Integral and Vector Calculus
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Lecture - 58
Gauss divergence theorem

Hello students. So, in the previous class, we started with volume integral and then we gave I gave you the different the statement of Gauss divergence theorem and then we practiced one or two examples. But since it is a very important topic in vector calculus, we will continue practicing few more examples on Gauss divergence theorem. So, let me start with our first example.

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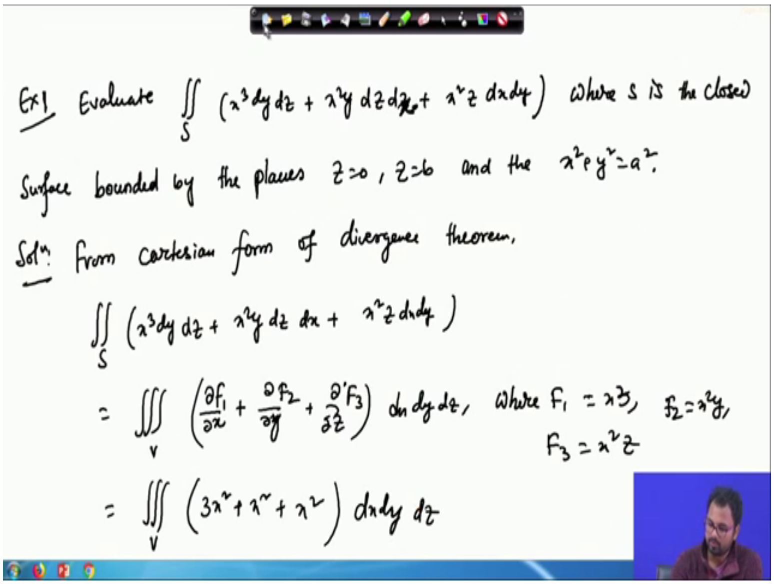

So, example one for today.

So, our first example is evaluate surface integral x cube dy dz plus x square y dz dx plus x square z dx dy where S is the closed surface bounded by the planes z equals to 0, z equals to b and x square plus y square equals to a square. So, basically we have a cylinder. So, here the given surface integral is this one. Of course, if you want to evaluate the surface integral ah; it would be very complicated because we have to evaluate these three triple these three terms actually three terms and we have to calculate the limits for whenever we are integrating.

So, here we have to calculate limits for xy and z; for z it is given. So, its relatively complicated, but if we look into the Cartesian form of Gauss divergence theorem there we had ; so, we can go back.

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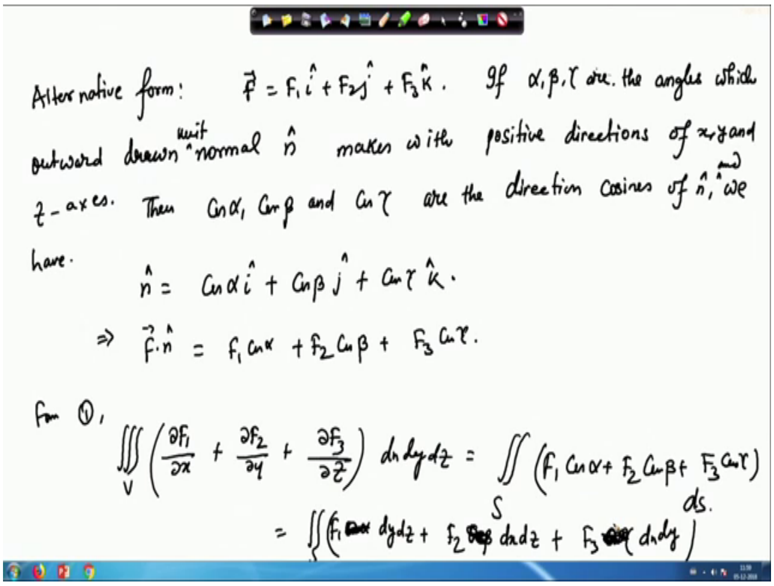

And there we had F1 F2 F3 and we can write. So, F1 F2 F3 and we can write this thing as a. So, let me go back and yeah in the previous form I believe when we are multiplying cos alpha by d S, then we are taking basically the dy the dy dz and then this one is our dx dz and this one is our dx dy yeah.

