Integral and Vector Calculus

Prof. Hari Shankar Mahato
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 59
Stoke's Theorem

Hello students. So, in the previous class, we started with Gauss divergence theorem and we also looked into the statement of Gauss divergence theorem and how we can convert a surface integral into a volume integral and then a volume integral back into the surface integral. So, whichever is given, if in your question if you are asked to evaluate the other one, you can use the Gauss divergence theorem. And since we have only two more lectures left, today we will start with one example on Gauss divergence theorem and then we move to our final theorem in vector calculus which is basically stokes theorem and unfortunately we do not have any more lectures left.

So, we can may not be able to practice examples from the integral calculus section or even from the introduction of the vector calculus part, but we will try to solve at least two or three examples on stokes theorem just to make the concepts clear. So, today, we will start with one example on Gauss divergence theorem.

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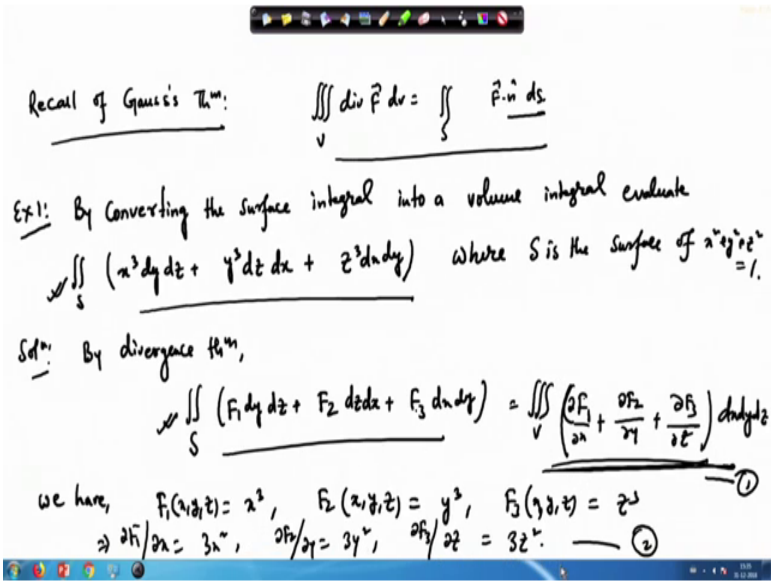

So, in the statement from the Gauss divergence theorem, you remember that you can write if F is any vector function, then you can write the volume integral over the divergence of let us say our vector function is F, then in that case I can write divergence of F dv equals to surface integral and here you have a curl of F dot n d S. So, this is basically our, I am sorry. So, this one is just F dot n d s.

So, this is basically our Gauss divergence theorem. And with the help of Gauss divergence term, you can be able to convert or if you are given to evaluate a surface integral, you can just convert it into a volume integral and the things will become very straightforward. So, we will solve one example based on this Gauss divergence theorem.

So, the first example is for today, so the first example is let us say by converting the surface integral into a volume integral into a volume integral, evaluate surface integral over S x cube d y d z plus y cube d z d x plus z cube d x d y where S is the surface of x square plus y square plus z square equals to 1. So, this is basically our sphere. And we have to convert the surface integral which is to be evaluated on the surface of the sphere into a volume integral and then we have to calculate the value of this surface integral. So, we know that from Gauss divergence theorem, we know that, so we have learnt this formula.

So, there are like two or three expressions for Gauss divergence theorem and one of them is this one another one is with d x, d y, d y, d z and d y d x form. So, we are going to write that form here. So, by divergence theorem we know that, surface integral over S F 1 d y d z plus F 2 d z dx plus F 3 d x d y equals to volume integral over V. We will have F 1, we have del F 1 del x. So, this is a bit bigger bracket del F 2 by del y and then we have del F 3 by del z and then here we have d v, so which is basically d x, d y, d z.

So, this is the required how to say form in terms of for Cartesian coordinates like x, y and z. So, now, we can compare. So, you see this surface integral here and this surface integral here, if we compare these two forms, then in that case our F 1 is x cube F 2 is y cube and F 3 is z cube right. So, we have where V is the, so here V is the volume enclosed by the surface S.

