Integral and Vector Calculus
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Lecture – 55
Surface integral (Contd.)

Hello students. So, yesterday we were looking into the concepts of or in our previous class, we were looking into the concepts of surface integral and circulation of function around a given curve and things like that. So, today we will continue practicing those examples and then if time permits, then we will move on to our next topic which is basically a Green's function and then or Green's theorem actually. So, the Green's function is actually in some other topic.

So, we will look into Green's theorem in the vector calculus context. So, first of all, let us start with our example on circulation actually.

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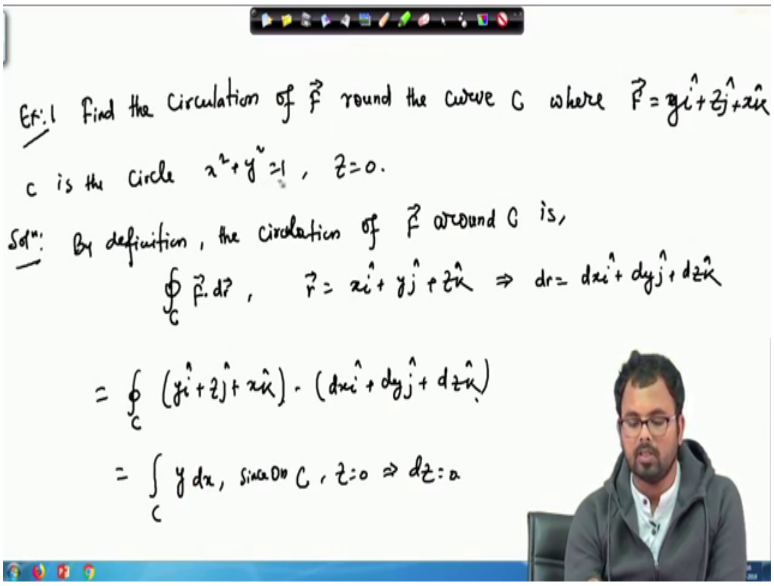

So, our first example is find the circulation of F around round the curve C where F is equals to y i z j plus x k and C is the circle x square plus y square equals to 1 and z equals to 0. So, basically here we have to find out the circulation of this function F around curve C where we have the given function F x, y, z it goes to this.

So, of course, F is a function of x, y and z. So and C is the circle given by x square plus y square equals to 1 and z equals to 0. So, basically our curve is in the x, y plane and the given equation of the curve is basically a circle. So, by definition we know that, so by definition of circulations, so by definition the circulation. So, since it is a circle which is a simple closed curve, so any line integral along that simple closed curve will be circulation. So, by definition, the circulation of F around C is this notation F dot d r. So, this can be written as where r is basically x i plus y j plus z k. We know that r is always given by this way and therefore, from here we will have d r is equals to d x i plus d y j plus d z k, right.

Now, here if I write integral over the curve c, y i plus z j plus x k and then this can be written as d x i plus d y j plus d z k. Now, since we are in the x, y plane, so z is 0 and d z is 0. So, if I substitute z is 0 here and these are 0 here, then this will lead to integral over c this is 0, this is 0. So, ultimately we will have y d x because on C, since on C z is 0 and this implies d z is also 0. So, we will basically substitute z equals to 0 here and d z equals to 0 here. So, we will end up with integral over C y d x.

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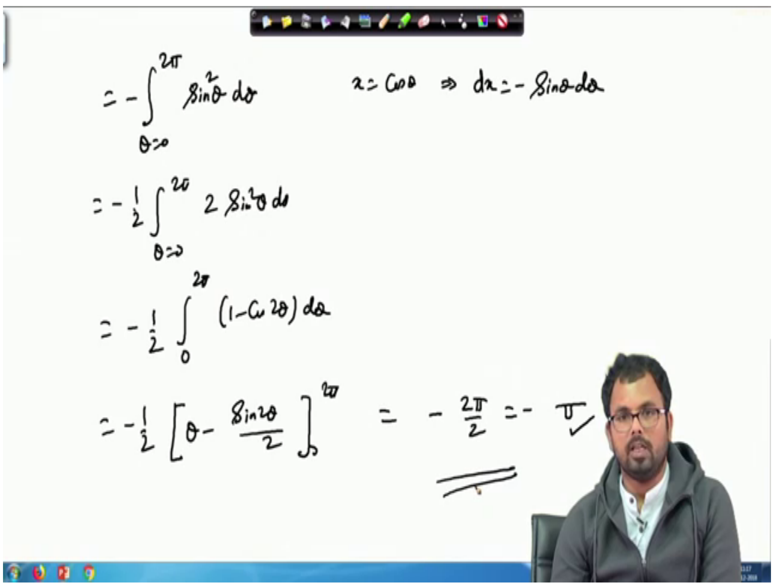

