Integral and Vector Calculus
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Lecture – 56
Green's Theorem & Example

Hello, students. So, in your previous class we started with Green's theorem in vector calculus.

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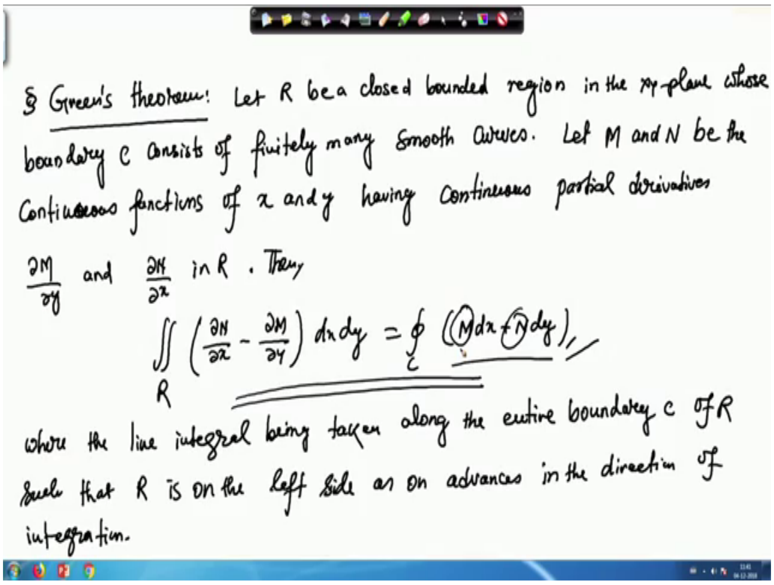

So, we saw that if we if you have two functions say M and N which has continuous partial derivatives with respect to x and y then in that case then in that case you can be able to write this identity del N del x minus del M del y dx dy is equals to is equals to line integral m dx plus N dy and you take the integration this line integral in such a way that the region R which is being enclosed by the curve c is always on the left hand side; so, that means, in a way you are walking in an anticlockwise direction, alright.

So, that was the statement of for Green's theorem today we will verify if you Green's theorem with some examples. The thing is why do we have to study these theorems like Green's theorem, Gauss theorem or Stokes' theorem; so, you can see that on the left hand side you have a surface integral, but on the left hand and the right hand side you have a line integral. So, sometimes it is not easy to evaluate this line integral and if your M and N are have they have continuous partial derivatives then you can basically evaluate you just take their partial derivatives and then you can evaluate their surface integral and most of the time it becomes a very simple expression or you have a let us say surface integral and it has a complicated region and other things. So, what you do you basically identify your M and N and then you evaluate the line integral.

So, it is a very handy tool to convert a surface integral into a line integral or a line integral into a surface integral and sometime. And, most of the time it actually make our life easier to evaluate either one of them when the either one of when their respective integrals are in surface or line and they are complicated in a way. So, just to do this conversion it sometimes we may be able to evaluate the integral easily, but right now we will just verify this theorem.

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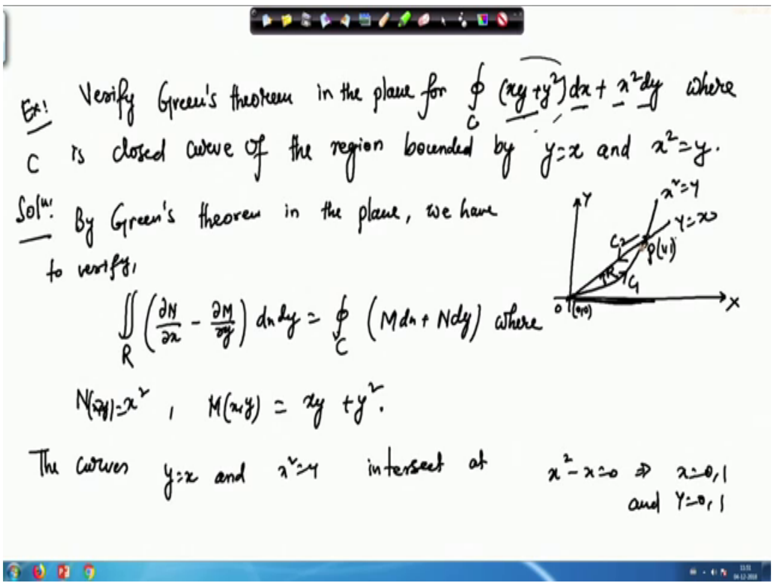

So, let us see how we can do that. So, in order to verify let me start with this example. So, verify Green's theorem verify Green's theorem in the plane for integral over the curve C xy plus y square dx plus x square dy where C is the closed curve of the region bounded by y is equals to x and x square equals to y, alright.