So, we have F 1 x, y, z as x cube F 2 x, y, z as y cube and F 3 x, y, z as z cube. Since F is a function of x only, y is F 2 is a function of y only and F 3 is a function of z only, it makes our life a lot easier because it may happen that here F 1 could be a product of x y z and their power and things like that. So, then it would have become a little bit complicated. But since here everything is just like a function of one variable only, then this is a fairly simple example to do. So, now, by the divergence theorem let me name this equation as 1. So, from equation 1 if we convert this surface integral, if we convert this surface integral into a volume integral, then it will take this form like an equation 1. And in equation 1, V is the volume enclosed by this fair S which is given here and we have to evaluate del F 1 by del x.

So, if I do the partial derivative, then this will be del F 1 by del x 3 x square del F 2 by del y 3 y square and del F 3 by del z 3 z square, alright. So, let us substitute all these values here.

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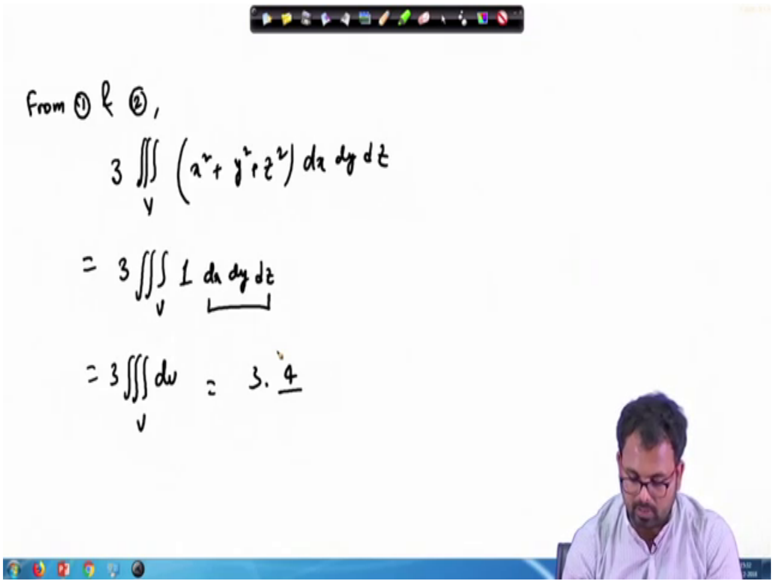

So, from 1 and let me call this equation as 2. So, this is as 2. So, from 1 and 2, we have volume integral over V del F 1 by del x square. So, basically 3 x square plus y square plus z square d x d y d z. Now, this is basically volume integral and if we substitute this as x square plus y square plus z square as here as 1, so then in that case this will become 1 d x d y d z and this is volume integral.

So, 1 and then d x d y d z is basically d v, so that is the volume element. So, I can write it as d v alright and this means that we are basically doing or calculating the volume of that sphere right. So, if your vector function is 1 like in this case and I mean not vector function. So, if your vector, if your function overall is 1 in a way, then in that case here this means that you are actually doing a volume or calculating the volume of the given surface. So, that is what this mean and here in our case, the given surface is a sphere of unit radius.

So, in that case we can write 4 by 3 or what we can do is instead of substituting, there is another method which we can follow.

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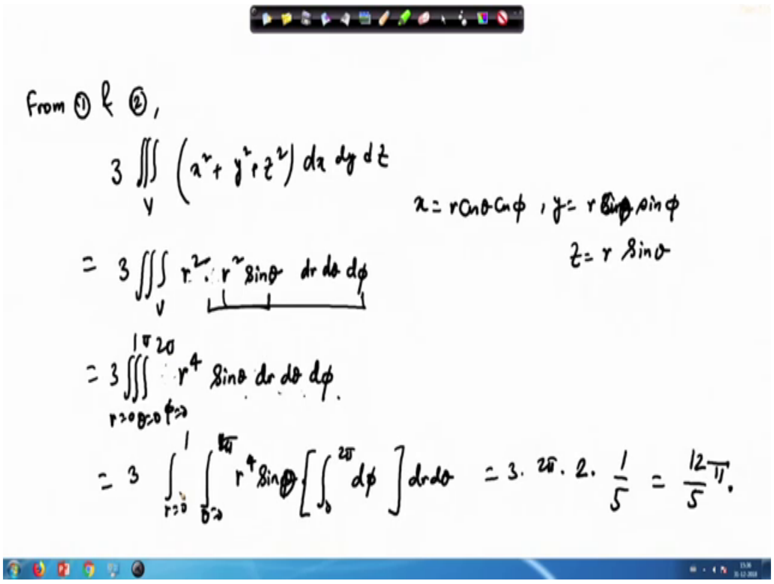

So, what we can do? Instead of substituting this one, I can substitute just to make things a little bit more clear, I can substitute x equals to r cos theta cos phi, y equals to r sin theta sin phi and r cos theta sin phi and z equals to r sin theta. Then in that case, this will be x square plus y square. So, cos theta cos theta cos square theta will get out and then this will be cos square phi plus sin square phi and then that will be 1 and then it will be r square sin square theta yes. So, this is correct. So, we substitute x equals to r cos theta cos phi y equals to r cos theta sin phi and z equals to r sin theta.

