Integral and Vector Calculus
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Lecture – 06
Examples (Contd.)

Hello students. We will continue our lecture with the examples on Riemann integration. So, in the last class the we looked into the theorem on if a function has finite number of discontinuity or if the set of points of discontinuity of the function has only a finite number of limit points then the function f is also Riemann integrable.

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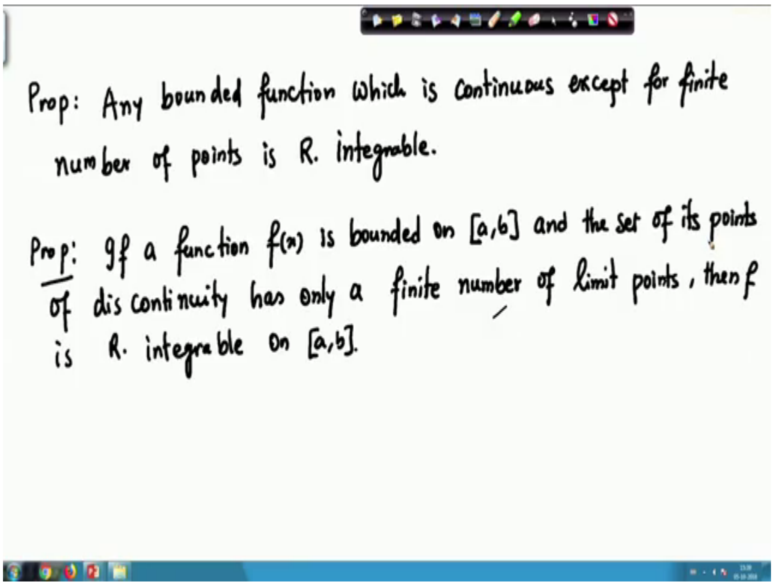

So, what do we mean by these 2 theorems we will look into by an example.

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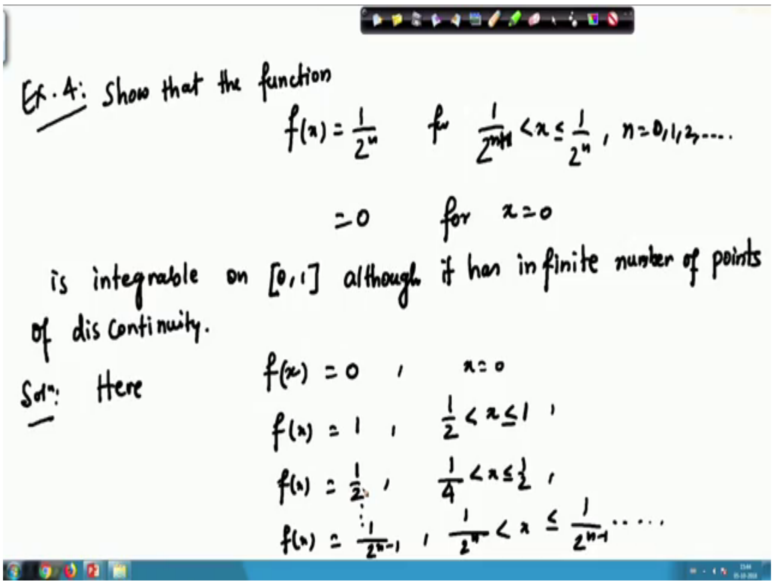

So, let us consider this example here.

