Integral and Vector Calculus
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Lecture - 60
Overview of Course

Hello students. So, in today's lecture we will continue with some examples on Stokes theorem and since today is our last lecture I apologize that I could not be able to cover more examples on integral calculus section as I promised or showed you in our syllabus. But I try to compute as many examples as I could also because the time is very limited and because of that sometimes I also had to rush through and maybe on one or two occasions we could not be able to do more examples.

But I suggest you to follow any one of the textbooks or if possible, then follow all the textbooks which I suggested in the reference what are the beginning of this course. And I am pretty sure there you will be able to find a lot of examples just for you to practice and get ready good at it and I am also providing a lot of assignments. So, try to solve those assignments that will definitely help you clear out your doubts and would also help you to become how to say very good at this particular subject.

So, today let us start with our last topic which is stokes theorem and I am going to solve some examples and let us do that.

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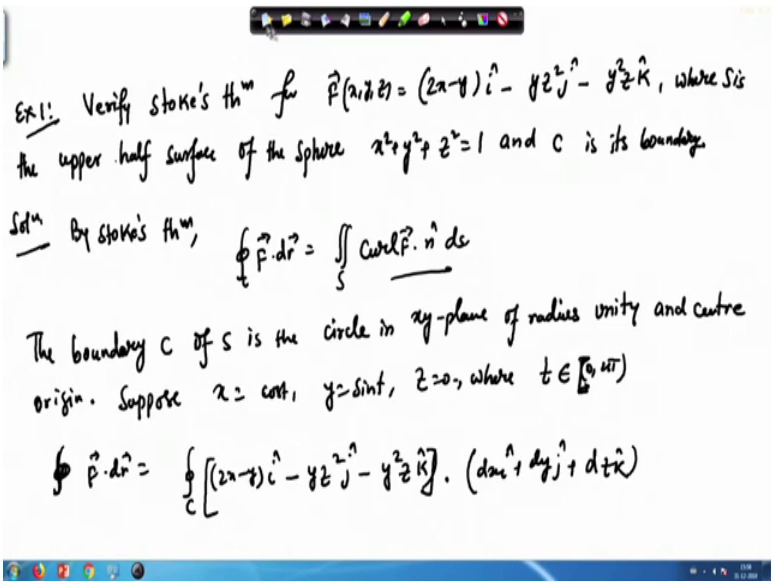

So, verify Stokes theorem for F equals to 2 x minus y times i minus y z square j minus y square z k where S is the upper half or upper half surface of this sphere and C is its boundary. So, what we have basically is a sphere and we have to verify the stokes theorem on the upper half of that sphere so; that means, if you consider let us say x axis y axis and z axis, then it will it we can consider the upper half as from z equals to 0 till z equals to 1 so; that means, that could be one upper half or x equals to 0 till x across to 1.

So, that can be another upper half. So, any one of the upper half basically and in this case to do that first of all to verify actually the Stokes theorem; we write the statement. So, by Stokes theorem what do we have? We have line integral F dot d r equals to surface integral curl of F dot n ds right. So, we have to so that left hand side is equal to the right hand side.

So, let us start with the left hand side. So, its line integral F dot d r. So, here C is the boundary in the xy plane and we are considering basically z equals to 0 part. So, z equals to 0 to z equals to one and if C is the boundary then in xy plane then it is basically that circle right. So, that is basically the circle and if you walk along that circle then in the anti clockwise direction, then your surface is falling on the left hand side, so, that makes sense.

So, let me invite one or two lines before I actually calculate this line integral. So, in order to do that; so, in order to do that the boundary so, let us write the boundary C of the surface S is the circle in xy plane of radius unity and center origin and the parametric representation of that circle would be x equals to cos t y equals to sin t and z equals to 0. So, now, the line integral and t is basically where t is between 0 to 2 pi alright. So, now, we have line integral F dot dr. So, F is basically 2 x dot dr; dr is basically dx i plus d yj plus dz k.