So, if I substitute all these values here, then in that case this will be r square times r square sin theta and then we will have d r d theta d phi, all right. So, now for this V, what we can do is we can write the range for r. So, r is basically varying from 0 to 1, theta is varying from 0 to pi and phi is varying from 0 to 2 pi. And if we evaluate this integral, then this will be r to the power 4, r to the power 4 sin theta dr d theta and d phi. So, we can integrate because they are not product they are not in a way involving theta and r together. So, we can integrate with respect to r for the integral with respect to r and we can integrate for theta for the integral with respect to theta.

So that means, this will reduce to integral r running from 0 to 1, integral theta running from 0 to 2 pi. So, first we will integrate with respect to phi. So, this will be sorry theta. So, this will be integral from 0 to 2 pi d phi and here we will have d or d theta. So, this will give us 2 phi a 2 pi. So, this is three times 2 pi and then we integrate sin theta. So, it will be cos theta and cos theta and theta is from 0 to 2 pi. So, then we will have cos pi as minus 1. So, this will give us a 2 and when we integrate r to the power 4, then this will give us 1 by 5.

So, ultimately it will be 12 by 5 pi. So, you see initially we had a very how to say complicated expression where it is not that much complicated but still we had to calculate the surface integral. So, in the question itself it says that convert the surface integral into a volume integral. So, if we are asked to convert this surface integral to a volume integral, we just have to use this divergence theorem. And in that divergence theorem if you compare, then this is basically our F 1, this is basically F 2 and this is basically F 3. And on the right hand side of that divergence theorem, we had to calculate the partial derivatives. So, we calculated the partial derivatives, we substituted these partial derivatives here and then we are just substituting taking the help of a spherical polar coordinate system.

So, we substitute r equals to cos theta cos phi, y equals x equals to r cos theta cos phi, y equals to r cos theta sin phi and z equals to r sin theta. So, the volume element will be d r d theta d phi multiplied by r sin theta and then we have here are square basically. So, we so this is our basically volume element and that is the function with what we are getting from this x is square plus y square plus z square. So, all together this is the r to the power 4, this is sin theta d r d theta d phi and the limit for r is 0 to 1 for theta is 0 to pi and phi is 0 to 2 pi. Then when you integrate, you obtain this integral the value of this surface integral. So, that is how we use the Gauss divergence theorem.

So, whenever you are given a surface integral, you can use Gauss divergence theorem to convert it into a volume integral and it will become very easy. Because when you are evaluating surface integral, in some cases you may have a parallelepiped, cube or a cube and then in that case, you have to do the surface integral on every surface. However, if you convert that surface integral into a volume integral and the light will become much easier, then in that case we just have to get the limits for x, y and z and do that calculate that divergence and hopefully the whole thing will become very easy. So, this is one of the examples motivated from Gauss divergence theorem. And now we move to our final theorem which is basically stokes theorem. So, I am going to write the statement first.

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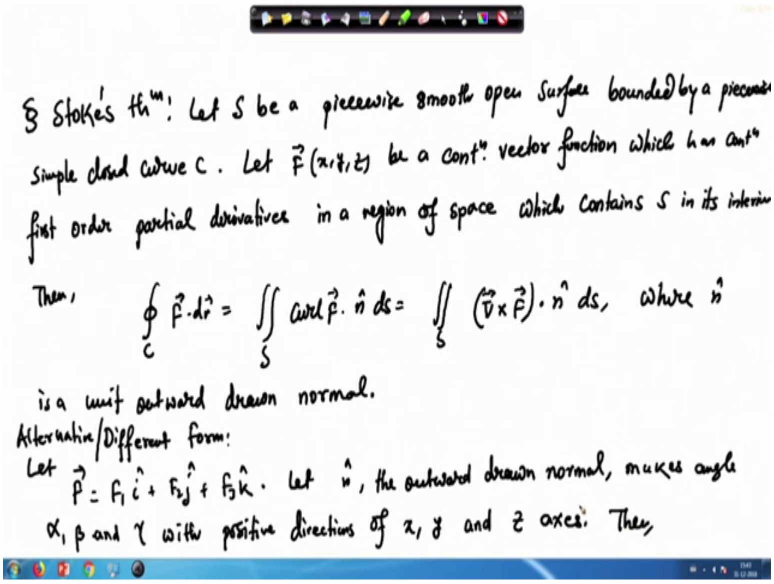