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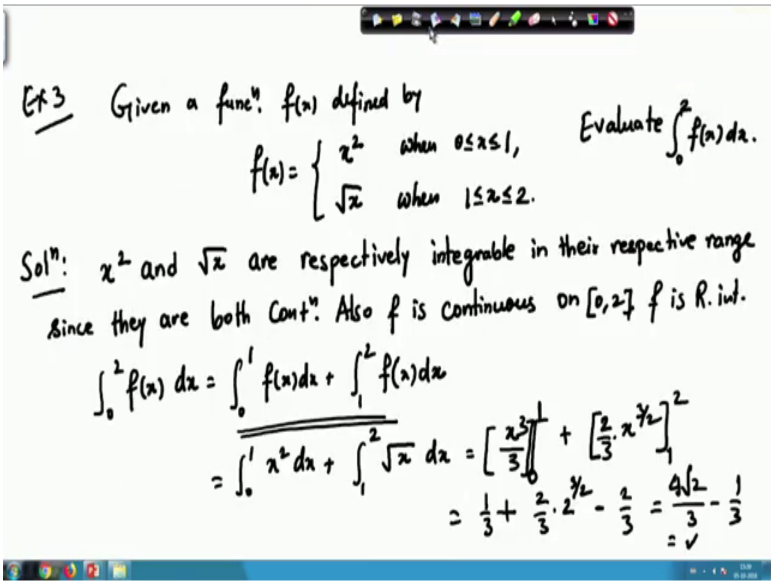

So this will be our example 4.Show that or show that the function f x equals to 1 by 2 to the power n for 1 by 2 to the power n minus 1 is less than x less than equal to 1 by 2 to the power n and equals to 0 for x equals to 0 and here n is running from 0, 1, 2 dot dot and so on.

Now, is integrable on 0 comma 1 although it has infinite number of points of discontinuity. So, here the problem states that if you have a function f x given in this fashion then we have to show that it is Riemann integrable although it has an infinite number of points of discontinuity. So, let us see whether it has infinite number of discontinuity or not. So, here let us say when n is 1 so first of all f 0 is 0 when n is so when n is 0 so sorry here it is 2 n plus 1 to 2 n yes, so it is 2 n plus 1 to 2 n, so there was a small correction. So, when n is 0 then it is f x equals to 1 for x between half to 1 then f x is 1 by 2 for x between 1 by 4 to 1 by 2 and dot dot and so on it will be f x equals to 1 by 2 to the power n minus 1 for x between 1 by 2 to the power n less than equal to 1 by 2 to the power n minus 1 and so on. So, it will continue.

So that means if we see f when x equals to 0 when x equals to 0 the function f x is 0 actually. So, here instead of having 0 you can write it as f x just to keep the same notation. So, when x is 0 f is 0 when x is between half and 1 then f x is 1 when x is between 1 by 4 to half then f x is half and so on. So that means, the function f it is how to say monotonic. So, we are getting the value 1 half and so on, so that means, the value is continuously decreasing. So, the function f is basically monotonic. And secondly, we can see that at x equals to 0 the function is discontinuous at x equals to half it is again this continuous at x equals to 1 by 4 it is again discontinuous and so on. So that means the function has infinite number of points of discontinuity because n is running up to infinity. So, the function is always discontinuous at the point this 1 by 2 to the power n type.

Now, we know that the function f is monotonic, we can see that from here and also the function is bounded, because the maximum value it can attain is 1 and the lowest value it can attain is 0. So, when n goes to infinity then this will become 1 by 2 to the power n and so sorry when n goes to infinity then 1 by 2 to the power n will go to 0, and consequently the maximum value of this function f can attain is 1 and the minimum value it can attend is 0. So, it is also bounded.