Now, if I substitute x x equals to cos theta and y equals 2 sin theta, so this is basically integral over the curve C. So, in this case the parameter theta would vary from 0 to 2 pi. I am substituting x equals 2 cos theta y equals 2 sin theta. So, this is sin theta and then if x equals 2 cos theta, then d x is minus of sine theta d theta. So, this will be minus of sine square theta d theta. So, I can adjust a minus half here and then this will be theta running from 0 to 2 pi to sin square theta. And then, this can be written as 1 minus cos 2 theta, 1 minus cos 2 theta and if we integrate d theta, then this will be minus half this will be theta minus sin 2 theta by 2 integral from 0 to 2 pi.

So, sin 2 pi is sin 4 pi 0, sin 0 is 0. So, this will be ultimately minus of 2 pi by 2. So, that is minus pi. So, this is the required circulation of this function F around this curve C given by this circle. So, circle is a simple closed curve. So, the line integral will be a circulation. So, this is how we calculate the circulation of the function F, alright. Now, we will continue with our examples on a surface integral.

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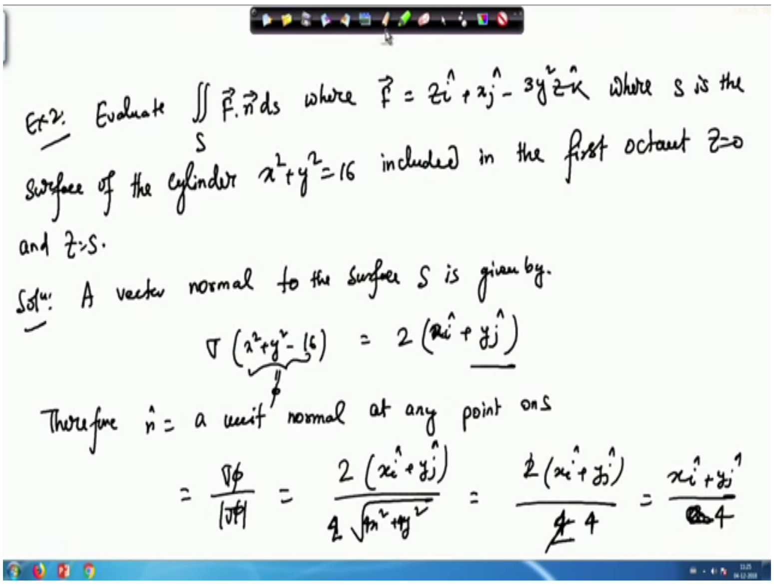

So, let me write an example, let me write a problem here. So, evaluate integral over the surface S F dot n d s where F of x, y, z is given by z i x j minus of 3 y square z k where S is the surface of the cylinder x square plus y square equals to 16 included in the first octant z equals to 0 and z equals to 5. So, basically x squared plus y squared equals to 16. So, that is a circle and then you have z equals to 0 and z equals to 5. So, it is a cylinder.

Now, if you consider the part which is included in the first octant; that means, the circle in the first octant and the length z equals to 0 to z equals to 5 at the, I mean like the height of that cylinder. So, whatever lies in the first octant of that cylinder; so, that is the surface S and our given vector function F is this one. So, first of all if we have a given vector function F, so from there we have to and if you have the surface s, then from there we have to find out the normal n. So, a vector normal to the surface we, if you remember then yesterday we basically calculated the gradient of this x square plus y square minus 16. That is our given surface.

So, a vector normal to the surface S is given by gradient of x square plus y square minus 16 and this will be 2 times del x. So, this is basically x i plus sorry y j, yes. So, this is the required normal and from here the unit normal. Therefore, the unit normal therefore, n cap a unit normal basically, unit normal at any point on S; at any point on S will be given by a gradient of let us say this one was our phi. So, if this one was our phi. So, grad phi divided by mod of grad phi. So, this is 2 x i plus y j and divided by x square plus y square.