So, Stoke's theorem, so let me have a look at the statement in my lecture note. So, the Stoke's theorem says let S be a piecewise smooth open surface bounded by a piecewise simple closed curve C and let F x, y, z be a continuous vector function which has continuous first order partial derivatives in a region of space which contains S in it is interial, then line integral F dot d r equals to surface integral curl of F dot n d s or we can write it as where n is a unit outward drawn normal alright. So, this is and so this is the required statement and of course S so if you are walking along the curve C, then your surface S must lie on the left hand side.

So, that is how we choose the curve C that the orientation is always in the anticlockwise direction. So, that is the property of this curve C and n is outward drawn normal. So, if you are walking in the anticlockwise direction, then your surface will always lie on the left. That is how we mean by closed, that is what we mean by this piecewise smooth bounded by a simple closed curve piecewise, simple closed curve C. And n is actually the outward or normal on the surface S and this then in that case you can write the, so line integral F dot d as surface integral of F dot nds. So, this is also a very important theorem in vector calculus and also in applied it several other branches of applied mathematics where we take help of this theorem.

So, for example, in partial differential equations or even in fluid mechanics and these theorems have a very wide application. So, this is required statement and we can also write this theorem in a Cartesian form. So, I have it in my lecture notes here. So, if I write it as a Cartesian form, then in that case we can write it as let. So, alternatively or a different form basically, so alternative or a different form a different form. So, basically let F is a given vector function which has three components F 1 i, F 2 j and F 3 k and let n the outward drawn normal, the outward drawn normal makes angle alpha, beta and gamma with positive directions of x, y and z axes.

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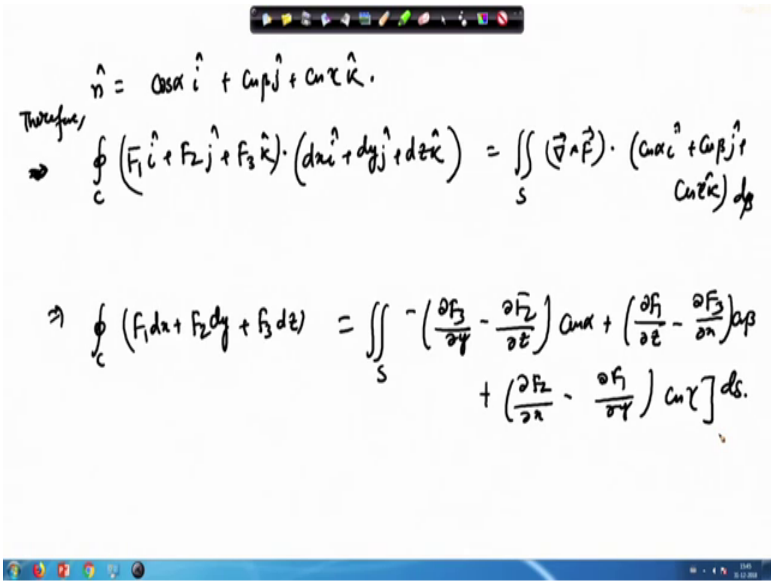

Then, we can write our normal n; then we can write our normal n as cos alpha i plus cos beta j and cos gamma k and we know that curl of F, so curl of F; if F has three components and curl of F is basically a cross product which can be determined by that determinant i, j, k del del x, del del y, del del z and then F 1, F 2, F 3 and then we calculate the determinant.

So, we basically take a dot product with this normal n which is given here. So, on the left hand side, basically we will have line integral. So, therefore, F 1 i is F 2 j plus F 3 k, dot product with d r d r is basically d x i plus d y j plus d z k equals to surface integral. So, here we have curl of F.