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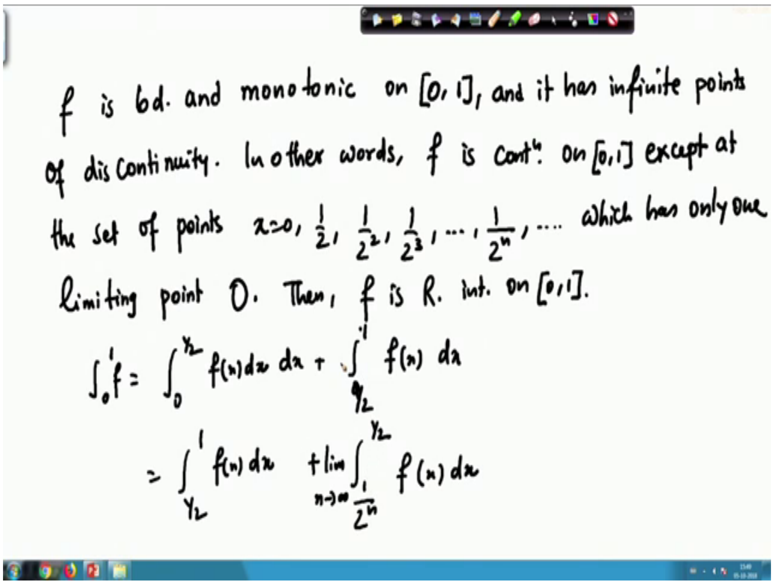

So, the function f f is bounded and monotonic right f is bounded and monotonic on 0 comma 1 the given interval in the problem and it has infinite points infinite points of discontinuity alright which we just saw.

So, it has infinite point of discontinuity that means, we can write it in other words in other words f is continuous on 0comma 1 except at the set of points what are those set of points? x equals to 0 x equals to half x equals to 1 by square x equals to 2 by 2 whole to the power 1 by 2 whole to the power 3 up to x or x equals to 1 by 2 to the power n and so on, which has only one limiting point so this our limit point. So, this sequence 0 half 1 by 2 1 by 3 dot dot and so on up to 1 by up to a 1 by 2 to the power n and so on it has one limiting point which is 0. And since in the previous theorem in this theorem it says that that the if the function has the set of set of points of discontinuity if for the function f the set of points of its discontinuity has only 1 finite number of limit has only a finite number of limit points then in that case the function is Riemann integrable

So, here in our case the function f has only 1 limit point which is 0 and this is obviously, a finite number of points. So, then in that case we can use that property or we by the help of that property whatever number we can give this function f is Riemann integrable on the interval 0 comma 1. So, just using this property this property 2 in this case we can be able to say that the given function f here is Riemann integrable. And now we can calculate the value of this Riemann integral. So, the value is integral from 0 to 1, we have 0 to half, 0 to half f x d x and 0 to sorry half to 1 f x d x, next we will divide this we will continue doing this. So, what we will do here is basically we now write; so now, we write this as half 2 sorry m integral from. So, here f x is missing. So, integral from half to 1 f x d x then we can write this one as half and here I can write 1 by 2 to the power n f x d x and I can put a small limit.

So, limit n goes to infinity integral from 1 by 2 to the power n to half f x d x. So, when n goes to infinity this whole thing will go to 0 and therefore, it will be this integral 0 to half.

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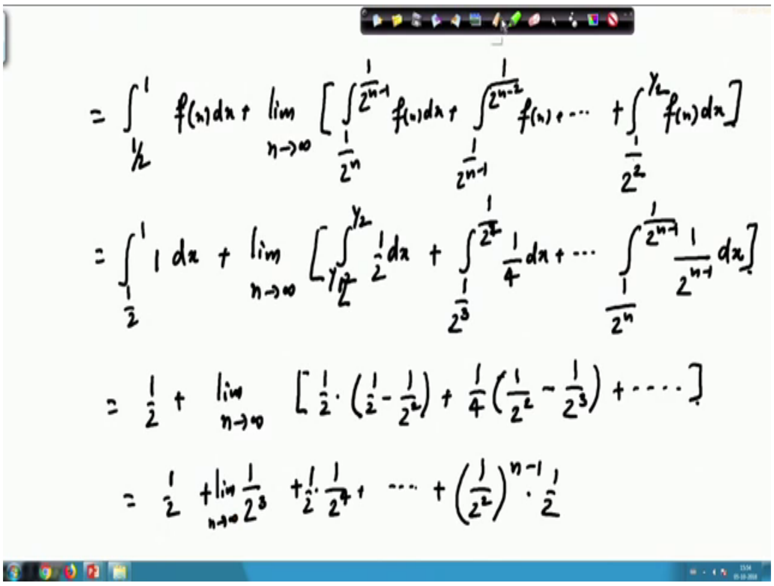