So, there was a small mistake in that formula. So, this one has to be F1 cos alpha F2 F1 dy dz F2 dx dz and F3 dx dy. So, that is the small correction in our previous previous formula. So, just make sure you do that. So, here we have F1 dy dz F2 dx dz and F3 dx dy. So, now, if I look into the formula where is that if I look into this function here. So, this is our F1, this is our F2 this must be dz dx and this is our F3.

So, from Cartesian form of divergence theorem of divergence theorem divergence theorem, we have basically a surface integral over S x cube dy dz plus x square y dz dx plus x square z dx dy. So, this will be volume integral del del x of f 1plus del del x of del del y of F2 plus del del y of F3. So, del del z of del del z of F3 dx dy dz where our F1 is I am just writing them at first of all x cube F2 is x square y and F3 is x square z.

So, if I substitute all these things here. So, this will be del del x of F1 so; that means, three x square plus del del y of F2. So, this is x square plus del del z of F3. So, this is simply x square right and then we have dx dy and then d z right. So, now, this will be basically 3 plus 4 or 5.

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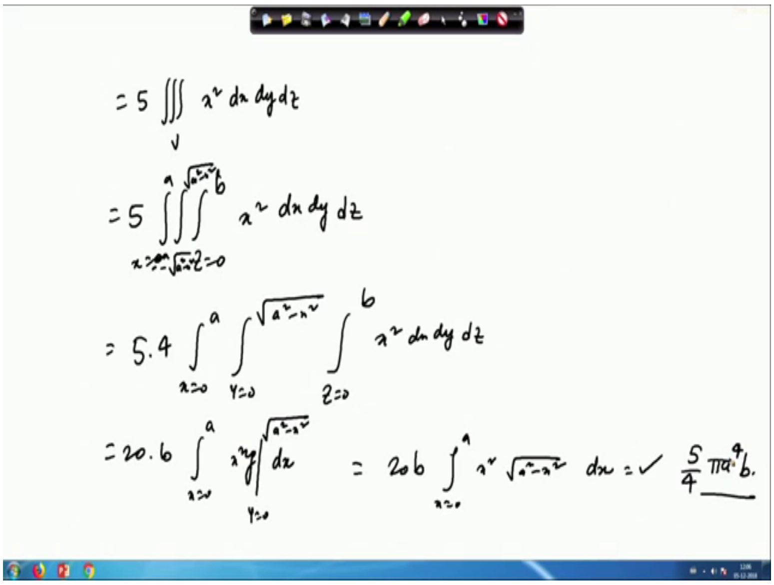

So, this will be 5 volume integral x square dx dy and dz. So, this is my this is Cartesian form of Gauss divergence theorem actually made this term or this integral this integral to reduce to a fairly simple integrand which is basically 5 x squared dx dy dz.

Now, we have to guess the limits. So, in order to guess the limits z is actually varying from 0 to b. So, guessing the limits for z is not complicated. Now y will vary from minus of a square minus x square to square root of a square minus x square and x will vary from 0 to sorry x will vary from minus a to a.

So, minus a to a and y is varying from minus alpha square root of a square minus x square to a square root of a square minus x square. So, basically we take both plus and minus value and x is varying from minus a to plus a. So, first of all we can integrate we can integrate with respect to with respect to x, but here we can see that we can see that there is symmetry in a way. So, instead of calculating the area in the whole in the whole circle, we can actually calculate in one of the halves actually.

So, this will become 5times 2 to 4 and then we have half and then this one is one quarter. So, this is basically x running from 0 to a and this one is y running from 0 to a square minus x square and then z is running from 0 to b x square dx dy dz. Because whatever you get the area in this one half in a way is actually the four time and multiplied by four times and that will be there that will be the whole area in a way or volume integral in this case in the whole cylinder. So, basically we took the four times of that volume in that one particular quadrant in a way. So, therefore, we had to add a 4. Here now we can integrate.