Now, x square plus y square is basically 16. So, this will be 2 x i plus y j divided by 4. So, this is x i plus y j divided by 2. So, this is our unit normal n. Now, since we have a cylinder which is whose base is x squared plus y square equals to 16 and height is 5. So, we take a projection either on x, z plane or y, z plane.

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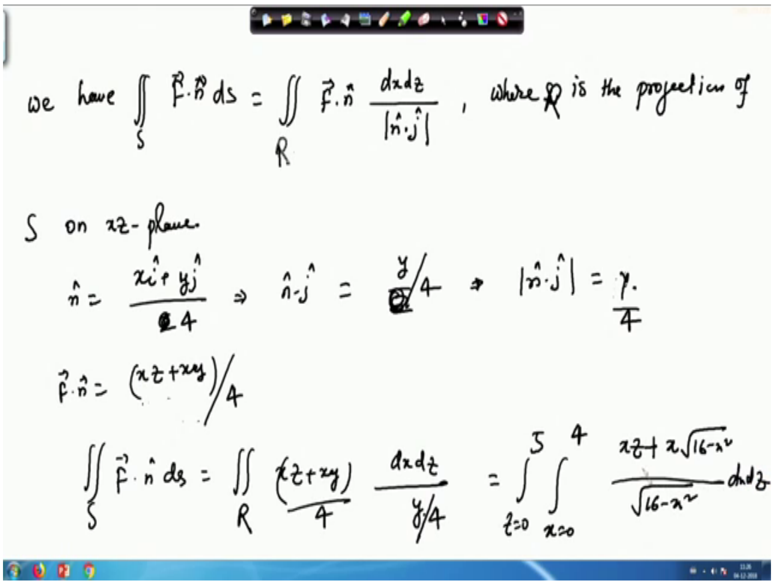

So, if we take the projection on x, z plane, so we have in surface integral over S F dot n d s or let us say n cap. So, this can be written as surface integral over S F dot n. And if I take the projection on x, z plane, so d x d z and then dot product will be n dot j, right.

So, this is our required dot product where R is the and this one can be let us write as S 1, so where S 1 is the projection or let us call it as R, so S for the surface and R for the region, so where R is the region, where R is the projection of S on x, z plane. So, that is the projection on x z plane and we cannot take the projection. So, we cannot take the projection of S on x, y plane as the surface S is perpendicular to the x, y plane. So, we simply cannot take the projection because since the cylinder is perpendicular to x, y plane. So, if we take the projection, then it will be just a circle. So, I mean we exclude that cylinder part here.

So, that is why we take the projection either on x x, z plane or y, z plane. So, that is that and now our n is, our n is x i plus y j divided by 2. So, from here n dot j would be just y by 2 and n dot j mod will be simply y by 2 right. And F dot n is basically F dot n is our F is given by z i plus x j minus this. So, when we take the dot product, so the k component is 0. So, that will be 0 and therefore, this will be x, z plus x, y by 2 right. So, this will be x, z plus x, y by 2 and now, x, z plus x, y by 2 right. So we have this here, so this is 4, so here I am getting 4.

So, 2 2, did I do this calculation correctly? So, this will be 4 x square 4 y square. So, this will be a square root of 4. So that means, 2. So, there will not be any 2 here. So, just x i plus y j and that means, here we will have just simply y. So, this will be just y I am sorry. So, this will be just y and then here we will have x, z plus x, y.

So, when we take the dot product, now we substitute all these values here. So, our required surface integral S equals to F dot n d s which is equal to integral over region R, F dot n is x z plus x y and this one will be d x d z n dot j. So, n dot j is simply y. And now, if we take the projection on r, then this will be integral over the region r, z will be running from 0 to 5 and x will be running from 0 to 4 x, z plus x, y. So, y will be 16 minus x square. Wait, so this is here 4; so, this is 2 and this will be x square plus y square. So, here it will be 4 right, yes. So, it will be 4 here and then there will be a 4 here. So, y by 4, this is also y by 4 and this is y by 4. So, this will be by 4 and this is by 4. And next, we will have x, z plus x 16 minus x square divided by y. So, y is also 16 minus x square d x d z.