Now we can write this as integral from half to 1. So, we can write half to 1 f x d x plus limit and goes to infinity we can write this as 1 by 2 to the power n can be written as integral from 1 by 2 to the power n to 1 by 2 to the power n minus 1 f x d x plus integral from 1 by 2 to the power n minus 1 to 1 by 2 to the power n minus 2 f x d x plus it will continue until integral from 1 by 4 to 1 by 2 or I can write it as just for the sake of notation I can write it as 1 by 2 square f x d x. So, let us close this interval. Now from half to 1 what is the value of the function from half to 1 from half to 1 the function is 1. So, from half to 1 the function is 1 and here it will be integral from.

Now, we can write this whole thing in terms of other way around. So, integral from 1 y 2 square to the power from 1 by 2 square to 1 by 2 f x d x. So, between 1 by 4 to 1 by 2 the value of the function is between 1 by 4 to 1 by 2 the value of the function is half. So, let us write half here d x plus from 1 by 2 to the power 3 to 1 by 2 to the power of 2 the value of the function is 1 by 4 d x plus it will continue 1 by 2 to the power n to 1 by 2 to the power n minus 1 this will be 1 by 2 to the power n minus 1 d x. So, this can be written as half. So, the first integral is half, and here it will be limit n goes to infinity this will be 1 by half times half minus 1 by 2 to the power half 1 by 2 to the 1 by 2 to the power 2 sorry; then this is 1 by 4 times 1 by 2 square minus 1 by 2 to the power 3 plus it will continue. So, if we calculate this whole thing then after simplification we will end up with 1 by 2 plus 1 by 2 to the power 3 plus 1 by 2 1 by 2 to the power 4 plus dot dot and so on. This will be 1 by 2 square to the power n minus 1.

So, this is what we end up with 1 by 2 to the power and then 2 to the power 4 there will be a 1 by 2 here and there will be 1 by 2 here now it is ok.

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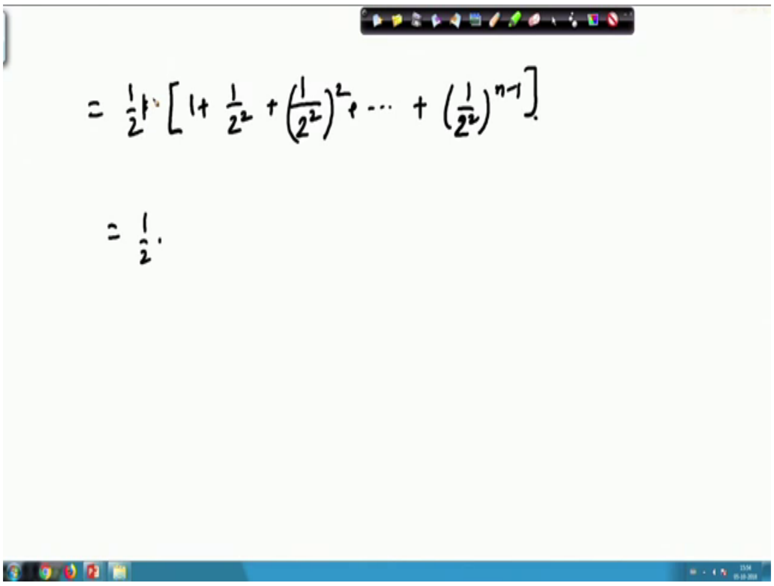

And if we calculate the whole thing then this will be half then 1 plus 1 by 2 to the power 2 plus 1 by 2 to the power 2 whole to the power 2 plus dot dot and so on 1 by 2 to the power 2 whole to the power n minus 1 and this is basically a geometric series, and here our limit is missing. So, this will be a limit is missing. So, limit n goes to infinity and so plus.