So, this is basically 20. So, we can first integrate with respect to z and this will be basically b and then we can integrate with respect to y and then this will be integral x running from 0 to a. If we integrate with respect to y, then this will be x square y and so, this will be x square y or we can integrate with respect to x first; sorry y also we are integrating with respect to y. So, this will be x square y times d x and then y is varying from 0 to square root of a square minus x square right and then this will be 20 b times.

So, this will be integral x running from 0 to a x square is square root of a square minus x square and then the second limit is 0 times dx. Now we have to evaluate this integral. So, evaluating this integral is not complicated, you just have to substitute x equals to sum a sine theta and then convert the whole thing in some polar coordinates and then you will basically be able to evaluate this integral; it just that it is slightly lengthy. So, I am leaving this task up to the students. I am pretty sure you can be able to do that and finally, the answer would be 5 by 4 pi a to the power 4 b. So, this will be the answer.

So, other than guessing the limits the rest of the things are pretty much same what we have studied before. So, this obtaining this limit is not complicated. So, I am pretty sure you can be able to do that. So, this is how we just using the Gauss divergence theorem, you see the whole integral is simplified to some volume integral a simple volume integral where of course, now here you have to do some complicated calculation a slightly complicated. So, it is simple from here to here and yeah this is one of the applications of Gauss divergence theorem we will practice the next example now.

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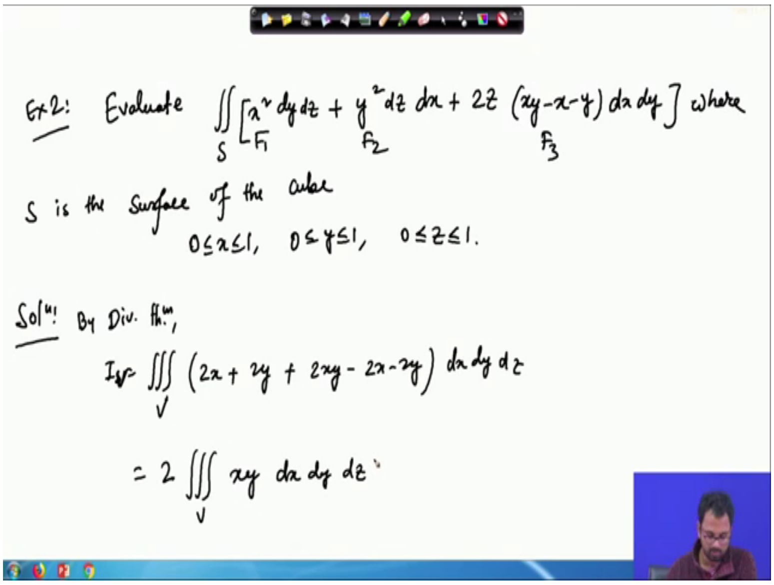

So, let me consider an another example. So, evaluate example 2, I guess. So, evaluate surface integral x square dy dz plus y squared dz dx plus 2 z xy minus x minus y dx dy where S is the cube, S is the surface of the cube 0 less or equals to x less or equal to 1, 0 less or equal to y less or equal to 1 and 0 less or equal to z less or equal to 1. So, here of course, its it again falls into that Cartesian form category of Gauss divergence theorem. And if we compare like previous example, then this is our F1, this is our F2 and this is our F3. So, U am not writing all those things; I am just writing by Gauss divergence theorem or simply by divergence theorem sometimes I am using small d. So, do not get confused. So, you can write small d or capital D; it is up to you.

So, our I s which is basically the surface integral will reduce to our volume integral I V and del f del f del F1 del x. So, this will be 2 x then del F2 del y. So, this will be 2 y and then del F3 del z. So, this will be 2 xy minus of 2 x minus of 2 y dx dy and dz.

So, 2 x 2 x will go will be cancelled and then this will be 2 times volume integral xy dx dy and dz. So, now, here we have to substitute the values for x y and z.