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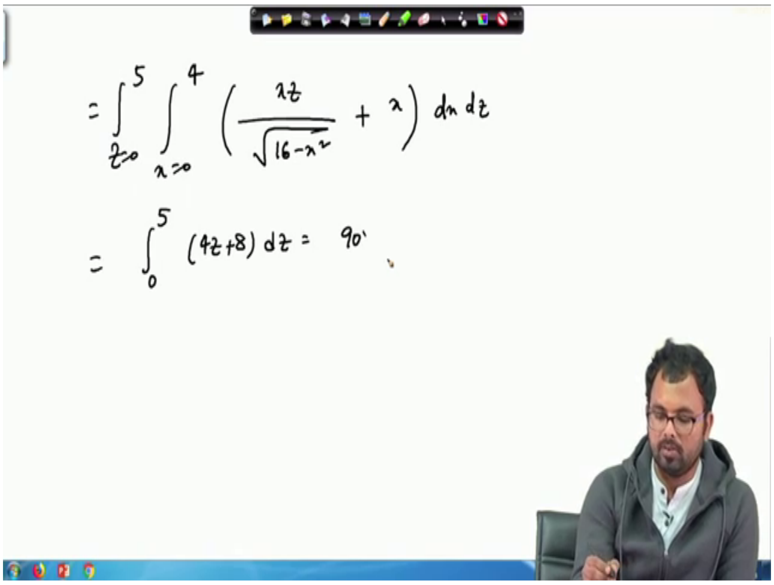

Now, this can be written as if we simplify, then this will be written as z running from 0 to 5, x running from 0 to 4 right in the x, z plane and this will be x, z divided by 16 minus x square plus x d x d z. So, we did the separation. So, basically the partial fraction and now we can integrate this very easily. So, we basically do the integration with respect to x first and so, this one will be x square by 2 and this one will be if we substitute this equals to z. So, that will be just square root of z and all that.

So, doing a simplification from here is not complicated. So, we will basically at the end obtain 0 to 5 4 z plus 8 times d z which is equals to 19. So, this is the required surface integral. So, here there is a small mix up. So, it is just that here it will be x square plus y square, this 2 will come outside. And since x square plus y square equals to 16, so it will be 2 times 4. So, 2, 2 will get cancel. So, we will obtain simply 4 yeah sorry about that mix up. And then when we take the dot product, so then this will be y by 4 and this will be x j plus x, y by 4. So, ultimately they get cancel. So, they really do not affect anything. It is just that yeah, we have to be a little bit careful.

And now, substituting y equals to 16 minus x square and here also, then we do the partial fraction and then this is ultimately 19. So, this is how we calculate the surface integral of a given surface integral of a given function F whose surface S is given by that circle and cylinder. So, that was another interesting example. We can consider another example on surface integral before we move to the Green's theorem.

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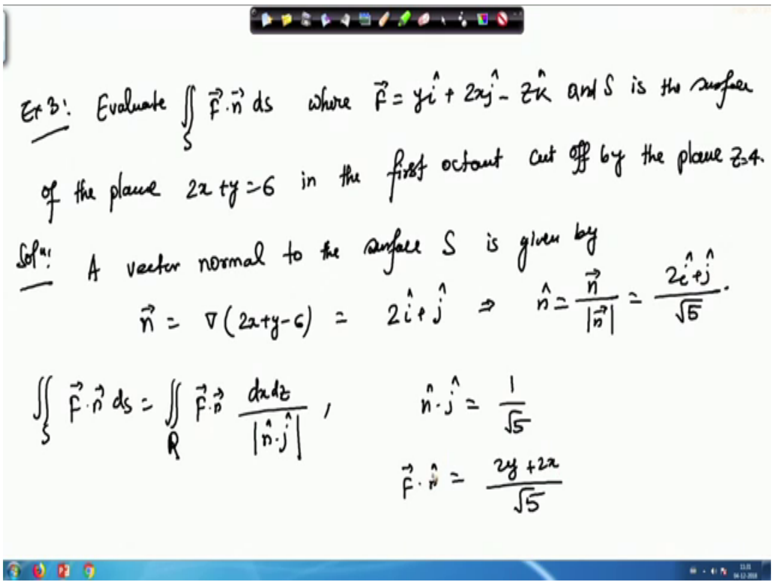

So, another example, evaluate where F is equals to y i 2 x j minus z k on S and S is the surface of the plane 2 x plus y equals to 6 in the first octant cut off by the plane z equals to 4. So, here we have a given equation of the given this equation vector function actually and S is the surface of the plane in the first octant cut off by the plane z equals to 4 right. So, that is the given equation of the surface.