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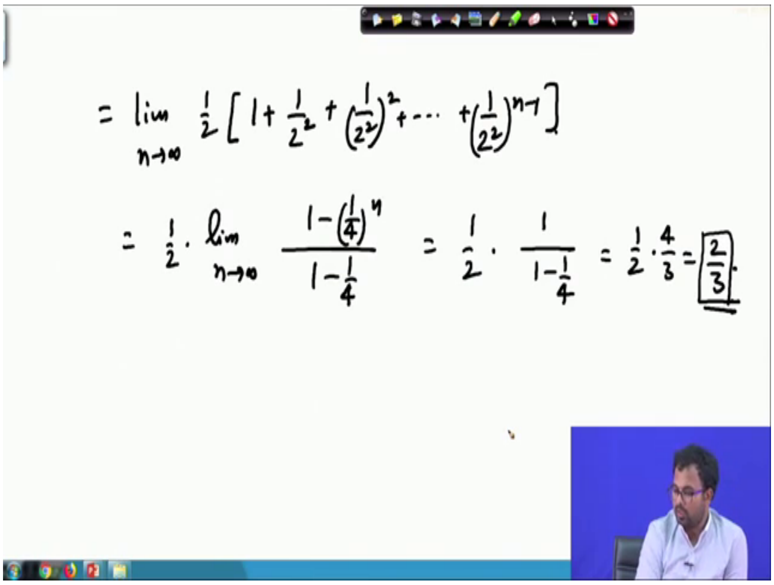

Next we can write half plus if I take half common then in that case sorry So, I can write limit here. So, I can write limit here, limit n goes to infinity, I take half common and then it will be 1 plus 1 by 2 square plus 1 by 2 square whole to the power whole square plus dot dot and so on, 1 by 2 square whole to the power n minus 1. So, this is 1 by 2 limit n goes to infinity here it is basically a geometric series. And therefore, the sum can be given by 1 by 1. So, it is geometric series then as sorry a geometric progression and in a way. And therefore, the sum can be given as 1 minus 1 by 4 to the power n divided by 1 minus 1 by 4 and if n goes to infinity then in that case this term will go to 0 and here we will have 1 by 2 times 1 by 1 minus 1 by 4. So, this will be 1 by 2 times 3 1 by 2 times basically 4 by 3. So, here we will obtain 2 by 3.

So, this will be the answer of this problem So, here the given function has infinite point of discontinuity and even though it has so many points of discontinuity all those points of discontinuity has only 1 limiting point. Therefore, the function is Riemann integrable with that property. And now we can calculate this integral the definite integral by using a simple geometric sum formula. So, if we use that formula and when we do n go to when we go n go do n go to infinity then in that case the value of the integral is 2 by 3. So, this is one such result where we use that previous theorem.

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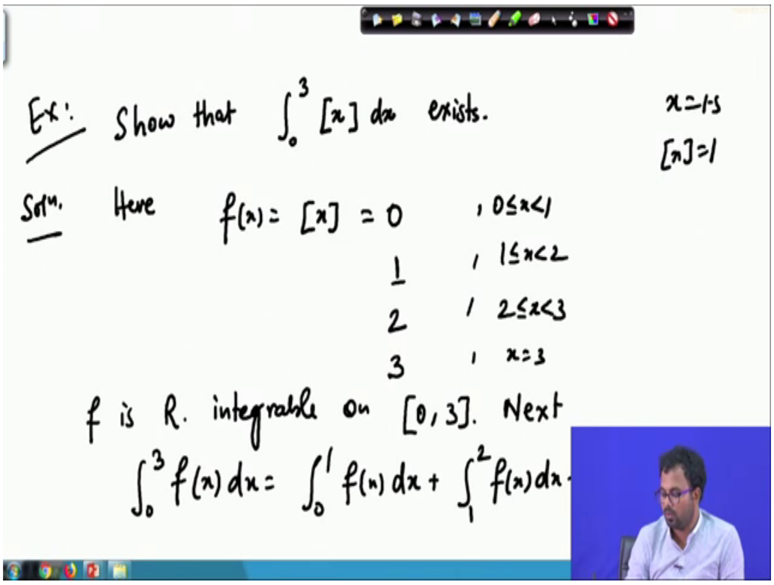

