Integral and Vector Calculus
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Lecture – 57
Volume integral, Gauss theorem

Hello, students. So, in the last class we practiced surface integral and we also learnt about Green's theorem where we could connect the line integral with the surface integral in a way and if we have a certain form given let us say in the line integral then from there we can guess them m and n part of this integral. And from there we can convert this line integral, if the if m and n have continuous partial derivatives then we can convert this surface integral this line integral into a surface integral and more how to say in more situations where you can actually convert a complicated line integral into a simple surface integral and vice versa.

So, you may have a how to say complicated surface integral could be and if you see that you can actually integrate it back to m and n then that line integral might become easier. So, it works in both ways it is just that your m and n needs to have continuous partial derivatives. So, we practiced a few examples motivated from this on Green's theorem part and now we will start with volume integral. So, we did surface integral Green's function and then now today we will practice some volume integral examples and then we will move to Gauss divergence theorem, alright.

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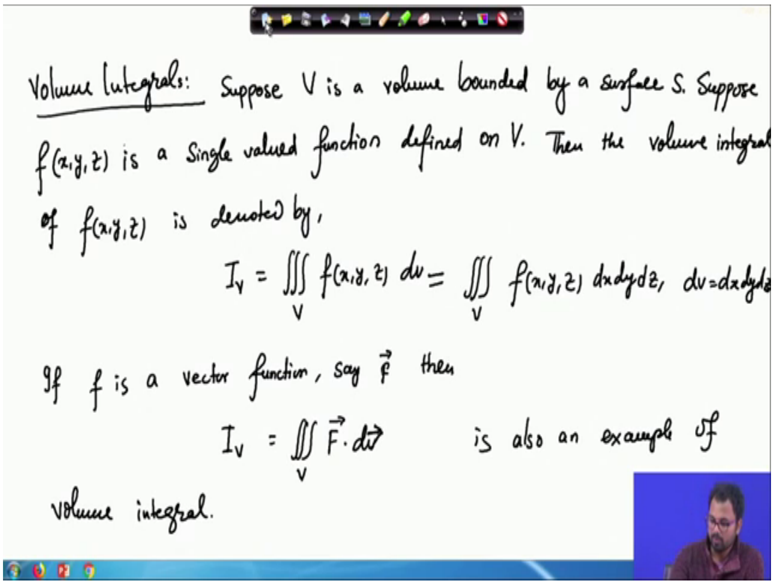

So, volume integral basically mean that suppose the statement goes like this. So, suppose you have suppose V is a volume bounded by a surface S bounded by a surface S suppose f x, y, z be are is a single valued function defined on V. So, that means, f x, y, z is defined on capital V. So, it maps actually from V to set of all real numbers in a way.

And if when we write then the volume integral basically then the volume integral then the volume integral; volume integral; so, it follows the similar motivation what we have learned for the function of one variable. So, in case of function of one variable you have an interval then you divide this interval into several sub n number of sub intervals and then sum of the; sum of the areas of each of these sub intervals when you sum them and when you make n goes to infinity then that gives you actually the integral of that function defined in that interval a to b.

So, its also the same here. So, you basically consider sub volumes on V and then you consider the volume of those how to say sum of those sub volumes times f and then you basically take n tends to infinity and that gives you actually the volume integral. So, the motivation is pretty much same what we have learnt in case of function of one variable or function of two variable. So, I am not writing all those things. I am just writing the notation for the volume integral because our main target here to practice few examples because we have very less number of lectures left.

So, then the volume integral of f x, y, z is denoted by; is denoted by or given by let us say I am writing I V. So, I is for the integral, V is for the volume and then you write triple integrals. So, triple integral is for the volume integral and we write f x, y, z dv, alright. So, here f x and suppose if you if our dv, so, if we divide this V into three into sub into sub volumes then in that case on one of those sub volumes we can take the length breadth and height as dx dy and dz. So, that is basically the volume element in a way.