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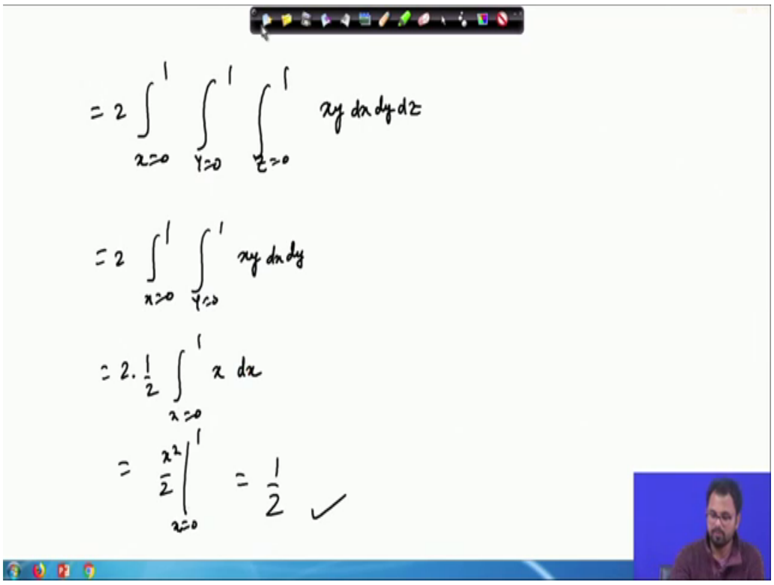

So, since it is a volume integral, our x will vary from 0 to 1, y will vary from 0 to 1 and z will vary from 0 to 1; then we have x y dx dy dz. So, first we integrate with respect to z and the value will be 1 1 z is 1. So, this is just 2 times surface integral from x running from 0 to 1, y running from 0 to 1 x y dx dy. Now we integrate with respect to y and then this will be 1 by y square by 2.

So, this is basically 2 times 1 by 2 and then we integrate with respect to with respect to x and then this will be x square by 2. So, x square by 2 at sorry x square by 2 and x is 0 to 1. So, this is basically half. So, you see initially we had a very how to say complicated expression and we had to evaluate the surface integral, but all of these things are just algebraic expression and therefore, they must have continuous partial derivatives.

So, just applying the Gauss divergence theorem, we were able to obtain this form here and from there just substitute the values of xy and z and then that will give us the required answer which is half in this case. So, like this we can practice many examples. So, I have some examples in my lecture note we can. So, let me let me consider and another example alright.

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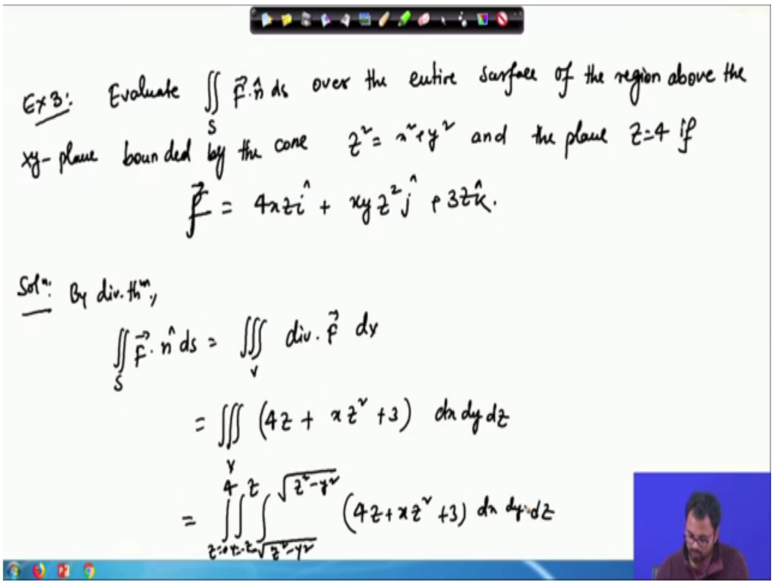

So, here we have an another interesting example; so, example 3 I believe. So, evaluate surface integral of F dot n ds over the entire surface over the entire surface of the region of the region above the xy plane above the xy plane bounded by the cone by the cone z square equals to x square plus y square and the plane that equals to 4. If this capital F capital F is 4 x jet times i plus xy z square times j plus 3 z k. So, here we have to evaluate the surface integral where we have the given vector function and the given volume or the given surface basically for this surface integral is actually the region above xy plane bounded by this cone and the plane jet is equals to 4 actually.

