus 1. And then on the right hand side, we will have sine m plus 1 cos n minus 1 divided by m plus 1 plus n minus 1 m plus 1 I m n minus 2. So, next we will divide by m plus 1 on both sides. And we will take m plus n on the left hand side on the denominator on the right hand side. So, this will become 1 by m plus n sine m plus 1 x cos n minus 1 x plus n minus 1 I m n minus 2.

So, this is basically our required reduction formula for product of two trigonometrical functions of this type. And we will use this formula to calculate how to say the product of two trigonometrical functions integral of the product of two trigonometrical functions. And we will see that in our next class.

Thank you.