Next we can look into another example. So, an another example would be show that integral from 0 to 3 box x d x exists. So, first of all I hope most of you know what is a box function is. So, let us write here, so let us write the box function. So, here f x is box of x. So, what would be a box of x if x is between 0 and 1 So, box function is basically the integral part of the function; that means, it is sort of like a floor function.

So, if the value of x is for example, if the value of x is I do not know 1.5 then in that case box of x will be 1. So, you always take the integral part of an out of the function. So, here if x is between 0 and 1 then in that case box of x will be 0 and if it is between if it is between 1 and 2, then the value of the box function would be 1 and if it is between 2 and 3 then the value of the box function will be 2 and if x is 3 actually then in that case the value of the box function will be 3. So, now, here in this case what we have is a box function whose Riemann integral t we have to confirm. So, obviously, the function is Riemann integrable because it is bounded between 0 to 3 not only that it is also monotonic 0 to 3.

So, it is always a Riemann integrable function. So, f is Riemann integrable on 0 to 3 right. Next we will calculate integral from 0 to 3 f x d x. So, we divide the interior until the interval into 3 sub intervals 1 to 2 f x d x plus 2 to 3 f x d x let us go to new page integral from 0 to 1.

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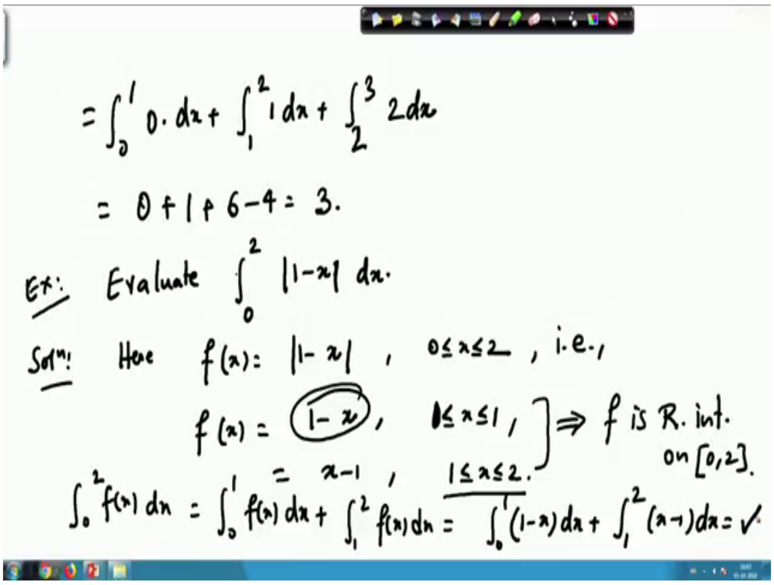

So, between 0 to 1 f x is 0 between 1 to 2 f x is 1 and between 2 to 3 between 2 to 3 f x is 2. So, if we calculate if we calculate then this will be 0 plus 1 plus 2 x, so 6 minus 4, so, ultimately 3. So, that is the value of the integral. So, it is Riemann integrable and the value is 3 alright. Next, we can look into an example of type this here. So, let us consider an another example evaluate 0 to 2 mod of 1 minus x d x. So, here our function f x is mod of 1 minus x for x between 0 to 2.