So, first of all we in order to verify the Green's theorem first of all we have to identify what is our M and what is our N. So, let us look at the statement. So, in the statement it says that you have a circulation or let us say line integral Mdx plus Ndy. So, here you have dx and then this thing and then you have dy and then this thing, so; that means, this must be our M and this must be our N. So, by Green's theorem by Green's theorem in the plane; so, what is what is the plane? So, we have x y. So, this is our line y equals to x and then we have a parabola x square equals to y. So, this must be our parabola alright.

So, we have to walk in such a way that the region R must be on the left hand side. So, that means, we are walking in the anticlockwise direction and then this point of intersection let us say P can be obtained. So, they basically intersect at two points; first at origin and then the second one at P. So, by Green's theorem in plane this we have to verify we have to verify del N del x minus del M del y dx dy is equals to surface sorry, line integral C Mdx plus Ndy, where N is x square and xy basically and M xy as xy plus y square, right. So, these are our M and N.

Now, the curves y is equals to x and x square equals to y intersect at; so, we can solve these equations these two equations to obtain the point of intersection. So, if you solve them then basically what we have to do is substitute x square equals y equals to x. So, this will be minus x. So, the two possible points are 0 and 1 and so, y is also 0 and 1 therefore, the intersect at 0, 0 and 1, 1. So, we just solve these two equations and will be able to obtain these the point of insertion intersection at 0, 0 and 1, 1 alright. So, we have these two points of intersection.

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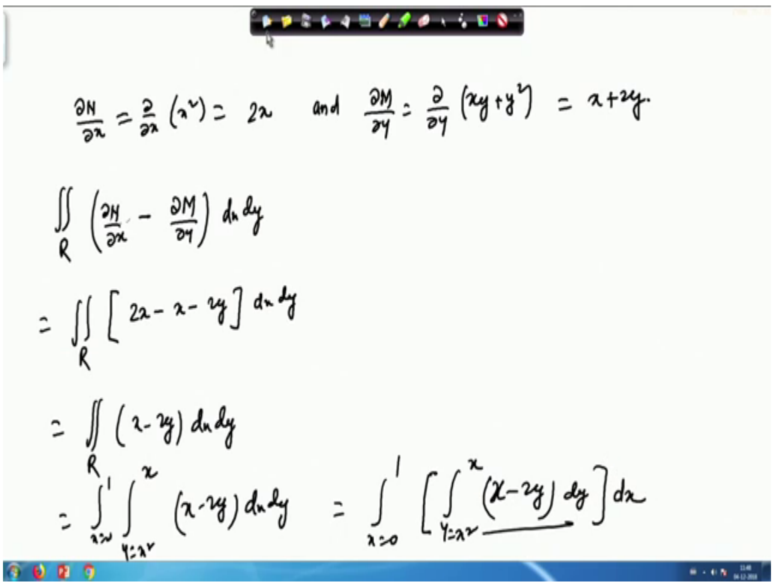

Now, in order to verify the Green's theorem we first calculate del N del x and del M del y, alright. So, del N del x would be del N del x would be del del x of x square. So, this is basically 2x and del M del y would be del del y of xy plus y square, so, this will be x plus 2y, alright. Now, we will calculate the surface integral. So, del N del x which is 2x minus del M del y. Let me write the expression. So, we have del N del x minus del M del y dx dy.

So, surface integral over the region R del N del x would be 2x minus x minus 2y and then this is dx dy. So, this will be ultimately integral over R x minus 2y dx dy, right and now in the region R in the region R our x is varying from 0 to 1, if we see x is varying from 0 to 1 and y is varying from x square to x, right. So, y is varying from x square to x x is varying from 0 to 1. So, let me write those limits here. So, x is varying from 0 to 1 and y is varying from x square to x x minus 2y dx dy.

So, we first integrate with respect to y. So, this will be x running from 0 to 1, then integral y running from x square to x x minus 2y dy and then dx. So, we integrate with respect to y first, substitute the limit and then we integrate with respect to x.