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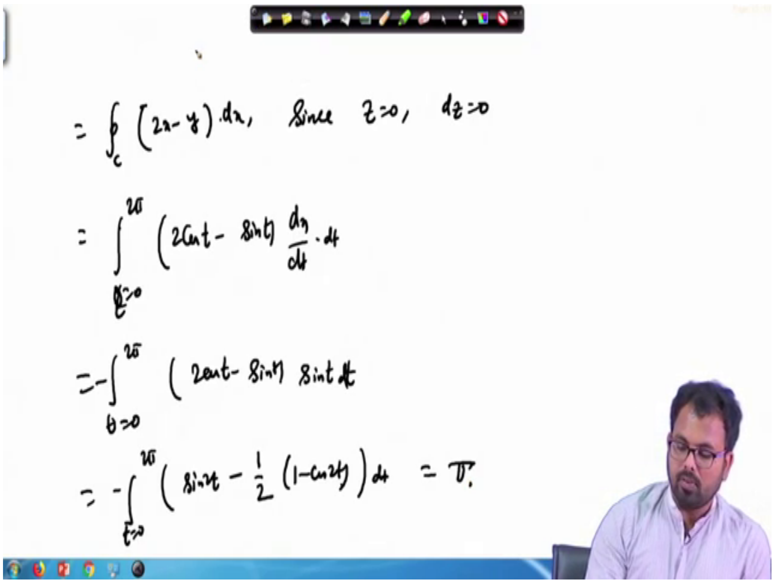

Now if we take the dot product then this will be basically 2 x minus y dx since z equals to 0 and dz is 0. So, here d z will be 0 because on the curve C, we have taken z as 0 right. So, on the curve C, we have taken z as 0. So, we substitute that equals to 0 here and here and we substitute dz equals to also 0. So, ultimately we will get 2 x minus y dot times dx. So, that is what we are obtaining here. And now we substitute x equals 2 cos t and y equals to sine t. So, this will be t running from 0 to 2 pi 2 x means 2 cos t minus sin t dx dt times dt. So, this is sorry this is t t running from 0 to 2 pi 2 cos t minus sin t and x is cos t. So, dx will be minus of sin t.

So, I can write the minus sign here sine t dt and then with multiply this sin t here. So, this will be 2 sin t cos t so; that means, sin 2 t and then we will have a minus minus plus 2 sin squared t which can be written as sorry this is dt so, which can be written as 1 minus cos 2 t. So, that is cos and sin formula we have to remember and then you substitute the then you substitute those calculations here. So, this will be t running from 0 to 2 pi; we have sin 2 t minus half 1 minus cos 2 t.

So, we just are just a 2 here dt and then we integrate term by term and ultimately we will obtain pi right. So, ultimately we will obtain pi. Now we calculate; now we calculate the right hand side.

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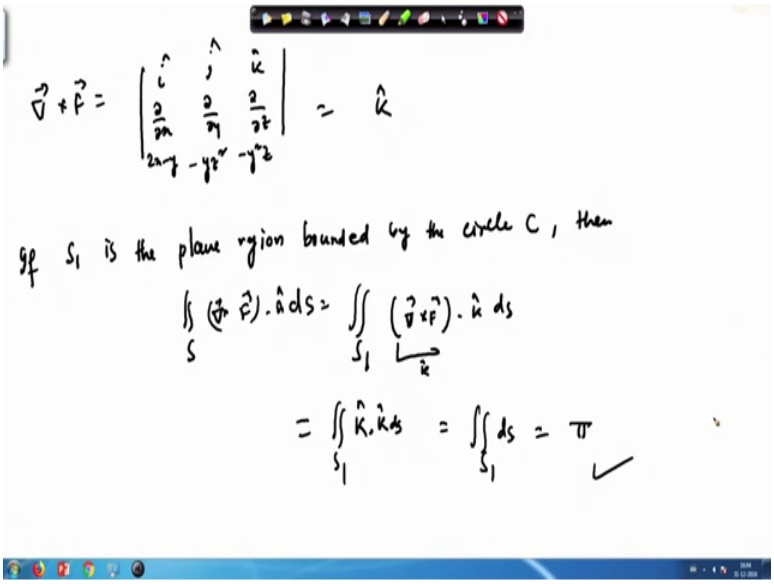

So, first of all we have to calculate curl of F. So, this we have i j k and F would be 2 x minus y as of y z square minus of y square z. So, when we do the partial derivative, we will ultimately obtain at k because rest of the term will be either cancelled out or they will either be 0. So, they do not play any role and therefore, the component of ij and k will be i j will be 0 and k will be one. So, that is why we are only writing the vector unit vector k and n is the unit outward wrong normal on the surface S.

So, if our n c is the curve basically that is enclosing that surface and if n is the outward or normal so, you have a sphere on the upper half of the z axis and then if you have a unit on normal n, then you can choose k axis itself as your unit normal right because k axis is normal to that surface of the sphere and we can actually choose it instead of calculating we can actually choose k as the unit vector.