So, the terms in this vector function, they are all algebraic functions; they are all algebraic functions and basically in a way they are product of xy and z square. So, they have continuous partial derivatives. So, we can use Gauss divergence theorem. So, by divergence theorem by divergence theorem; we have surface integral F dot n ds equals to divergence of V divergence of F dv.

So, divergence of F would be volume integral divergence. If we take the divergence here then this will be 4 z plus x z square plus 3 right. So, 4 z plus x z square plus 3 dx dy and dz. You just have to take the divergence of this function and then you obtain this limit here. Now in this case we have to get the limits for x y and z and we have to remember that the volume enclosed by this cone and the plane has to be above xy plane in a way. So, the limit for z would be then; so, the limit for z would be actually first of all for x would be a square root of.

So, for x would be as let me write as square root of y as so, x would be minus of z square minus y square to square root of z square minus y square. Then for y is for y is when x is 0 so, its minus z to plus z and z is 0 to 4. So, this will be 4 z plus xz square plus 3 dx dy dz. And now we can instead of calculating in every 4 this whole this whole integral, I can calculate in the in the upper half.

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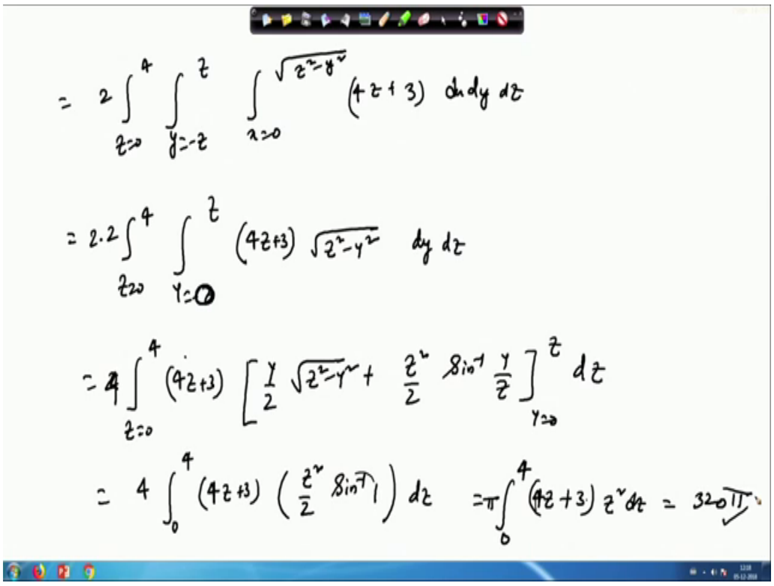

So, then in that case this will be 2 times z running from 0 to 4, y running from minus z to plus z this will be x running from 0 to square root of z square minus y square 4 z plus 4 z plus 3 dx dy dz.

Now if we if we assume that our if so, this function basically ah; so, this function basically is an odd function in x. So, this function is basically in the is an odd function in x. So, therefore, that will be 0 and these to some of these to function and so forth that plus 3 is an even function. So, basically I have written a 2 here. So, that x that is square is vanished because of being an odd function right.

So, that is taken care of here and now what we are going to do? We are going to integrate first with respect to x. So, this will reduce to 0 running from 0 to 4, y running from minus z 2 plus z, and if I integrate with respect to x. So, this will be for z plus 3 times x and I substitute the value. So, this will be z square minus y square dy dz. Now next we integrate with respect to y. So, this will be integral z running from 0 to 4 if I integrate with respect to y.

So, this will be or I can write this as 0 to z and then put a 2 here again. So, this is 4 and then I am integrating with respect to z. So, this will be 4 z plus 3 and then I have y by 2 the square root of z square minus y square plus z square by 2 sin inverse y by z and then this is y running from 0 to z dz. So, ultimately we will obtain 4 integral from 0 to 4 4 z plus 3 times z square by 2 sine inverse 1 right dz.