So, now, first of all we will calculate the normal. So, a vector normal to the surface S is given by let us say, n equals 2 gradient of 2 x plus y minus 6. So, this is ultimately 2 i plus j and therefore, from here our unit normal is n by mod of n. So, this will be 2 i plus j square root of 5, right. So, this is our required normal, unit normal actually. And the given surface integral S can be written as so, since we have to take a, we have to take a projection. So, in this case also we take the projection either on x, z or y, z plane.

So, we cannot take the projection on x, y plane because the surface S is, because the surface S is perpendicular to the x y plane. So, here we take the surface integral and we take the projection. So, that is the region R where the projection will be F dot n and d S will be d x d z divided by n dot j.

So, now n is given in this fashion. So, n dot j would be our n dot j would be 1 by square root of 5 right and F dot n would be, F dot n would be 2 y plus 2 x divided by square root of 5; the k component is 0. So, dot product after taking the dot product, the k component will not occur here. So, that is what we will get F dot n and that is what we will get as n dot j. So, we substitute everything here.

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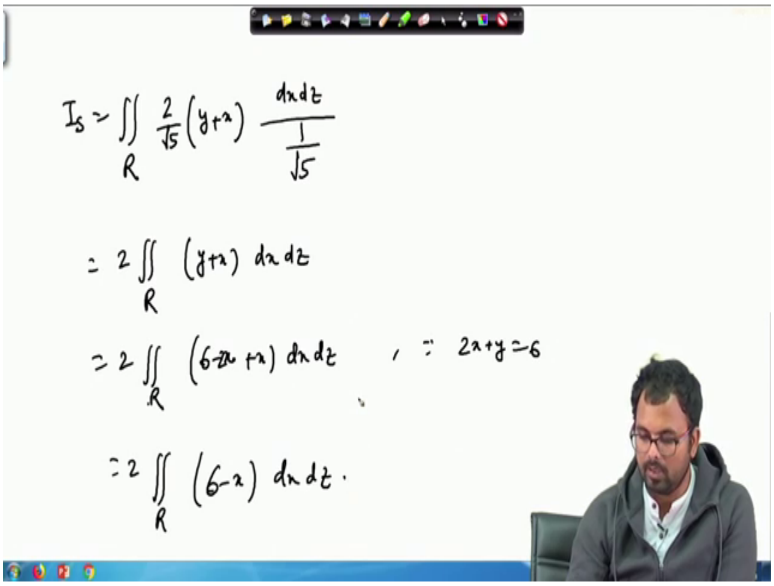

So, it will be let us say let us say I S, I S is our surface integral. So, over the region R F dot n is 2 square root of 2 by square root of 5 y plus x times. So, it is y plus x time; so, 1 d x d z divided by 1 by square root of 5, right. So, square root of 5, square root of 5 will get cancel and this will become one by sorry two times in integral over the region r y plus x d x d z. Now, y can be written as over region R, y can be written as 6 minus x right. So, here where is that? So, our y is 6 minus 2 x actually. So, this is basically 6 minus 2 x plus x times d x d z since 2 x plus y equals to 6. And now this will be 2 times integral over the region R 6 minus x times d x d z. Now, x if we take the projection on x, z plane, then in x, z plane basically y is 0. So, x will vary from 0 to 3 and z will vary from 0 to 4.

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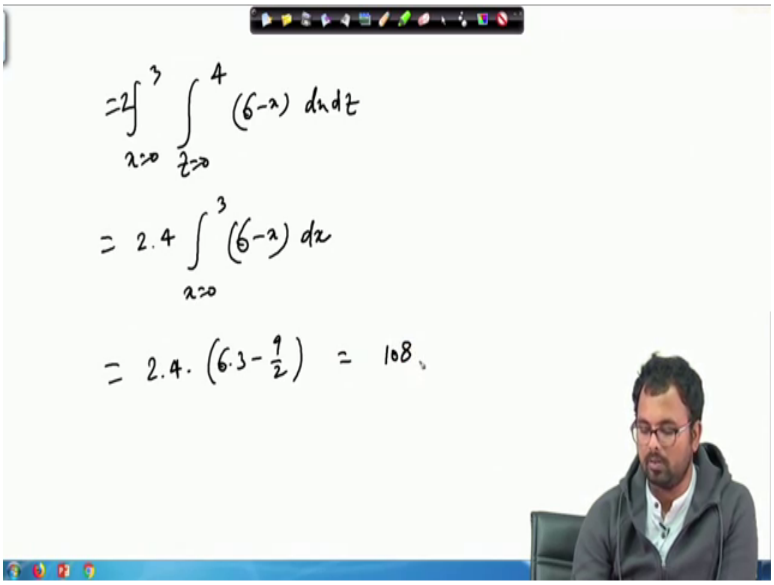