And therefore, we can write if S 1 is the plane region as the plane region bounded by the circle C, then so, basically we have this as surface integral over S 1. Let us say we have gradient of curl of F dot n n is basically k and now this thing is also k. So, k dot k ds therefore, k dot k is 1 this will be surface integral over S k square and this since k is a unit vector. We can write k dot k as 1. So, this is basically ds and a surface integral over S 1 and this can be written as d s pi r square r is 1.

So, then we have basically pi. So, our left hand side was pi and our right hand side is also pi. Therefore, if the left hand side and right hand side are both same then by Stokes theorem, we can be able to say that the given integral or the given problem satisfies the Stokes theorem which means that the surface integral and the line integral both are same or both hold true all right. So, that is one way we can solve our problems in Stokes theorem motivated from stokes theorem.

Similarly you can have several other problems from several other problems from Stokes theorem where you either convert your stove your surface integral into a line integral or you can convert your line integral into a surface integral and just calculate either one of them. So, if one expression is given to be complicated try to use the Stokes theorem to convert it into a simpler one and see if you can calculate that one or not alright.

So, let me give you a quick recap of these labels. So, before I proceed any further.

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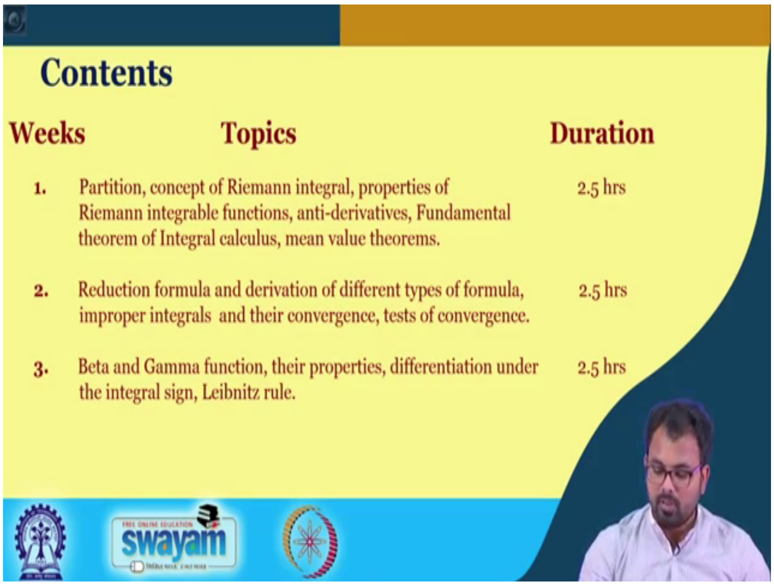

So, we started so, we initially we started this integral and vector calculus course from Riemann integral or from the concepts of Riemann integral. So, I know that in your plus 2 or even when you were preparing for other competitive exams, you must have learnt a little bit about indefinite integrals and definite integral. Those are all very nice form of integrals or sometimes they are also called as Newtonian integral. So, where you look for the anti derivative and you either by method of substitution or whatever methods are unknown, you just use them to calculate the integral.

So, more or less since you were familiar with the integral of a function or integral of single variable. We started with concepts of Riemann integral I gave you introduction of partition concepts of Riemann integral; Riemann integrable function fundamental theorem of integral calculus mean value theorems and things like that and based on which we will also provide you some examples or assignments to practice.

Then we looked into reduction formula and derivation of different types of reduction formula. So, sometimes you may be asked to calculate sin to the power 13 x dx. So, then in that case you just have to use that reduction formula for sin to the power nx dx and from that formula you can be able to calculate whatever power is given there.

So, the deduction formula is a very nice tool where you just see if there is some kind of liquidity formula there, so, that you can come to a generalized formula. And then we learnt about improper and proper and improper integral their convergence, we also learnt about some tests to do the convergence all right. And afterwards we moved to beta and gamma function, we looked into their properties we learnt about differentiation under the integral sign; Leibnitz theorem and things like that.

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Afterwards we looked into double integral, we also learnt about change of order of integration in Jacobian transformation, then we moved to the applied path a part of integral calculus where we learnt about rectification and the surface integral. We learned about volume integrals and several other types of volume integrals the center of gravity and moment of inertia. I did not cover that part because from integral calculus perspective I think it, it might go a little bit in the different direction. So, if anyone of you are interested you can look into those references; I was since I was focused to also cover the vector calculus part I gave my attention to those two topics.