So, sine inverse one is pi by 2. So, I will take pi by pi by 2 outside and pi by 2 and then there is so, this 4 will be gone. And therefore, we will have simply this integral 0 to 4 we have 4 z plus 3 times z square and this can be written as 4 z 4 z plus 3 plus z square and then this will be reduced to z to the power 4 by 4 and that cube by 3. So, ultimately this whole thing will be z to the power 4 plus z to the power 3 times pi here.

After integration so, ultimately if we do the integration, then this will be 3 where 320 pi. So, you see initially we had this surface integral to evaluate, we could have done that we could have calculated the normal for this surface and then we could have taken the projection and things like that, but it would have lead to a little bit complicated calculation. So, instead of doing that, we took the help of Gauss divergence theorem and with the help of Gauss divergence theorem, we can be able to see that we just have to calculate these limits and then do this simple calculation use the odd and even property of this function here. So, since it is an odd function that is why this term is vanished here and there and the rest of the functions are even function.

So, we I put a 2 here, then I integrated with respect to x, then integrate it with respect to y and then integrate it with respect to that. So, this is actually fairly simple to do instead of doing that complicated surface integral. So, this is another example or application of Gauss divergence theorem. You might also be asked to verify. So, when you are asked to verify that is when the examples becomes very lengthy and hopefully let us assume that still you will be not asked to verify. But sometimes you might and then in that case you have to evaluate the both surface integral and the volume integral.

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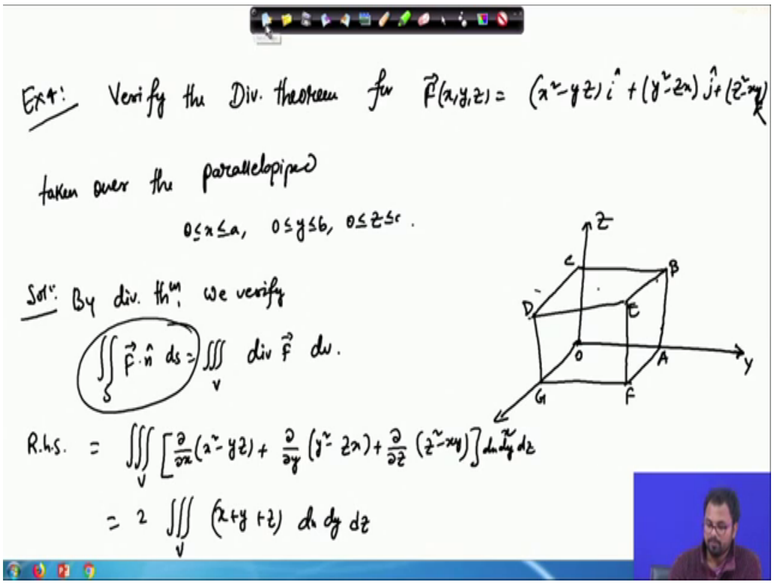

So, let me give you one example where you have to verify example 4; I believe. So, verify the divergence theorem verify the divergence theorem for F x y z equals to x square minus y z times i plus y square minus z x times j plus z square minus xy times k taken over the parallelepiped 0 less or equals to x less or equal to a 0 less or equal to y less are going to b and 0 less or equal to z less or equal to c. So, the solution; so, first of all let us draw our parallelepiped.

So, this is the origin that is my x axis that is y axis, this is z axis and if I draw then this is all right and this is G, this is F A B. So, this is A B C D E F and G alright. So, we have 8 faces right yes all right. So, now, we have to verify.

So, by divergence theorem by divergence theorem, we verify that surface integral F dot nds is equals to volume integral divergence of F dv right. So, let me calculate the right hand side. So, the right hand side Rhs equals 2 volume integral over V divergence of f; so, divergence of F that is our F.

So, we basically have del del x of x square minus y z plus del del y of y square minus z x and del del z of z square minus xy dx dy dz. So, this will be 2 x 2 y 2 z. So, we will have a volume integral x plus y plus z dx dy dz. Now x is varying from 0 to a, y is varying from 0 to b and z is varying from 0 to c.