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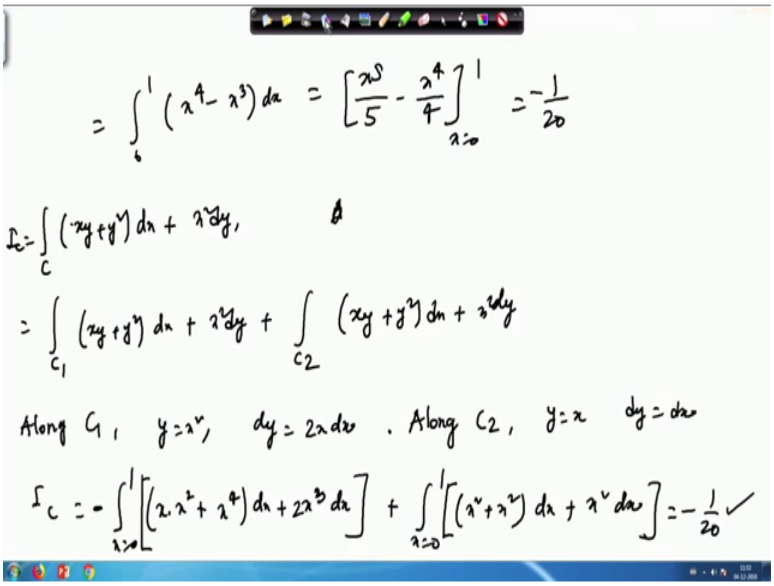

So, ultimately we will obtain after integrating with respect to y and putting the values x to the power 4 minus x cube dx. So, this will be x to the power 5 by 5 minus x to the power 4 by 4 and x will be running from 0 to 1. So, this will be basically 1 by 20 minus 1 by 20.

Now, that is this surface integral, now we will evaluate the line integral to match the values whether they are same or not. So, and for the circulation part we see that of course, it is a closed curve, but it is a piecewise smooth curve actually. So, it is continuous from here to here and then here to here. So, we actually evaluate the line integral along these two paths. So, first we will evaluate along this path and then we will along this path. So, we did some an example like this.

So, if we go back. So, this is our C, M is basically M is basically xy plus y square. So, xy plus y square dx and N is x square dy. So, along we can let us write C 1 xy plus y square dx plus x square dy and then we have another integral C 2 xy over C 2 xy y squared dx plus x square dy. So, what do we have is basically is I am calling this one as C 1 and this one as C 2. So, the part of the parabola is considered as to be C 1 and the part of the straight line is considered as to be C 2.

Now, along C 1 basically now along along C 1 we have y is equals to x square and therefore, dy is equals to 2x dx. So, I will substitute dy is equals to 2x dx here and x will vary from 0 to 1 and similarly, along this line we have y is equals to and along C 2 and along C 2 we have y is equals to x. So, dy is equals to dx. So, we substitute y is equals to x and dy is equals to dx. So, let us call it as I C. So, I C is basically integral over C 1; that means, x is running from 0 to 1 xy y is basically x square plus x to the power 4 dx plus x square 2x to the power 3 dx plus integral x running from 0 to 1, along C 2 y is x. So, this is basically x square plus x square dx plus x square dy; so, dy is basically dx.

Now, we have to evaluate this this line this integral where the limit is from 0 to 1. Let me put everything in the bigger bracket. So, this is our given integral and if you evaluate this whole thing, if you evaluate this whole thing then it will be actually you will obtain as minus 1 by 20. So, this is not very complicated to obtain. So, of course, there will be a minus sign here because forgot to tell you because we are going in the reverse direction. So, if we are going in the reverse direction; that means, we are going from 1 to 0.

So, if we are going from 1 to 0, then I want to reverse the direction I want to go from 0 to 1, so, I have to put a minus sign here. So, this is this is an obvious thing that we have to remember and keeping a minus sign here because initially it was supposed to be 1 to 0. So, now, we are doing 0 to 1. So, a minus sign and then you evaluate and then basically obtain minus 1 by 20. So, you see this is how we verify the Green's theorem.