So, this can be written as integral over the volume V f x, y, z dx dy dz. So, that is basically our dv. So, this is the volume element dx dy and dz. So, this is the required volume element and if capital F if F is a vector function if F is a vector function say capital F then I V is integral capital F dv. So, capital F is a vector function when we are doing the volume integral, then we have to take this dot product is also an example of volume integral.

So, basically instead of dv we take a capital dv and this dv vector, alright and this is also an example of volume integral. So, that is how we write the volume integral. Now, let us see if we can solve some examples. So, I am just looking for an example in my lecture note, yes.

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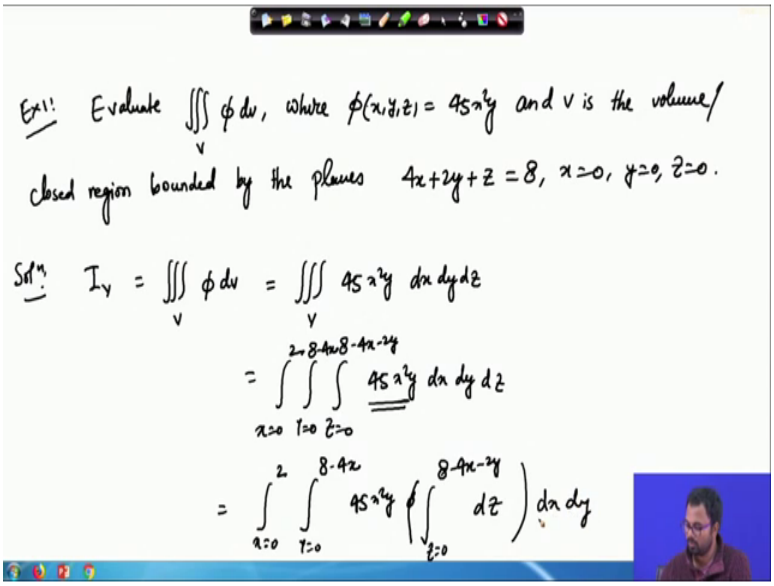

So, to start with let me consider this example evaluate integral over V phi dv where phi is 45 x square times y and v is the volume or a closed region bounded by the plains 4x plus 2y plus z equals to 8 and x equals to 0, y equals to 0 and that equals to 0.

So, obviously, drawing this plane is very easy because it is actually x equals to 0, y equals to 0 and z equals to 0 and bounded by that plane 4x plus 2y plus z equals to 8. So, you can actually be able to draw this in 3D this plane and it is fairly a simple domain basically or a bounded volume all right. So, here we if you want to evaluate the limit for let us say when y and z are 0, then x is varying from 0 to 4. So, x intercept will be 0, 0 and 4 0 sorry x intercept would be 0, 0 and 2, 0.

So, there is a 4 here excuse me, y intercept would be 0, 0 and 0, 4, 0 and z intercept would be 0, 0, 8 in a way. So, you are getting the idea what I am trying to say. So, when you draw this you can actually how does they get the x the point where it is intersecting the x axis, the point where it is intersecting the y axis and the point where it is intersecting the z axis based on that you can be able to draw this plane. So, it is a bounded domain and now, if we want to evaluate the volume integral.

So, as I was saying we can write it as I v and this is volume integral phi dv. So, volume integral over v phi is 45x square y and dv is the volume element dx dy dz. Now, to write the limits for x, y and z how do we calculate the limits? So, first of all our z is varying from 0 to 8 minus 4x minus 2y, then y is varying from if z is 0 then 8 minus 4x and when y and z both are 0, then x is varying from 0 to 2, right.

So, now, to integrate first of all we integrate with respect to z. So, x is varying from 0 to 2y is varying from 0 to 8 minus 4x and then this is 45x square y because the integrand this integrand is independent of z and here z is varying from 8 0 to 8 minus 4x minus 4y here dz and then we have dx dy. So, this will be dz and at z equals to 0 at 0.