So, these are the two topics which I initially proposed, but then over the time I realized that this might lead to problems from a different parts of mechanics or something. So, it is better that we should not do it at the moment, but if any one of you are interested you should just look into those books and you will be able to find these two topics there. And then we moved to our vector calculus part where we looked into the limit continuity differentiability of a vector function, arc length, partial derivative of vector function, directional derivative, gradient divergence and curl operators. So, we learned about different types of notions in vector calculus. We also practiced some examples.

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Then we learnt about tangent normal and by normal and then we derived one of the important formulas in vector calculus which is Serret Frenet formula we also looked into the applied part of vector calculus where it is used and how to derive some equations of motions and things like that. We then looked into 3 important theorems like Gauss divergence theorem, Stokes theorem and the Green's theorem.

So, up until previous class I did try to cover Green's theorem, Gauss theorem and today we are we learnt about Stokes theorem. We also tried to solve one example motivated from the Stokes theorem. And in our 12th week or in our last week, I thought maybe I could practice more examples from integral calculus or vector calculus, but due to the time constraint unfortunately we cannot be able to practice more examples. But I would suggest you to look into those books, there you will find plenty of examples for you to practice and work out problems from our assignment sheet that will also help you get a sense of this topic. And this is what we tried to cover in a 12 week session actually. Although I personally feel that it might need a little bit more time to cover these two topics in detail; however, we tried our best to do that.

So, just go through these lectures and try to get sense a feeling of the topic whatever we learned and talked about and if you have any questions or doubts you, I believe we have a weekly sessions where we can clear out the doubts. Also if you have any questions, you can just go in to my home page and write me an email I will be; I will be more than happy to answer your questions and try to help you with your queries. So, we do not have to just worry about things you do not have to just worry about the fact that you can only contact me why this NPTEL format or something.

So, if you have any in general any questions or doubts on integral calculus or vector calculus or which you could not be able to follow in this lecture series, then just write me an email go to the IIT Kharagpur website, try to find out my profile there. You find my email address and write me an email and I will be more than happy to answer your queries.

So, from my side I tried to cover as many topics as I could and I hope you will be able to learn these things more clearly and I look forward to your questions your queries and yeah good luck. Since we still have at least a 5 minutes left so, let me just state one more example from Stokes theorem and then we will move I will close this lecture. So, let me go back to my alright.

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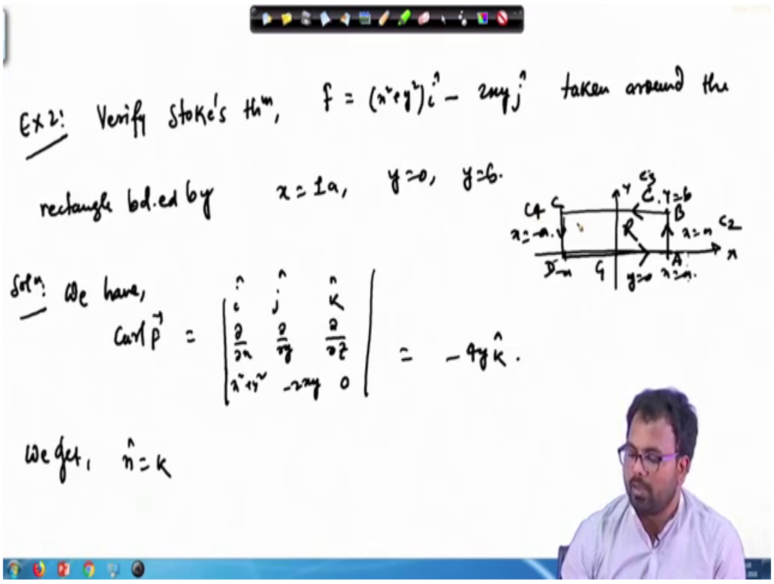

So, since we have one more some time we can focus on one more example. So, example 2 from Stokes theorem. So, verify stokes theorem F is equals to x square plus y square times i minus 2 xy j taken around the rectangle bounded by x equals 2 plus minus a and y equals to 0 and y equals to b. So, this is just one more example for us to practice. It is really not that much complicated it is a lengthy problem, but it is really not complicated.

So, if we want to draw this rectangle, we can just do that x equals 2 minus a x equals to a and this is our x axis, this is our y axis and then we have y equals to 0 and we have y equals to b sorry. So, this is our x equals to n and our rectangle this is our rectangle. So, we our curve C will move like this. So, this is our curve C alright and the equation of this line is x equals to a, this line is y equals to b, this line is x equal y equals to 0 and this is x equals to minus a. So, these are the 4 equations and let us call it as A B C and D. So, first of all in order to verify we have to calculate the surface integral and the surface integral can be calculated by so, we have.