So, it will be slightly lengthy because all these theorem. So, Green's theorem, Stokes' theorem or Gauss theorem if you want to verify then it will be lengthy because you have to check both sides of the identity. So, here in this case we had to guess what is our M and N and from there we had to calculate del N del x del M del y and then substitute here and simplify the surface integral. Similarly, for the line integral I had to calculate the line integral along two different paths and substitute here and just see whether the two sides are equal or not. So, since the both sides are equal that is; that means, that actually that Green's theorem hold true in this case, alright. So, this was an interesting example where you verify the Green's theorem.

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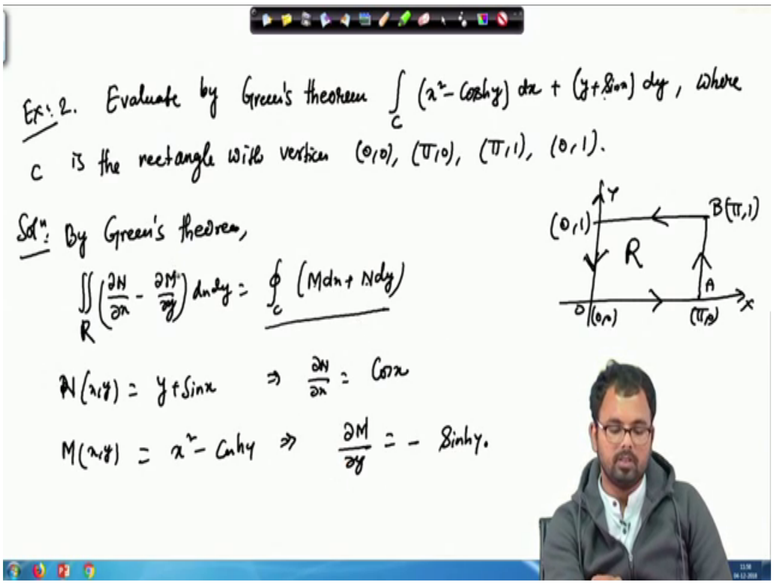

Let us consider an another example where we can use the Green's theorem to simplify a given integral which is slightly complicated.

So, example 2 evaluate by Green's theorem by Green's theorem line integral over C x square minus cos hyperbolic y dx plus y plus sin x dy, where C is the rectangle with vertices 0, 0; pi, 0; pi, 1 and 0, 1, right. So, we basically have a region R. So, let me draw this region R xy. So, this is our rectangle. So, we have vertices 0, 0; pi, 0 which is say A, then b is pi, 1 and then B is pi, 1 and then this one is 0, 1 alright. So, this is our region R this is the curve C along which we have to calculate this line integral.

So, by Green's theorem we know that now we will see first whether by Green's whether the Green's theorem is applicable or not. So, first of all we have x square minus cos hyperbolic y which is obviously, differentiable and I have continuous first order partial derivative and then we have ys y plus sin x which is again having continuous partial derivatives. So, that means, these are our M and N and they both behave nicely. So, by Green's theorem we have to write or we have to when we have been use.

So, we when we use Green's theorem so, this will be del N del x minus del M del y dx dy integral C Mdx plus Ndy. So, that means, since our M and N have a continuous first order partial derivatives I can use Green's theorem and therefore, this line integral can be converted into a surface integral, alright. So, if I convert this line integral into a surface integral then our N is N xy is basically y plus sin x. So, from here del N del x would be cos x and our M xy is x square minus cos hyperbolic y. So, if we differentiate then del M del y would be minus of sin hyperbolic y, right, alright.

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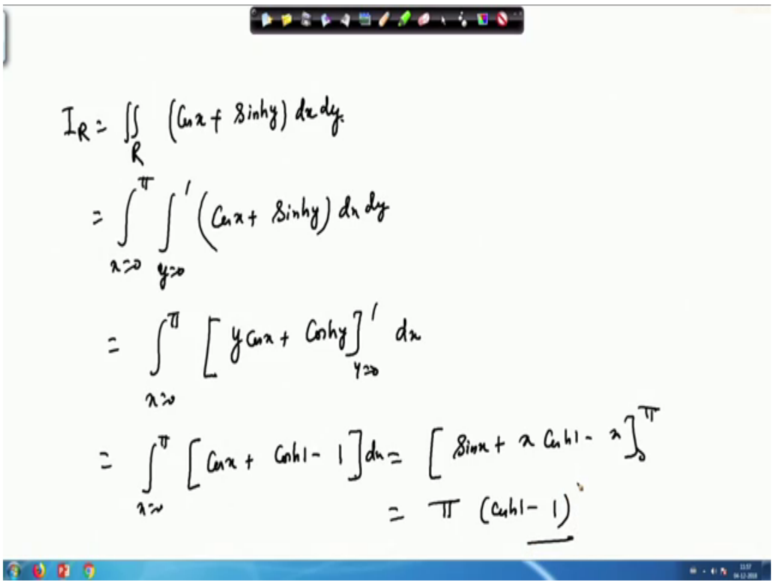