So, I can write the integral as x varying from 0 to 3 and z varying from 0 to 4, 6 minus x d x d z here there is a 2 and we can integrate with respect to is z first. So, it will be 2 times 4 and then we integrate with respect to x. So, this will be 6 minus x times d x and. So, this will be 2 times 4. It is 6 x minus x square by 3. So, this is 6 times 3 minus 9 by 2. So, ultimately, it will be 108. So, that is the required surface integral.

So, like this here you can practice many more examples where you have a given vector function. So, you take the projection on x, y plane y, z plane or z, x plane. You have to find out the normal n which is perpendicular to the surface and then everything is just doing some simple calculation. So, after doing the projection, you get to know what is the limit for x or y or z depending on your projection and then just perform the integration it is it is from then on onwards it will become very simple. So, I will move to our next topic because I think we have a practiced enough examples on surface integral. And next, we will move to our next topic which is basically Green's theorem.

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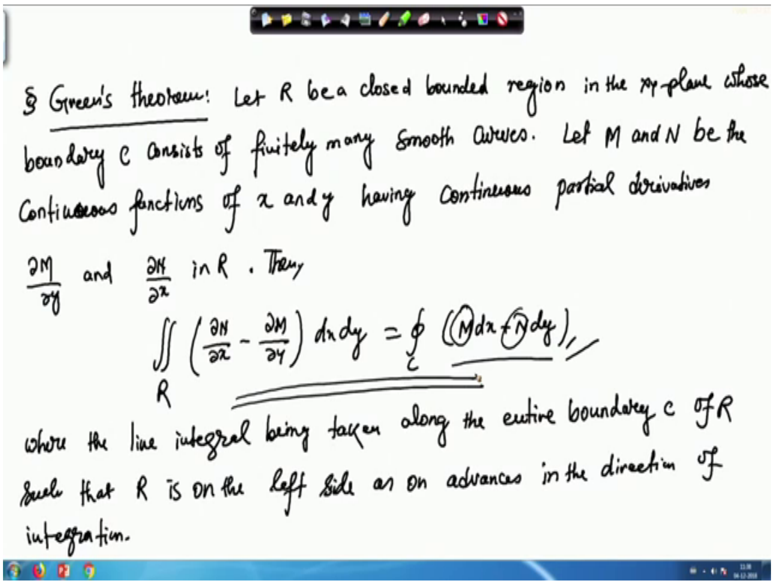

So, Green's theorem states that let R be a closed bounded domain or closed bounded region, closed bounded region in the x, y plane whose boundary C consists of finitely many smooth curves, smooth curves. And let M and N be the continuous functions of x and y having continuous partial derivatives, continuous partial derivatives del M del y and del N del x in R. Then so, basically then surface integral over the region R del n del x minus del m del y d x d y is equals to circulation over the closed curve C M d x plus N d y.

Where the line integral, where the line integral being taken along the entire boundary C of R such that R is on the left side as one advances in the direction of integration. So, basically R is a closed bounded region in the x, y plane. And suppose we have two functions M and N which are continuous functions of x and y which has continuous partial derivatives del M del y and del N del x in R.

Then, if it has continuous partial derivatives, then we can write del N del x minus del M del y d x d y is equals to line integral over the curve C M d x plus N d y where the line integral is taken in such a way that when you walk along this curve C, then your region are will always be on the left hand side. So, that means, it you always have a anti clockwise direction. So, you the way you are walking along the boundary C is such that the region R which is being closed by the boundary by the curve C is always on the left hand side. So that means, you always walk along how to say along the anti clockwise direction and this is what this theorem says and all you need to have is that M and N should have continuous partial derivatives of order 1 and then this identity would hold.

So, we can actually verify this identity, it is also one of the important theorems in vector calculus. And we can be able to verify this by using some by using some formulas and some calculus, calculus results. So, I will stop here from for today and in the next class we will work out one or two examples in Green's theorem and we will see how it is, how it is being solved. So, thank you for your attention and I look forward to you in your next class.