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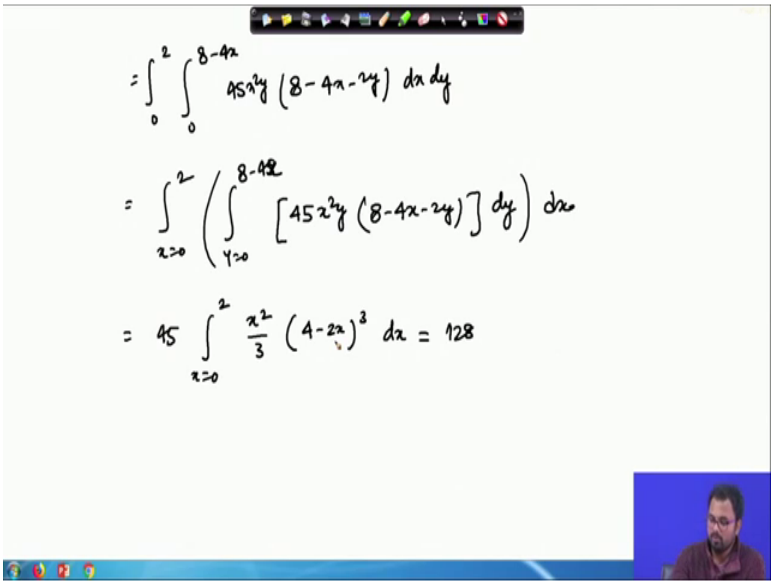

So, ultimately we will obtain the integral 0 to 2 integral 0 to 8 minus 4x and this will be 45x square y 8 minus 4x minus 2y dx dy and now, we integrate with respect to y. So, this will be integral x running from 0 to 2, 45x square or let me write the limit for y. So, I will write the limit for y; y running from 0 to 8 minus 4x and then 8 minus 4x and then this will be 45x square y and this one will be 8 minus 4x minus 2y and this is dy and then we are integrating with respect to dx. Sorry, about the brackets a, but you can use the correct bracket. So, here it should be curly bracket and then the big bracket, but using bracket is not the concern here, so, I am not focusing on that thing.

And now we can integrate this and integrate when we integrate with respect to y. So, this will become. So, we multiply by 45 inside this bracket and then we do the integration. So, everything is pretty straightforward and ultimately we will obtain 45 times x running from 0 to 2, x square by 3 and this will be 4 minus 2x whole to the power 3 dx and when we integrate with respect to x then this whole thing will reduce to 128. So, the required answer is 128.

So, other than doing some complicated arithmetic calculation there is not there is really nothing much in this example. So, what you really had to do just substitute the value of the wall of this integrand here and then guess the and then calculate the limit for z, y and x or z is running from 8 minus 4x minus 2y similarly for y similarly for x and then integrate individually and just do some complicated calculations here. So, that is the only difficult part in here other than that this example is pretty straightforward. So, this is an example or interesting example where we calculated the volume integral next is the statement of Gauss divergence terms. So, let me go to this statement, yes.

So, now if you have given as we saw in this example if you are given a function phi or whatever it is and if you are given the volume V then you can be able to calculate the volume integral. Similarly, if it is a vector function then you take the dot product f dot dv and just calculate the limit substitute the limit and then that will be; that will be pretty much it. So, in either cases it is a very generic example.

Now, we go to a very important theorem in vector calculus and not only in vector calculus it is used in other fields as well. For example, in partial differential equations in some context you use these theorems in particular discourse divergence theorem.

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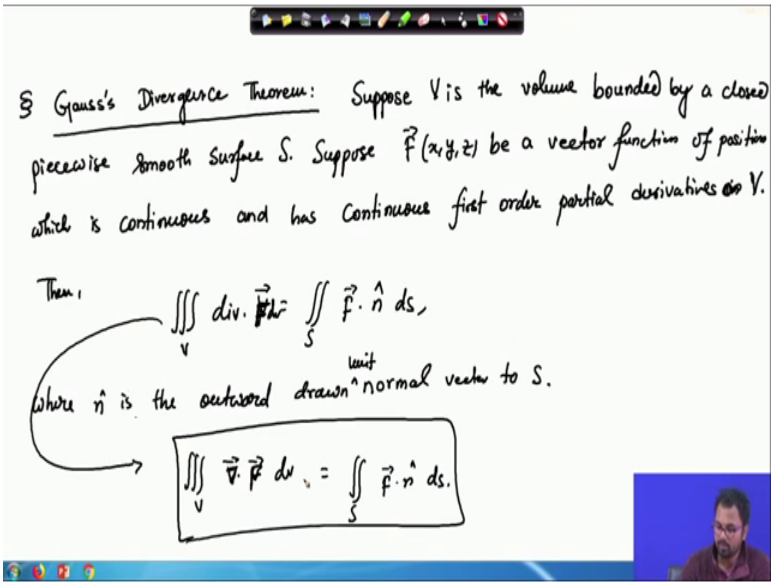