So, then we substitute in this surface integral let us say I R is equal to surface integral over R del N del x is cos x minus plus sin hyperbolic y dy dx and then dx dy. So, then R for the region R, x is varying from 0 to pi and y is varying from 0 to 1 and then we have cos x plus sin hyperbolic y. So, since we have we do not have any product of x and y or something, so that means, here we do not have a function of y or here we do not have a function of x.

So, it is fairly easy to integrate because then we separate the terms and once we separate the terms then this will be basically integral x running from 0 to pi we first integrated with respect to y. So, this will be y cos x plus cos hyperbolic y; y is from 0 to 1 dx. So, this will reduce to integral x running from 0 to pi cos x plus cos hyperbolic 1 minus 1 because cos hyperbolic 0 is 1. So, cos hyperbolic 1 minus 1 and when we integrate then this will be sin x x cos hyperbolic 1 minus x 0 to pi. So, this will be sin pi 0. So, pi times cos hyperbolic 1 minus 1.

So, you see initially we had a very complicated line integral to evaluate, but we just took help of Green's theorem which says that you can be able to use this identity only when M and N have continuous first order partial derivatives with respect to y and x, I believe and since our functions are behaving nicely I just calculated the partial derivatives and put it on the left hand side of this formula and that gave us this nice result, alright. So, this is one such situation, where we can use that theorem and obtain the required obtain the required surface or the required result.

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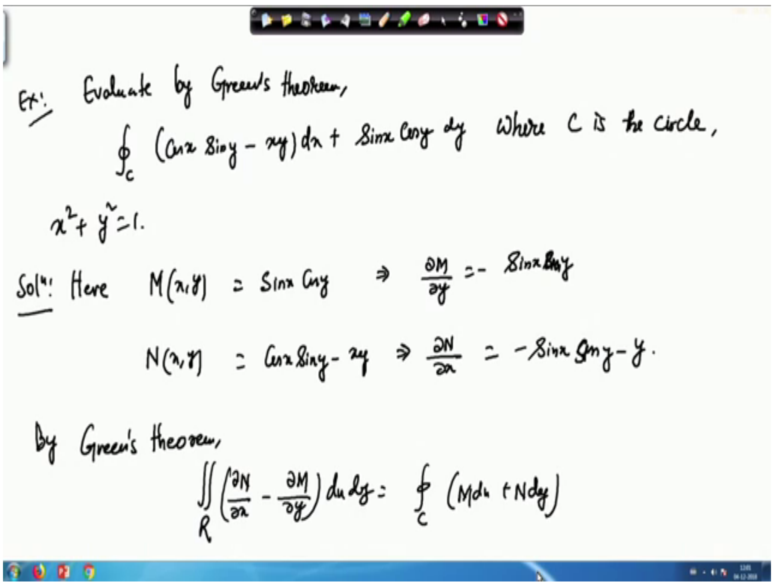

Similarly, we can have an solve an another example. So, let us consider evaluate this example evaluate by Green's theorem. So, here it says specifically Green's theorem cos hyperbolic sorry cos x sin y minus xy dx plus sin x cos y dy where C is the circle, x square plus y square equals to 1, alright. So, here again we are given a very complicated expression to evaluate and obviously, when you are told that you have to use Green's theorem then you really do not have any way out. So, we see how we can use the Green's theorem.

So, first of all we have cos x sin y x y sin x cos phi. So, they are all very nicely behaving functions. So, obviously, they have continuous partial derivatives. So, we do not have to worry about that. So, then we can write here M xy is sin x cos y. So, what will be our del M del y? del M del y will be minus of sin x cos y, alright and cos x sin y sin x cos y. So, if I differentiate, so, this will be minus of sin x minus of sin y sin y. So, this is sin y and N xy is cos x sin y minus xy. So, this will be del N del x and del N del x would be minus of sin x cos sin y minus of y.