So, that is in other words f x is 1 minus x for 0 less or equal to x less or equal to 1 and it is x minus 1 for 1 less or equal to x less or equal to 2 because your mod function is sort of like a positive function and if our x is between 0 and 1 then in that case this mod function will remain as 1 minus x and if our x is between 1 and 2 then in that case 1 minus x mod of 1 minus x will be written as x minus 1 So, just to keep the value of positive or the value absolute in a way. So, let us see if the function is continuous or not. So, here the function f is definitely continuous between 0 to 1 the first part this part here and the second part x minus 1 this 1 is also continuous between 1 to 2

Now, the only thing which we have to make sure here in this case is that whether a function is continuous or monotone or something and then we can talk about certain integrate built in. So, since the function is continuous between 0 to 2 it is obviously, Riemann integrable on 0 to 2. So, we can say that since it is continuous I am not writing that part it is it is obvious I guess that the function is continuous and from here we can say that the function is Riemann integrable 0 to 2. And in order to evaluate the Riemann integrable in order to evaluate the Riemann integral of f x we just write the function into 2 sub interval a sub integrals and the integral into 2 sub integrals So, 1 to 2 f x d x and what is the value of the function from 0 to 1? The value of the function is 1 minus x and the value of the function between 1 to 2 is x minus 1 and I believe you can be able to evaluate this integral

So, that will be the value of the Riemann integral for the function f. So, this is this was another example where you have to how to say break the mod and get the individual functions and check whether that whether those individual functions are continuous or bounded or monotonic and then you can talk about the Riemann integralbility. Alright, the next the next example which we will look into as is to apply the mean value theorem.

So, in order to apply the mean value theorem in order to apply the mean value theorem so, I am sorry another example.

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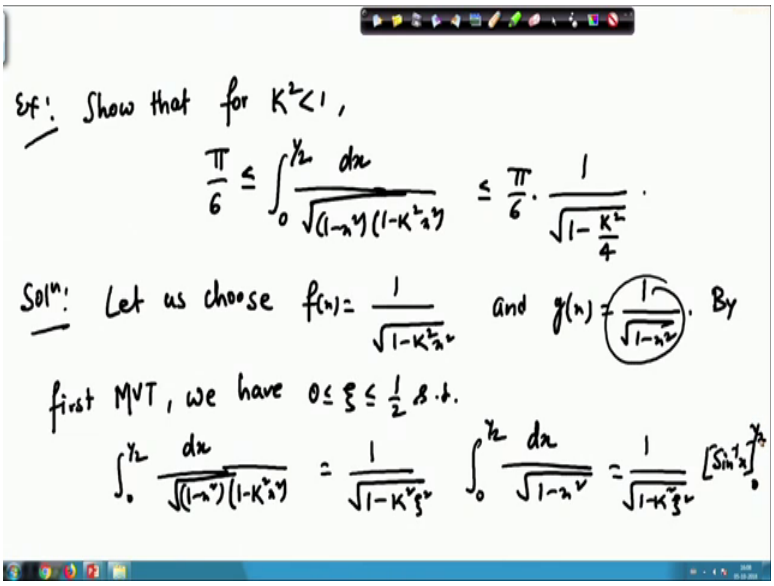

So, another example show that for K squared less than 1 pi by 6 is less or equal to integral from 0 to half d x divided by square root of 1 minus x square times 1 minus K square x square is less or equal to pi by 6 times square root of 1 minus K square by 4. So, here we have to show that this integral this integral is bounded between these 2 points.

So, in order to apply in order to apply the mean value theorem we will start with the first mean value term. So, let us see whether we can apply the first mean value theorem or not. So, the first mean value theorem says that if we have how to say, if we have 2 functions f and g. And if the function g maintains the same sign on the interval a to b then in that case you can write the integral of 2 functions as the value of the one function multiplied by the integral of the second function. So, here our 2 functions 1 function can we can choose as, so 1 function we can choose as, so let us choose f x as 1 by square root of 1 minus K square x square and g x as 1 by square root of 1 minus x square.