So, let me give you the statement Gauss divergence theorem. So, the statement goes like this. Suppose V is the volume bounded by; bounded by a closed piecewise smooth surface S. So, V is bounded by closed piecewise smooth surface S. So, it can be a parallelepiped.

So, parallelepiped if you consider or a cube then in that case cube is like only piecewise smooth in a way and you have the volume V enclosed inside the cube. So, examples like that or you can have a sphere which does not have any discontinuity or something. So, you can always consider a smooth surface as well or you can also consider a piecewise smooth.

Now, suppose capital F x, y, z be a vector function be a vector function of position which is continuous and has continuous first order partial derivatives partial derivatives; first order partial derivatives in V sorry on V on V then on V then we can write volume integral of divergence of V is equals to surface integral of F dot n ds, where n is outward drawn normal unit normal actually unit normal vector to S.

So, that means, if you have a given volume V which is bounded by a piecewise smooth surface actually say S and if you have a vector function F x, y, z which is defined on V then in that case of course, it also needs to have continuous first order partial derivatives so, that you can write this thing. Then in that case the volume integral of divergence of V is equals to F dot n ds or in terms of notation you can write volume integral the gradient operator times V dv.

So, there is a dv missing dv is a cost to surface integral of F dot n ds. So, this is the required Gauss divergence formula and you need to have just a continuous first order partial derivatives for this vector function sorry this is V F right, so, this is F. So, it is not V this is F.

So, divergence of F dv is equal to F dot n ds. So, similarly this is the divergence of f I thought it is the function V, but it is actually F. So, here you take a divergence of F dv is equal to the surface integral F dot n ds and similarly in terms of notations you can write this divergence of F dv is equals to F dot nds. So, this is a very important theorem in vector calculus and if it has also different how to say varieties in a way I mean you can reformulate these the this equation in several other forms.

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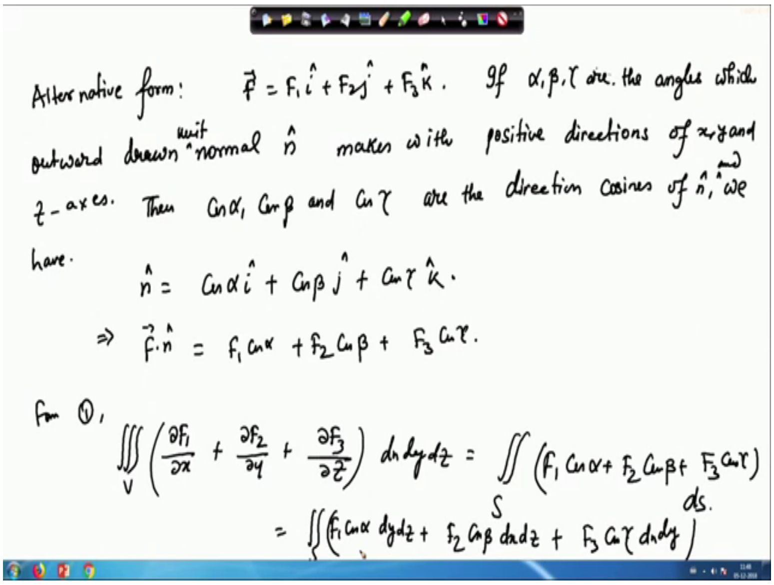

So, if we assume that F has 3 parts; so, alternatively alternative forms alternative form, so, if I assume that our vector function F has 3 components F 1 i, F 2 j and F 3 k let us say and if alpha, beta and gamma be the angles be the angles or are the angles alpha, beta, gamma are the angles which outward drawn normal or unit normal unit normal n makes with positive direction with positive directions of x, y and z axes.