So, therefore, by Green's theorem by Green's theorem we have over region R del N del x minus del M del y dx dy is equals to line integral M dx plus N dy. So, del N del x is minus of sin x sin y minus y and this one is plus sin x sin y. So, you see we will basically obtain a very simple expression to solve. So, instead of working with this complicated one we will obtain a very simple expression to sort or to work with.

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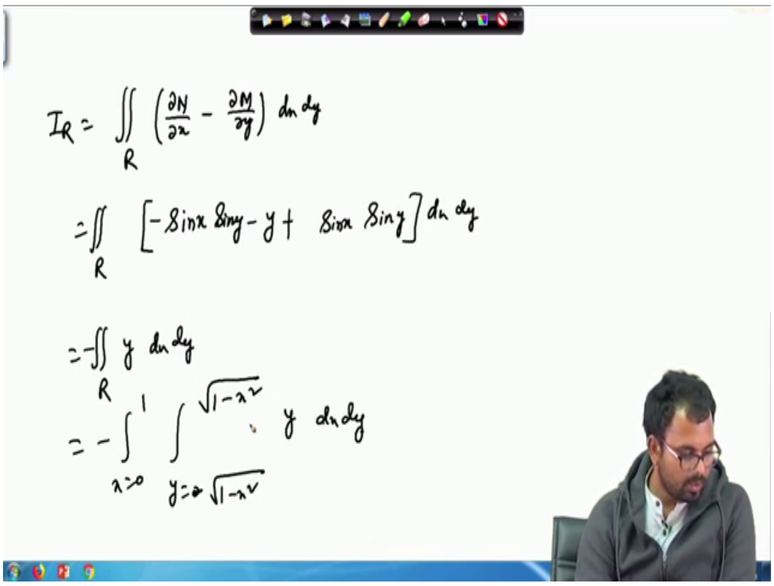

So, let me write IR; IR is over R del N del x minus del M del y dx dy. So, what is our del N del x. So, del N del x is minus of sin x sin y minus y minus of sin x sin y minus y and minus del M del y del M del y minus minus plus. So, this will be plus sin x sin y. So, the both will get cancelled dx dy and then we will basically obtain over region R minus of y dx dy, right.

So, now for this circle R our x will vary from 0 to; so, the equation of this circle is 1, so, our x will vary from 0 to 1 and y will vary from 0 to sorry minus of 1 minus x square 2 plus 1 minus x square y dx dy, right because our that is the range for the y and the range for the x is. So, the range for the x is 0 to 1 or what we can do. So, here you can integrate and then we basically obtain so, we basically obtain here so, del N del x. So, this is my M oh sorry. So, this is our M and this is our N. So, that M; so, this is my M. So, del M del y is cos x sin y.

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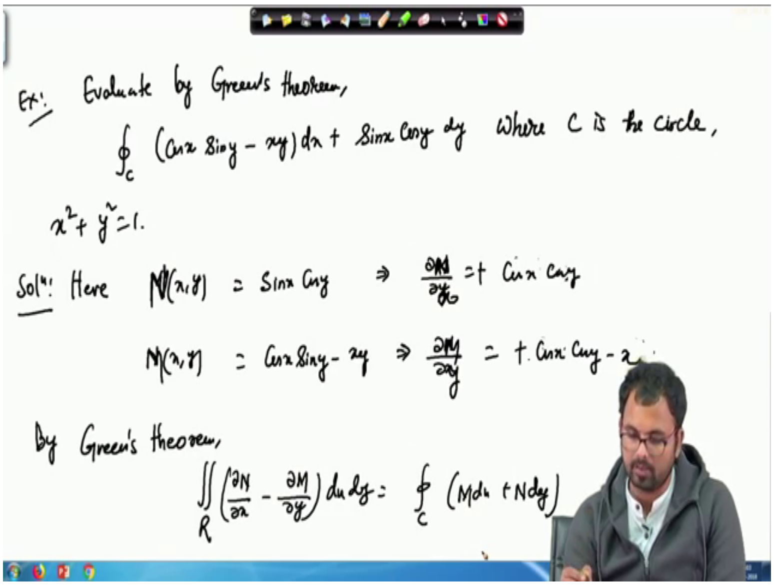

So, this is cos x, excuse me. So, this is cos x. So, this will be I think I took the wrong M actually. So, this will be cos x cos y minus of x and this is my N, yeah this is my N. So, this will be del N del x. So, this is del N del x and del N del x is cos x cos y, yes. So, this will be cos x cos y. So, this will be cos x cos y. So, if now if I substitute then this will be minus minus and then plus, yes. So, this is basically. So, ultimately here we will obtain cos x cos y.