So, we first need to calculate curl of F. So, curl of F can be calculated using that determinant form which is basically i j and k and then we have del del x del del y and del del z. And we can write the x component x square plus y square y component 2 x y and then that component is 0. If you calculate this determinant, then ultimately you will be able to obtain minus 4 y k. And unit normal since this whole curve this sorry this whole surface lies in the xy plane and out of drawn unit normal would be any cost all right.

Because z axis is obviously perpendicular to the xy plane and you can choose an escape. So, we choose or we have or we get n equals to k since the given surface is in xy plane.

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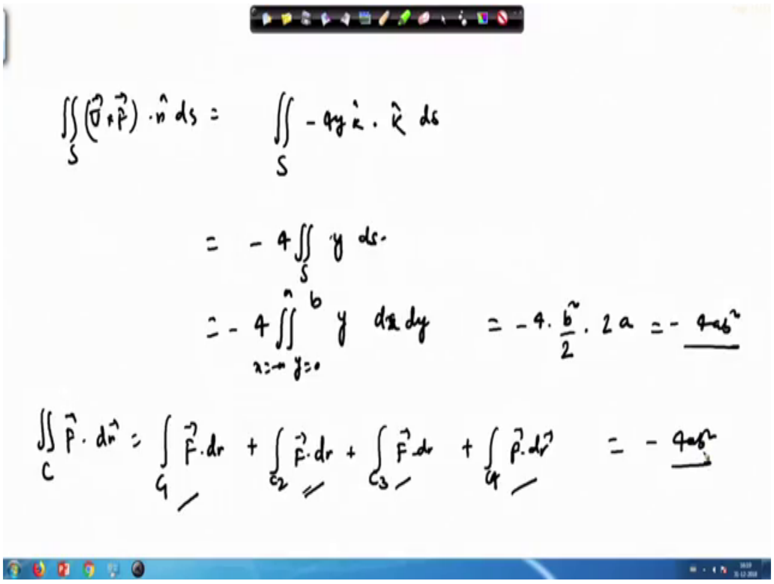

So, therefore, if we want to calculate the surface integral curl of F dot nds. So, curl of F is basically our minus of 4 yk dot product with k ds and this is basically minus of 4 y. So, this is basically minus of 4 s y d s and d s is that surface element in xy plane. So, this can be written as minus of 4 surface integral x.

So, x is varying from minus a 2 plus a and y is varying from 0 to b y d s is basically dx dy and if we integrate this whole thing, then in that case we first integrate with respect to y. So, this will be minus of 4 times b square by 2 and then we integrate with respect to x. So, this will be; so, this will be this will be 2 a y. This will be ultimately 2 a so; that means, we will have minus of 4 a b square.

So, that is the value of the surface integral. Now for the line integral; for the line integrals C here, we have to be a little bit careful because our curve is composed of 4 curves; our curve C is composed of 4 mini curves. So, this is our C 1, this is our C 2, this is our C 3 and this is our C 4 so; that means, we can write this integral over c as the union of C 1 C 2 C 3 C 4 and union of C 1 C 2 C 3 C 4 can be written as sum of integrals on C 1 plus C 2 plus C 3 plus C 4. So, we can write C 1 F the dr plus C 2 F dot dr plus C 3 F dot d plus C 4 F dot dr. So, we have to evaluate all of these 4 integrals right.

So, when you first evaluate over C 1, you have to take care of the fact that the equation on C 1 we have y goes to 0; then dy is 0 and you just; and you just integrate from minus a to plus a. So, that will be the value on C 1. Similarly when you are integrating on C 2, you have to take x equals to a and y is varying from 0 to b. So, and so, dx will be 0 x is a.

So, we calculate the second integral. Similarly we calculate third and fourth, then you sum all of them and you will be able to see that the required answer is minus 4 a b square. And therefore, your Stokes theorem will be verified. So, this was one more example where you could be able to verify the Stokes theorem.

So, I hope these examples have cleared out your doubts based on Stokes theorem and I believe we have exhausted the time. So, I will stop here for now and I hope you enjoyed this course and I will look forward to your questions your suggestions, your remarks. If you have anything and based on this course and I will try to answer them, I will try to reply to you as soon as possible and of course, I am always reachable by email. So, do not hesitate to contact me.

So, thank you for attention and have a good time.