Then obviously, we know that cos alpha, cos beta and cos gamma are the direction cosines then are the direction cosines. This is from our 3D geometry you can have a look in any standard book on 3D geometry. So, these are the direction cosines and of n and we can be able to write and we have; and we have; and we have and so, this is and we have our n as cos alpha i plus cos beta j plus cos gamma k.

So, we can be able to write our and in this fashion. So, therefore, from here F dot n would be; F dot n would be F 1 cos alpha plus F 2 cos beta plus F 3 cos gamma and our required Gauss divergence formula let us call it equation 1. So, there from equation 1, we will have from 1, we will have volume integral divergence of F. So, when I am charging the divergence operator on F then it will be del F 1 by del x plus del F 2 by del y plus del F 3 by del z times dx dy dz equals to right hand side surface integral F dot n ds.

So, this is F 1 cos alpha F 2 cos beta and F 3 cos gamma d s is the surface element. So, you can have ds as depending on the co cos alpha here you can have dy dz this one is dx dz and this one is d this one is dx dy. So, you can I can write here as surface integral F 1 cos alpha dy dz plus this one is F 2 cos beta dx dz plus this one is F 3 cos gamma dx dy. So, this is the right hand side or the surface integral part. So, yes, so, this is how we express this in terms of the Cartesian coordinate system.

So, now nothing is in the vector form everything is in the Cartesian coordinate system and based on this given form we can express this volume integral this in this way. So, this volume integral has a very as I was saying it has a very wide application in vector calculus and in other fields as well. So, will learn about those things later on, but today we will practice a few examples.

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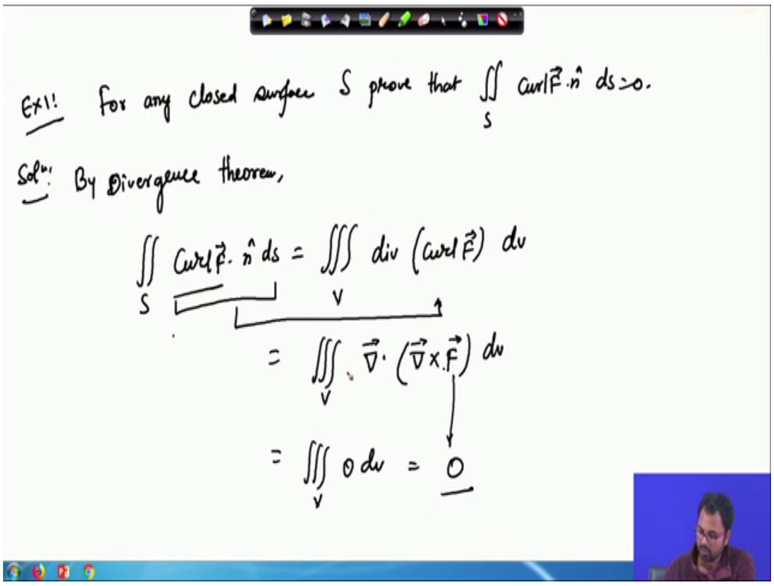

So, to start with let me consider our first example. So, for any closed surface let us consider the first example for any closed surface closed surface S prove that surface integral of curl of F dot n ds is equals to 0. So, this is what we have to prove. So, since we have curl of F; that means that the function F has continuous partial derivatives and the function itself is continuous. So, we have to prove this therefore, by divergence theorem by divergence theorem or gauss divergence theorem by divergence theorem what we will have curl of F dot n ds.