So, if we calculate the curl of F or let us just write curl of F dot n, so n is cos alpha i cos beta j plus cos gamma k. So, now, we can calculate the curl of F easily and then we take the dot product here also we take the dot product and therefore, the whole thing will reduce to and here it will be surface integral del F 3 del y minus del F 2 del z times cos alpha plus del F 1 by del z minus del F 3 by del x cos beta plus del F 2 by del x minus del F 1 by del y cos gamma d of S.

So, this is the required form of Gauss divergence, sorry Stoke's theorem in the Cartesian form. So, like these theorem or Gauss divergence theorem, Stoke's theorem also have also has very wide application in vector calculus and we will practice a few examples motivated from stokes theorem and then we will probably close this topic. So, let me just try to find out, ok. So, here I have some examples.

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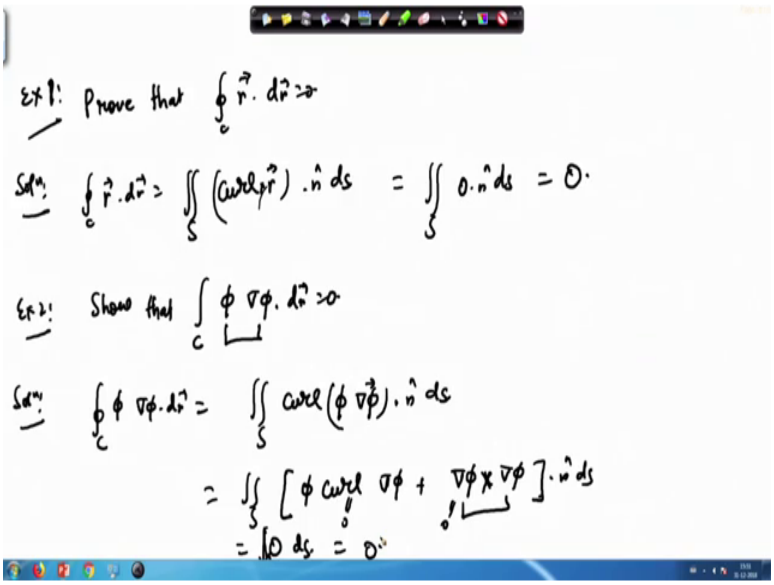

So, example 1, I guess. So, first example is prove that line integral of r dot d r equals to 0. So, we have to show that, so that line integral of this r is our given vector function in this case. So, F is basically r and line integral over this curve C where C is again any piecewise smooth curve closed curve of course, dot d a x equals to 0. So, we can write this as surface integral curl of r dot n d s. So, dot n n is basically the unit outward drawn normal and we have to calculate the curl of r, right.

So, curl of r if you calculate, then you will be able to see that this is 0 because r is x i plus y j plus z k and if you calculate curl of r, then this will be 0. So, surface integral over S 0 dot nds is equals to basically 0. So, this is how we use the Gauss stokes theorem. Next, we can apply can we can show that a similar type of result. So, show that, so there are several show that type of problems which you can also solve with this with the help of stokes theorem. So, that line integral over the curve C phi gradient of phi dot d r equals to 0. So, our given function F is basically phi times gradient of phi.

So, here we can have surface integral phi a sorry line integral C and gradient of phi dot d r equals to surface integral of curl of phi grad phi dot n d s. So, now, we have to calculate curl of phi times gradient of phi. So, if we do that, then in that case they are using the formula from vector calculus this can be written as phi times curl of gradient of phi plus gradient of phi curl with gradient of phi, gradient of phi cross product with gradient of phi dot n d s. So, now, this is gradient of phi cross product with gradient of phi.

So, this is basically 0 because a cross a is 0 and phi times curl of gradient of phi is also 0. So, this is also 0 and this is 0 by definition. So, ultimately we have this as 0, this as 0. So, the integrand is 0 and therefore, 0 vector dot n is again 0. So that means, this whole thing will reduce to 0 dot d S because this is 0 vector, this is 0 vector. So, 0 vector dot with n cap is as 0 or scalar and then we integrate over surface integral S let us say. So, this is ultimately 0.

So, you see we did not have to go through any complicated calculation or something, we just had to use Stoke's theorem. That means, that you can convert your line integral into a surface integral and then the rest of the simplification is pretty much straightforward. So, these were the, to prove that examples that we solved today, but in our next class we will consider at least one or two examples motivated from the stokes theorem where we might need to verify the stokes theorem and we will start such examples in our next class.

So, I thank you for your attention today and I will see you in your next class.