So, as for the statement of first fundamental theorem of integral invalid sorry as first mean value sorry as per the statement of the first we mean value theorem f and g both needs to be bounded. So obviously, they are bounded between 0 in the interval 0 to half because, K square is less than 1. So, this quantity is bounded and this quantity is bounded and of course, this function keeps the same sign. So, whatever the value is between 0 to half, between 0 to half this function g here keeps the same sign. And therefore, it satisfies that criteria of first mean value theorem. And by first mean value theorem by first mean value theorem I will write it as MVT first mean value theorem we have

So, it satisfies all the criteria of the first mean value theorem and therefore, by mean value theorem we have integral we have as psi between 0 to half such that the integral 0 to half d x 1 minus x square times 1 minus K square x square is equals to f of psi. So, f of psi is basically 1 minus k square is psi square integral from 0 to half d x 1 minus x square all right. So, but this integral here this integral here is nothing, but 1 by 1 minus K square psi square and this is actually sin inverse x sin inverse x from 0 to half now sine inverse half.

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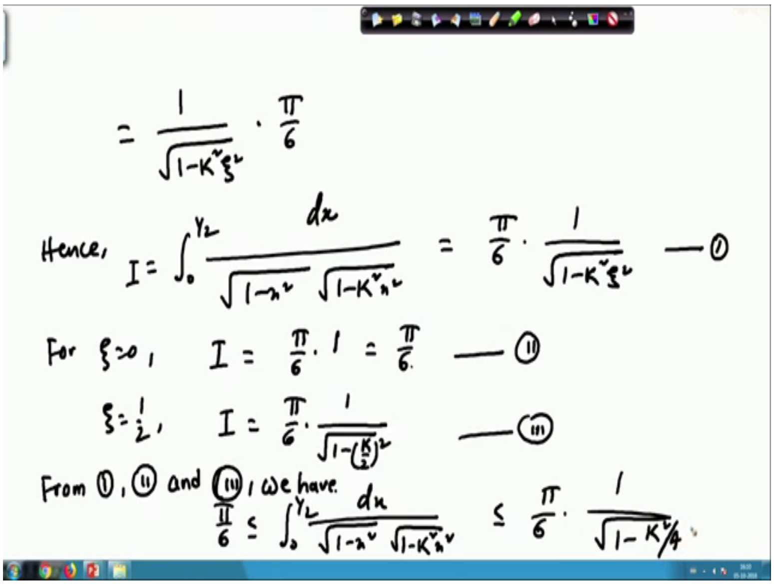

Now, this is equals to 1 by 1 minus K square is psi square and sin inverse half would be pi by 6. So, I write it as pi by 6. Next, so hence integral from 0 to half d x divided by 1 minus x square times 1 minus k square x square is equals to pi by 6 times 1 by 1 minus K square is psi square. And if I take psi equals to 1, so if I take psi equals to 1 and psi equals to a sorry if I take psi equals to 0, so for psi equals to 0 this integral, so let us call it as I, let us call it as I.

So, I equals to pi by 6 times psi equals to 0, so times 1. So, this is basically pi by 6 and if I take psi equals to half, so psi can attend only 2 can attend the balance between only 2 points 0 and half. So, if I take psi equals to half then this will be 1 sorry pi by 6 times 1 by square root of 1 minus K by 2 whole square. So, from let us say this one is equation 1 and this one is 2 and this one is 3. So, from 1, 2 and 3 we will have from 1, 2 and 3 we will have pi by 6 less or equal to integral from 0 to half d x by square root of 1 minus x square times 1 minus K square x square less or equal to pi by 6 times 1 minus K square by 4, which is what we needed to prove.

So, with the help of first mean values I am sorry with the help of first wind we can theorem we can be able to show that this integral is bounded between these 2 values. And we will look into in the next class probably one application of the second mean value theorem and then we will move to the reduction formula. So, for today we will stop here and in the next lecture we will start with some other examples.

Thank you.