So, now, if we go back to the divergence theorem it is divergence of F and that F is here F dot n ds. So, in this case that F is actually called of F, right. So, I can write volume integral over V divergence of curl of F because our F function is curl of F dv. So, if you calculate divergence of curl of F. So, we can write it in terms of notation. So, this is divergence and this is curl of F and if you calculate so, this one will actually give us a del y. So, if F has three components then this one will give F 3 del del z F 2 del del y and things like that, but we are taking the dot product with respect to with respect to x actually. So, this will be it will be we can show that this thing here will actually be 0. So, we can calculate this and this will actually lead to 0 dv and therefore, the whole thing is 0. So, since the divergence of curl of F is equals to 0 the whole volume integral will be 0.

So, you see instead of doing this complicated calculation we use a Gauss divergence theorem and we arrived from here to here and this is we know how to calculate. So, this we calculate and by doing that our answer is 0. So, of course, using Gauss divergence theorem is very how to say beneficial here.

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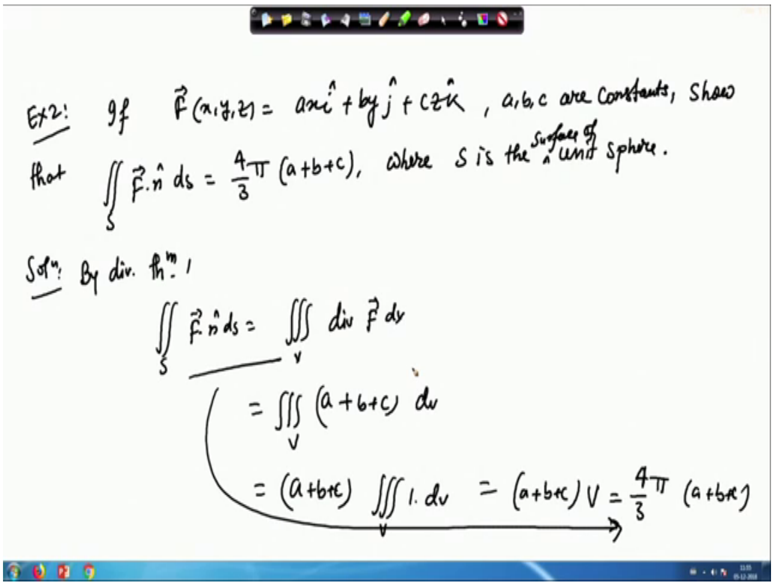

Now, let me consider an example another example. So, example let us say 2, if F x, y, z is equals to axi plus byj plus czk where a, b, c are constants are; constants are constants so that surface integral F dot n ds is equals to 4 by 3 pi a plus b plus c, where S is the unit sphere; where s is the unit sphere, alright is the surface of course, S is the surface of unit sphere S is the surface of unit sphere.

So, here we are given to evaluate the surface integral if we really want to do that we can do that by taking F dot n. So, we have to calculate n from the surface of the sphere. So, the equation of the sphere can be written x square plus y square plus z square equals to one from there we can calculate grad of that expression and then we can calculate n we can calculate n cap taking the projection and all that. So, of course, we can do that.

However, if we want to use the divergence theorem then by divergence theorem we can write surface integral of F dot n ds whatever it is here can be written as volume integral divergence of F dv and if we take the divergence then this will be divergence of F would be just a plus b plus c because every partial derivative will go to the every term and then this will be a plus b plus c times dv and this is basically a plus b plus c times volume integral 1.

Now, V is the volume enclosed V is the volume of that sphere. So, when you are integrating the constant function; that means, you are basically getting the volume V, right. So, this is this is what we will get actually. So, well get basically the volume V, but the volume V is the volume of that sphere and the volume of that unit sphere in this case would be 4 by 3 pi r cube r is 1. So, this is basically 4 by 3 pi a plus b plus c you see just a simple application of divergence theorem has reduced our effort in such an I mean nice manner.

So, we really do not have to calculate any n or do not have to take any projection or anything like that we just had to use a Gauss divergence theorem and by that cause divergence theorem everything reduced into a four line calculation. So, this divergence theorem is proven to be a very handy tool and we will practice few more examples just to show you how nice and convenient this and Gauss divergence theorem is.

So, today we will stop here and in our next class we will continue with the examples on they were just theorem. So, I thank you for your attention and I will see you in the next class.