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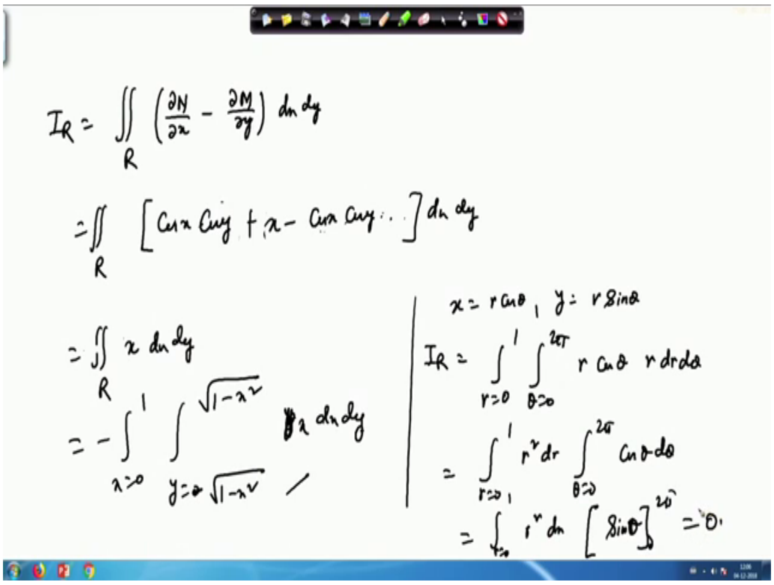

So, we will obtain cos x cos y cos x cos y plus x plus x minus of cos x cos y. So, ultimately we will obtain here is x dy dx, right. So, we will obtain basically x dy dx. So, either we can either we can have x 0 to 1 and y running from minus 1 minus square root of 1 minus x square to plus 1 minus x square x dx dy or we can substitute x equals to r cos theta and y is equals to r sin theta.

So, then in that case the surface integral I R would reduce to integral over the region R our r will vary from 0 to 1 and theta will vary from 0 to 2 pi x will be r cos theta and dx dy will be r dr d theta and then basically this will be if we integrate cos theta then this will reduce to r running from 0 to 1, r square dr and then we will integrate 0 to 2 pi cos theta d theta.

So, if I integrate cos theta then it will be a sin theta and sin theta theta equals to 2 pi. It will be basically it will be basically 0 because at t equals to 2 pi and theta goes to 0, this whole thing is 0. So, ultimately the value of the integral is 0. So, here you can see that for this given vector for this given expression here where we had to calculate the line integral for this curve C, I guess the function incorrectly. So, this is my M and this is our N. So, we have to calculate del N del x which is cos x cos phi and then here we have to calculate del M del y which is cos x cos phi minus x.

So, we substitute the whole thing in this expression, on the left hand side and then we do some calculations and either we can proceed in this way or we can we can proceed we can proceed in this way it is up to us and we just substitute x equals to r cos theta y equals to r sin theta and the ultimate answer is 0. So, looking at this integral it would not have been easy to guess the limit that it will because the value of the integral not the limit to the value of the integral that it will be 0. But, it is just that taking help of Green's theorem, we can be able to show that the value of the integral is 0 by doing some simple partial derivatives.

So, you got the idea that what we have to do actually. There are there might be some errors here and there while doing the calculation. I hope you would understand. But, yeah the message is that whenever you have a complicated line integral given to you and if your M and N seems to be having continuous partial derivatives then try to use Green's theorem and there is a strong possibility if you use the Green's theorem then the whole integral will reduce to a very simpler one and you just like in this case or in the previous case you just do some simple known formula or known calculation to obtain the value.

So, today we tried to learn about Green's theorem and we also saw its how to say efficiency that how you can use it and how it makes our life easier while calculating the integral surface integral or line integral. So, I will stop here for today now and in our next class we will start with the Gauss divergence theorem. So, I thank you for your attention.