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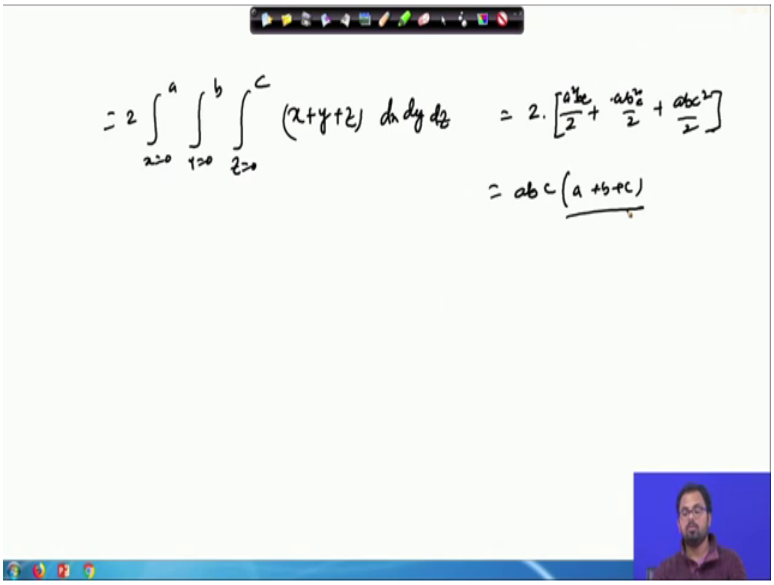

So, I can write those limits. So, x is varying from 0 to a, y is varying from 0 to b and z is varying from 0 to c x plus y plus z and then this is dx dy dz all right. And then we integrate first with respect to x, then with respect to y and then with respect to z.

So, it is a fairly easy thing to do and therefore, this will ultimately give you 2 times is quite 2 times so, this will give a square by b a square b by 2 a b square by 2 plus we will obtain. So, basically we will obtain c a b square c and then a b c square by 2. So, 2 2 will get cancelled and therefore, we will obtain a b c times a plus b plus c. So, this is the required volume integral. Now we have to verify whether the left hand side whether this left hand side is also equal to that volume integral or not now here is the interesting part. This is not a very simple surface integral to evaluate. Here we have basically 8 surfaces.

So, on every surface of this parallelepiped, we have to evaluate the surface integral and on every surface we have to calculate this unit normal and for every surface, we just substitute that unit normal and a then we do the calculation of this surface integral. So, let me give you an example over the surface. So, basically our surface integral would be sum of 8 sub surface integrals.

So, surface integral on this part surface integral on this part, this part that other side that other side this side and the downside of that parallelepiped so, basically there will be 8 surface integrals sub surface integrals I would say and when you sum them then that is when you obtain this surface integral here.

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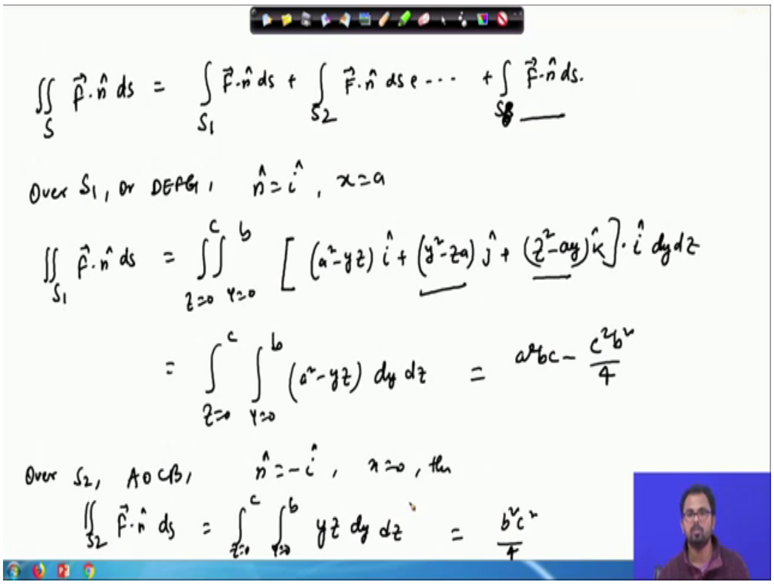

So, basically what would happen is let me give write this term is equals to there will be S 1 F dot nds. So, that is the first surface then the second phase is S 2 F dot nds then the third phase 4th phase dot dot up to S 8 F dot nds all right. So, let us call this phase facing front is our S 1. So, over S 1 or DEFG, our n is basically i and x equals to a right. So, n is i x equals to m. So, therefore, the surface integral over S 1 F dot n ds is basically z running from 0 to c, y running from 0 to b and then x is a.

So, basically a square minus y z times i and then we have y square minus z x. So, x equals to a times j and z square minus a y times k and then our normal is i and dx sorry dy dz. So, when we take dot product with this, these 2 terms will vanish and therefore, we will have we will have z running from 0 to c and y running from 0 to b. So, this will be a square minus y z times dy dz. Now we integrate in with respect to z first and then with respect to y it is up to us and therefore, ultimately we will obtain a square b c minus c square b square by 4. Next we will integrate on the other side.

So, if we so, if DFG was this face, then we will be now integrate with this on the on the surface a OBC and if we integrate on that other surface, then instead of taking I we will take minus I and proceed in the similar fashion. Of course, we have to take x equals to 0 because that is the plain x equals to 0 substitute here. So, over let us say I am calling that one as S 2 or AOCB our n would be minus of i and x would be 0 and then our surface integral would be F dot n ds and this will reduce to 0 running from 0 to c, y running from 0 to b; we will have basically here minus of y z right minus and then minus.

So, this will be basically y z dy dz and if we integrate then this will give us basically b square c square by 4. So, you see when you sum these integrals, then there will be some cancellation. So, surface integral over S 1, then S 2 sum them then there will be some cancellation similarly S3 plus S4. There will be some cancellation S5 S6 ah; there will be some cancellation. So, 1 2 3 4 5 6 oh sorrythere will be only 6 integrals not 8.

So, 12 1 2 then 3 4 and then 56 there will be 6 integrals. So, on these 6 surfaces we basically obtain 6 integrals and then you sum them you will have some cancellation.

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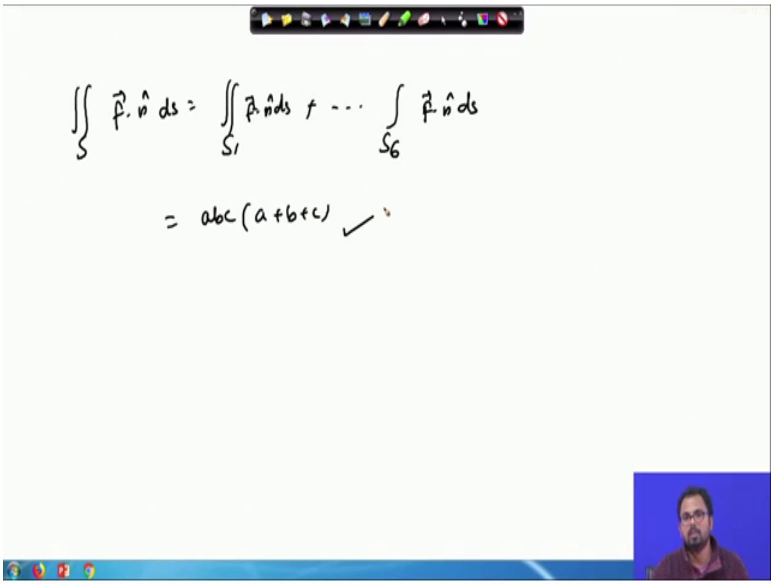

And the finally, our surface integral would look like this over S1 dot F dot nds then S 2 F dot n ds dot dot and so on. So, S 6 F dot n ds it is not S 8 its S 6 and when you write all these and then there will be some cancellation and when you sum them, then its abc plus that times a plus b plus c.

So, this is the required surface integral and therefore, this verification process shows that the Gauss divergence theorem is verified. So, we will stop here for today and will continue with our further integral so, in vector calculus terms. So, next we will look into stokes theorem. So, we will I think we have practiced enough examples on Gauss divergence theorem. So, next we will start looking into Gauss stokes theorem. So, I thank you for your attention and I look forward to you in your next class.